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On the Boolean Structure Generated by Q-Points of ω^*

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We prove that under $\mathfrak{p} = \mathrm{cf}(\mathfrak{c})$ is $\mathrm{RO}(\mathscr{P}(\omega)/\mathrm{fin}, \subseteq^*)$ isomorphic to the Boolean completion of the partial order of nowhere-dense subsets of ω^* defining Q-points (ordered downwards by inclusion).

Introduction and motivation

In this paper we study Boolean properties of a (naturally defined) ordering of the system of nowhere-dense subsets of ω^* which defines Q-points in the sense, that Q-points of ω^* are exactly those points of ω^* (ultrafilters) which are not in the union of these nwd sets. This study is a continuation of a work which originally arose from two different motivations.

The first motivation is that of [V2] namely to study natural partial orders (e.g. absolutely convergent and divergent series ordered as in comparison and ratio comparison test) from set-theoretic and Boolean-theoretic point of view. In [V2] it was shown that under $\mathfrak{p} = \mathrm{cf}(\mathfrak{c})$ ($\omega_1 = \mathrm{cf}(\mathfrak{c})$ resp.) Boolean completions of these ordering of divergent (convergent resp.) series are isomorphic to $\mathrm{RO}(\mathscr{P}(\omega)/\mathrm{fin})$ – the Boolean completion of the algebra of subsets of natural numbers modulo the ideal of finite sets.

The second motivation is that of [V1], namely a new type (besides topological and combinatorial) of definitions of points of ω^* as those outside of the union of a system of nowhere-dense subsets of ω^* (which leads to new existence theorems for points of ω^*). These systems of nowhere-dense subsets of ω^* are those connected to the definition of the very point, i.e. filters on ω which are connected to series, partitions, etc.

Moreover, in [V1] these two motivations met in an observation that the ordering of divergent series is the same as the ordering of nowhere-dense system induced

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by series. So, in general (see [V1]) having $\mathbb{F}$ a system of nowhere-dense subsets of ω^* it defines points, which we can call $\mathbb{F}$-points (in special cases these are rapids, Q-points, etc.) laying outside the union of $\mathbb{F}$. That is, $j \in \omega^*$ is an $\mathbb{F}$-point iff $j \in \omega^* \setminus \bigcup \mathbb{F}$. Considering $\mathbb{F}$ as being ordered by inclusion upwards the dominating number $\mathfrak{d}(\mathbb{F}, \subseteq)$ is the number of nwd sets necessary to cover the same portion of ω^* as the whole $\mathbb{F}$ does. By this way we get existence theorems of type $\mathfrak{n}(\omega^*) > \mathfrak{d}(\mathbb{F}, \subseteq)$ implies there are $\mathbb{F}$-points ($\mathfrak{n}(\omega^*)$ is the Novák number i.e. the minimal number of nwd sets necessary to cover the whole ω^*).

Further, the system $(\mathbb{F}_r, \subseteq)$ defining rapid ultrafilters was shown in [V1] to be Boolean isomorphic (after completion) to $\mathscr{P}(\omega)/\text{fin}$. So a new type of problems occured, namely, having a system $\mathbb{F}$ of nwd subsets of ω^* ordered by inclusion, look to it downwards and ask about the Boolean type of this ordering.

In this paper we investigate the Boolean structure of $(\mathbb{F}_q, \subseteq)$, where $\mathbb{F}_q$ is the (canonical) system of nwd subsets of ω^* defining Q-points and we show (surprisingly) it is again isomorphic to that of $\mathscr{P}(\omega)/\text{fin}$ (after necessary completion).

Notations

Let ω denotes the set of natural numbers, $[\omega]^\omega$ is the system of all infinite subsets of ω, $[\omega]^{<\omega}$ is the system of all finite subsets of ω, $\mathscr{P}(\omega)/\text{fin}$ is the Boolean algebra of subsets of ω modulo ideal of finite sets (sometimes seen as $[\omega]^\omega$). The Stone space of algebra $\mathscr{P}(\omega)/\text{fin}$ is denoted $\omega^* = \text{St}(\mathscr{P}(\omega)/\text{fin})$ and equipped with the topology generated by base consisting of sets of form:

$$A^* = \{j : j \text{ is a uniform ultrafilter on } \omega \text{ and } A \in j\},$$

where $A \subseteq \omega$.

For an ideal $\mathscr{I}$ on ω, $\mathscr{F}_{\mathscr{I}}$ denotes the dual filter. Filters on ω can be viewed (represented) as subsets of ω^* in the following way:

$$\delta(\mathscr{F}) = \bigcap \{A^* : A \in \mathscr{F}\}$$

is the closed set corresponding to $\mathscr{F}$. Note that $\delta(\mathscr{F}_{\mathscr{I}})$ is nowhere-dense iff $\mathscr{I}$ is tall (i.e. $(\forall X \in [\omega]^\omega)(\exists Y \in [X]^\omega)(Y \in \mathscr{I})$).

The set $\mathscr{R} \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ is said to be a (finitary) partition of ω if $\bigcup \mathscr{R} = \omega$ and elements of $\mathscr{R}$ are pairwise disjoint. $\mathbb{R}$ is the system of all (finitary) partitions of ω. (In following we omit the adjective finitary.) The set $\mathscr{A} \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ is said to be a partial partition of ω if elements of $\mathscr{A}$ are pairwise disjoint and $|\mathscr{A}| < \aleph_0$. $\mathbb{PR}$ is the system of all partial partitions of ω. Elements of $\mathbb{R}$ are denoted by $\mathscr{R}$, $\mathscr{S}$, $\mathscr{T}$, $\mathscr{W}$ and elements of $\mathbb{PR}$ by $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ respectively.

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For $\mathscr{R} \in \mathbb{R}$ we define the ideal

$$\mathscr{I}_{\mathscr{R}} = \{X \subseteq \omega : (\exists k \in \omega)(\forall R \in \mathscr{R})\, |R \cap X| \leq k\},$$

denote $\mathscr{F}_{\mathscr{R}} = \mathscr{F}_{\mathscr{I}_{\mathscr{R}}}$. For partitions $\mathscr{R}$, $\mathscr{S}$ we write $\mathscr{R} \preccurlyeq \mathscr{S}$ if $\mathscr{I}_{\mathscr{R}} \supseteq \mathscr{I}_{\mathscr{S}}$ and $\mathscr{R} \approx \mathscr{S}$ if $\mathscr{I}_{\mathscr{R}} = \mathscr{I}_{\mathscr{S}}$.

For $\mathscr{R}$, $\mathscr{S} \in \mathbb{R}$, $\mathscr{R}$ is said to be a refinement of $\mathscr{S}$ (denoted by $\mathscr{R} \sqsubseteq \mathscr{S}$) if $(\forall R \in \mathscr{R})(\exists S \in \mathscr{S})(R \subseteq S)$. For $\mathscr{A} \in \mathbb{PR}$ we denote $\mathrm{r}(\mathscr{A}) = \mathscr{A} \cup \{\{i\}: i \notin \bigcup \mathscr{A}\}$. Note that $\mathrm{r}(\mathscr{A})$ is a partition. For $\mathscr{A} \in \mathbb{PR}$, $\mathscr{R} \in \mathbb{R}$, $\mathscr{A}$ is said to be the partial refinement of $\mathscr{R}$ if $\mathrm{r}(\mathscr{A}) \sqsubseteq \mathscr{R}$. Denote $\mathscr{R} \cap\cap \mathscr{S} = \{R \cap S : R \in \mathscr{R},\ S \in \mathscr{S}\} \setminus \{\emptyset\}$ the roughest refinement of $\mathscr{R}$ and $\mathscr{S}$. For $\mathscr{R} \in \mathbb{R}$ and $X \subseteq \omega$ define $\mathscr{R} \restriction X = \mathrm{r}(\{R \cap X : R \in \mathscr{R}\ \&\ R \cap X \neq \emptyset\})$.

Recall for a Boolean algebra A, $\mathrm{c}A = \sup\{|X|: X$ is a pairwise disjoint family in $A\}$ (cellularity of A), $ca = c_A a = c(A \restriction a)$ and $\pi A = \min\{|X|: X$ dense in $A\}$ (density of A).

Basic facts

It is easy to see that $\mathscr{I}_{\mathscr{R}} = \mathscr{P}(\omega)$ exactly for $\mathscr{R}$ such that $(\exists k)(\forall R \in R)(|R| \leqq k)$. Denote $\mathbb{R}^0$ the set of such $\mathscr{R}$'s, let $\mathbb{R}^+ = \mathbb{R} \setminus \mathbb{R}^0$. Note that for $\mathscr{R}$, $\mathscr{S} \in \mathbb{R}^0$ is $\mathscr{R} \approx \mathscr{S}$. It can be shown easily that

$$\mathscr{R} \preccurlyeq \mathscr{S} \text{ iff } (\exists k)(\forall R \in \mathscr{R})(|\{S \in \mathscr{S} : R \cap S \neq \emptyset\}| \leq k).$$

Particularly, $\mathscr{R} \sqsubseteq \mathscr{S}$ iff $\mathscr{R} \preccurlyeq \mathscr{S}$ with $k = 1$. It is easy to prove that $\mathscr{R}$ and $\mathscr{S}$ are incompatible (denoted by $\mathscr{R} \perp \mathscr{S}$) iff $(\exists k)(\forall R \in \mathscr{R})(\forall S \in \mathscr{S})(|R \cap S| \leq k)$, i.e. $\mathscr{R} \cap\cap \mathscr{S} \in \mathbb{R}^0$. It is obvious that $\mathscr{R} \cap\cap \mathscr{S} \preccurlyeq \mathscr{R}$, $\mathscr{S}$. In the case $\mathscr{R} \preccurlyeq \mathscr{S}$, $\mathscr{R} \cap\cap \mathscr{S} \approx \mathscr{R}$ holds.

Define

$$[\mathscr{R}] = \{\mathscr{S} \in \mathbb{R} : \mathscr{I}_{\mathscr{R}} = \mathscr{I}_{\mathscr{S}}\} = \{\mathscr{S} \in \mathbb{R} : \mathscr{R} \approx \mathscr{S}\},$$

denote $\mathbb{R}^* = \mathbb{R}^+/_{\approx}$ with order

$$[\mathscr{R}] \leq [\mathscr{S}] \text{ if } \mathscr{R} \preccurlyeq \mathscr{S}, \text{ i.e. } \mathscr{I}_{\mathscr{R}} \supseteq \mathscr{I}_{\mathscr{S}}.$$

By [J], $\mathscr{R}$ and $\mathscr{S}$ are compatible iff $[\mathscr{R}]$ and $[\mathscr{S}]$ are compatible and $\mathrm{RO}(\mathbb{R}^*, \leq) \cong \mathrm{RO}(\mathbb{R}^+, \preccurlyeq)$ holds.

Denote

$$\mathbb{F}_q = \{\delta(\mathscr{F}_{\mathscr{R}}) : \mathscr{R} \in \mathbb{R}^+\}.$$

In [V1] it is shown that

$$j \text{ is a Q-point of } \omega^* \text{ iff } j \in \omega^* \setminus \bigcup_{\mathscr{R} \in \mathbb{R}^+} \delta(\mathscr{F}_{\mathscr{R}}) \text{ iff } j \notin \bigcup \mathbb{F}_q.$$

Observe that $\delta(\mathscr{F}_{\mathscr{R}})$ is nowhere-dense for all $\mathscr{R} \in \mathbb{R}$.

Since $\mathcal{R} \preccurlyeq \mathcal{S}$ iff $\mathcal{I}_{\mathcal{R}} \supseteq \mathcal{I}_{\mathcal{S}}$ iff $\delta(\mathcal{F}_{\mathcal{S}}) \subseteq \delta(\mathcal{F}_{\mathcal{S}})$, the Boolean algebras $\mathrm{RO}(\mathbb{F}_q, \subseteq)$, $\mathrm{RO}(\mathbb{R}^*, \leq)$ and $\mathrm{RO}(\mathbb{R}^+, \preccurlyeq)$ are all isomorphic.

The partial ordered set $(\mathbb{R}^*, \leq)$ is separative, because for $\mathcal{R}, \mathcal{S} \in \mathbb{R}^+$ such that $\mathcal{R} \not\preccurlyeq \mathcal{S}$, i.e. $\mathcal{I}_{\mathcal{R}} \not\supseteq \mathcal{I}_{\mathcal{S}}$, $\mathcal{R} \upharpoonright X \preccurlyeq \mathcal{R}$ and $(\mathcal{R} \upharpoonright X) \perp \mathcal{S}$ hold, where $X \in \mathcal{I}_{\mathcal{S}} \setminus \mathcal{I}_{\mathcal{R}}$. Hence $\mathbb{R}^*$ can be considered as the dense subset of its completion $\mathrm{RO}(\mathbb{R}^*, \leq)$.

The theorem. Now we are ready to state our main result.

Theorem. *If* $\mathfrak{p} = \mathrm{cf}(\mathfrak{c})$, *then the Boolean algebras* $\mathrm{RO}(\mathbb{F}_q, \subseteq)$ *and* $\mathrm{RO}(\mathbb{P}(\omega)/\mathrm{fin}, \subseteq^*)$ *are isomorphic.*

The idea of the proof is analogous to that of [V2], namely to construct isomorphic dense trees in algebras using the following.

Lemma 1 [BSV, BS]. *Let $\tau, \lambda \geq \aleph_0$, $\mu \geq 2$ be cardinals, A a $(\tau, \cdot, \mu)$-nowhere-distributive Boolean algebra having a λ-closed dense subset D. Let A be $(\kappa, \cdot, 2)$-distributive for each $\kappa < \tau$. If $\pi(A) = \mu^{<\lambda}$, then there is a dense subset $T \subseteq D$ of A such that $(T, \geq)$ is a tree of height τ and each $t \in T$ has $\mu^{<\lambda}$ immediate succesors.*

We show that presumptions of Lemma 1 are fulfilled for $\tau = \lambda = \mathfrak{p}$, $\mu = 2$, $A = \mathrm{RO}(\mathbb{R}^*, \leq)$, $D = \mathbb{R}^*$. Recall that (not only under $\mathfrak{p} = \mathrm{cf}(\mathfrak{c})$) $2^{<\mathfrak{p}} = \mathfrak{c}$ holds.

Lemma 2. *Below each $\mathcal{R} \in \mathbb{R}^+$ there are $\mathfrak{c}$-many pairwise incompatible elements from $\mathbb{R}^+$.*

Proof. We know that there is a system $\{A_\alpha : \alpha < \mathfrak{c}) \subseteq [\omega]^\omega$ such that every two sets of this system are almost disjoint (i.e. $(\forall \alpha, \beta < \mathfrak{c})(|A_\alpha \cap A_\beta| < \omega)$). Denote $\mathcal{R} = \{R_n : n \in \omega\}$, wlog assume that $\lim |R_n| = +\infty$. Define $\mathcal{R}_\alpha = \mathrm{r}(\{R_n : n \in A_\alpha\})$, clearly $R_\alpha \in \mathbb{R}^+$. Obviously every $\mathcal{R}_\alpha$ is a refinement of $\mathcal{R}$, hence $\mathcal{R}_\alpha \preccurlyeq \mathcal{R}$. For each $\alpha, \beta < \mathfrak{c}$, $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ are incompatible, because if $k = \max\{|R_n| : n \in A_\alpha \cap A_\beta\} + 1$, then $(\forall S \in \mathcal{R}_\alpha)(\forall T \in \mathcal{R}_\beta)(|S \cap T| \leq k)$, i.e. $R_\alpha \cap\cap R_\beta \in \mathbb{R}^0$.

Lemma 3. *For every $a \in \mathrm{RO}(\mathbb{R}^*, \leq)^+$ is $\mathrm{c}a \geq \mathfrak{c}$.*

Proof. Because $\mathbb{R}^*$ is dense in $\mathrm{RO}(\mathbb{R}^*, \leq)$, it is sufficient to prove it for $\mathbb{R}^*$. Since the compatibility in $\mathbb{R}^*$ corresponds to the compatibility in $\mathbb{R}^+$, below each $[\mathcal{R}]$ we can find $\mathfrak{c}$-many pairwise incompatible elements (namely $\{[\mathcal{R}_\alpha] : \alpha < \mathfrak{c}\}$, where $\{\mathcal{R}_\alpha : \alpha < \mathfrak{c}\}$ are those from Lemma 2).

Lemma 4. $\pi(\mathrm{RO}(\mathbb{R}^*, \leq)) = \mathfrak{c}$.

Proof. Because $\mathrm{c}\mathbf{1} \geq \mathfrak{c}$, there does not exist a dense subset of type $< \mathfrak{c}$, and for a dense subset $\mathbb{R}^*$, $|\mathbb{R}^*| = \mathfrak{c}$ holds.

Lemma 5. $\mathrm{RO}(\mathbb{R}^*, \leq)$ *is* $(\mathrm{cf}(\mathfrak{c}), \cdot, 2)$*-nowhere-distributive.*

Proof. $\mathbb{R}^* \upharpoonright a$ is dense in $\mathrm{RO}(\mathbb{R}^*, \leq) \upharpoonright a$ and $c(\mathrm{RO}(\mathbb{R}^*, \leq) \upharpoonright a) \geq \mathfrak{c}$ too, hence $|\mathbb{R}^* \upharpoonright a| = \mathfrak{c}$ too. Decompose $\mathbb{R}^* \upharpoonright a = \bigcup \{S_\alpha : \alpha < \mathrm{cf}(\mathfrak{c})\}$ so that $|S_\alpha|^+ < \mathfrak{c}$ for all

$\alpha < \mathrm{cf}(\mathfrak{c})$. By [BV] every S_α has disjoint refinement P_α. The system $\{P_\alpha : \alpha < \mathrm{cf}(\mathfrak{c})\}$ cannot have a common refinement as $\bigcup P_\alpha$ is dense in $\mathrm{RO}(\mathbb{R}^*, \leq) \upharpoonright a$ and $\mathrm{RO}(\mathbb{R}^*, \leq) \upharpoonright a$ has no atoms.

Lemma 6. $(\mathbb{R}^*, \leq)$ *is* $\mathfrak{p}$*-closed.*

Proof. Using the theorem of Bell [B], it is enough for every descending sequence in $\mathbb{R}^*$ of length $< \mathfrak{p}$ to find a σ-centered p.o. set and less than $\mathfrak{p}$-many dense sets such that any filter (in the ground model) that meets each of these dense sets produces a partition from $\mathbb{R}^+$ laying below the given descending sequence (in the ground model).

Let for $\kappa < \mathfrak{p}$, $\{\mathscr{R}_\alpha : \alpha < \kappa\} \subseteq \mathbb{R}^+$ be such that $\alpha \leq \beta$ implies $[\mathscr{R}_a] \leq [\mathscr{R}_\beta]$. Put

$$P = \{(\mathscr{A}, \mathscr{R}) : \mathscr{A} \in \mathbb{PR} \ \& \ (\exists \alpha < \kappa)(\mathscr{R} \approx \mathscr{R}_\alpha)\}$$

and the ordering $(\mathscr{A}, \mathscr{R}) \trianglelefteq (\mathscr{B}, \mathscr{S})$ if $\mathscr{A}$ is a prolongation of $\mathscr{B}$ (i.e. $\mathscr{A} \supseteq \mathscr{B}$), $\mathscr{R}$ is a refinement of $\mathscr{S}$ (i.e. $\mathscr{R} \sqsubseteq \mathscr{S}$) and the prolonging part $\mathscr{A} \setminus \mathscr{B}$ is a partial refinement of partition $\mathscr{R}$ (i.e. $\mathrm{r}(\mathscr{A} \setminus \mathscr{B}) \sqsubseteq \mathscr{R}$).

For fixed $\mathscr{A} \in \mathbb{PR}$, put $P_{\mathscr{A}} = \{(\mathscr{B}, \mathscr{R}) \in P : \mathscr{B} = \mathscr{A}\}$. $P_{\mathscr{A}}$ is centered, because for $\{(\mathscr{A}, \mathscr{S}_i) : i \leq m\}$, where $\mathscr{S}_i \approx \mathscr{R}_{\alpha_i}$ and $\{\alpha_i\}$ is non-descending, the element $(\mathscr{A}, \mathscr{S})$, where $\mathscr{S}$ is the roughest common refinement of $\{\mathscr{S}_i\}$ and $\mathscr{S} \approx \mathscr{R}_{\alpha_m}$, is below every $(\mathscr{A}, \mathscr{S}_i)$. Hence P is σ-centered, because $|\mathbb{PR}| = \aleph_0$.

The following ($< \mathfrak{p}$-many) sets are dense in P:

- for $k \in \omega$, $X_k = \{(\mathscr{A}, \mathscr{R}) : k \in \bigcup \mathscr{A}\}$, as $\{\mathscr{A} \cup \{\{k\}\} \mathscr{R}) \trianglelefteq (\mathscr{A}, \mathscr{R})$;
- for $k \in \omega$, $Y_k = \{(\mathscr{A}, \mathscr{R}) : (\exists A \in \mathscr{A})\, |A| > k\}$, as $(\mathscr{A} \cup \{\mathscr{R}\}, \mathscr{R}) \trianglelefteq (\mathscr{A}, \mathscr{R})$, where R is a set of $\mathscr{R}$ disjoint with every $A \in \mathscr{A}$ and $|R| > k$;
- for $\alpha < \kappa$, $Z_\alpha = \{(\mathscr{A}, \mathscr{R}) : \mathscr{R} \preccurlyeq \mathscr{R}_\alpha\}$, as $(\mathscr{A}, \mathscr{R} \cap \cap \mathscr{R}_\alpha) \trianglelefteq (\mathscr{A}, \mathscr{R})$.

Let $G \subseteq P$ be a filter that meets every X_k, Y_k and Z_α. Put

$$\mathscr{W} = \bigcup \{\mathscr{A} : (\exists \mathscr{R})\, (\mathscr{A}, \mathscr{R}) \in G\}.$$

$\mathscr{W}$ is obviously a partition of ω and $\mathscr{W} \in \mathbb{R}^+$ (because every $G \cap Y_k \neq \emptyset$ and if $W_1 \in \mathscr{A}_1$, $W_2 \in \mathscr{A}_2$ with $(\mathscr{A}_1, \mathscr{R}_1) \in G$, $(\mathscr{A}_2, \mathscr{R}_2) \in G$ there is $\mathscr{B} \supseteq \mathscr{A}_1, \mathscr{A}_2$ i.e. $W_1, W_2 \in \mathscr{B}$ i.e. $W_1 \cap W_2 = \emptyset$). We prove that $\mathscr{W} \preccurlyeq \mathscr{R}_\alpha$ for every $\alpha < \kappa$. Take $(\mathscr{A}, \mathscr{R}) \in G \cap Z_\alpha$, hence $\mathscr{R} \preccurlyeq \mathscr{R}_\alpha$. We show $\mathscr{W} \preccurlyeq \mathscr{R}$, i.e. there exists a $k \in \omega$ such that for every $W \in \mathscr{W}$, $|\{R \in \mathscr{R} : R \cap W \neq \emptyset\}| \leq k$ holds. If $W \notin \mathscr{A}$, then there exists a $(\mathscr{B}, \mathscr{S}) \in G$ such that $W \in \mathscr{B} \setminus \mathscr{A}$. Since G is a filter, there exists a $(\mathscr{C}, \mathscr{T}) \in G$ below $(\mathscr{A}, \mathscr{R})$ and $(\mathscr{B}, \mathscr{S})$. We have $\mathscr{B} \subseteq \mathscr{C}$ and $\mathscr{C} \setminus \mathscr{A}$ is a partial refinement of $\mathscr{R}$. Then $\mathscr{B} \setminus \mathscr{A} \subseteq \mathscr{C} \setminus \mathscr{A}$ is a partial refinement of $\mathscr{R}$ too, hence $|\{R \in \mathscr{R} : R \cap W \neq \emptyset\}| \leq 1$. As $\mathscr{A}$ is finite, it is sufficient to take $k = \max \{|\{R \in \mathscr{R} : R \cap W \neq \emptyset\}| : W \in \mathscr{A}\} + 1$.

Lemma 7. $\mathrm{RO}(\mathbb{R}^*, \leq)$ *is* $(\kappa, \cdot, 2)$*-distributive for all* $\kappa < \mathfrak{p}$.

Proof. λ-closedness of a dense subset implies κ-distributivity for all $\kappa < \lambda$.

Proof of the Theorem. By Lemma 1 there exists a dense tree $T \subseteq \mathbb{R}^*$ of height $\mathfrak{p}$ and each $t \in T$ has $2^{<\mathfrak{p}} = \mathfrak{c}$ immediate succesors. Denote P_α levels of T for $\alpha < \mathfrak{p}$. Obviously $D = \bigcup\{P_{\alpha+1} : \alpha < \mathfrak{p}\}$ is a dense subset of $\mathrm{RO}(\mathbb{R}^*, \leq)$ too. As D is clearly isomorphic to $\bigcup\{{}^\alpha\mathfrak{c} : \alpha < \mathfrak{p}\}$ ordered by the inverse inclusion, which is the (canonical) dense subset of complete Boolean algebra $\mathrm{Col}(\mathfrak{c}, \mathfrak{p})$, we have $\mathrm{RO}(\mathbb{R}^*, \leq) \cong \mathrm{Col}(\mathfrak{c}, \mathfrak{p})$. Using the results of [BPS] under $\mathfrak{p} = \mathrm{cf}(\mathfrak{c})$ the same is the case for $\mathrm{RO}(\mathscr{P}(\omega)/\mathrm{fin}, \subseteq^*)$. It means that under $\mathfrak{p} = \mathrm{cf}(\mathfrak{c})$, $\mathrm{RO}(\mathbb{F}_q, \subseteq)$ and $\mathrm{RO}(\mathscr{P}(\omega)/\mathrm{fin}, \subseteq^*)$ are isomorphic.

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