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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 35 (1994), No. 2, 45--50

Persistent URL: http://dml.cz/dmlcz/702013

Sandwich Type Theorems

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Received 15. March 1994

1. A subset $T \subset [0, 1]^{n+1}$,

$$T := \{t = (t_0, \dots, t_n) : \sum_{i=0}^{n} t_i = 1, t_i \geqq 0\}$$

is said to be the standard n-dimensional simplex. Let $D = [d_0, \dots, d_n]$ be n-dimensional simplex spanned by the vertices $d_0, \dots, d_n \in R^n$,

$$D := \{x \in R^n : x = \sum_{i=0}^{n} t_i \cdot d_i, \sum_{i=0}^{n} t_i = 1, t_i \geqq 0\}$$

where $t_i : D \to D$ means that the i-th barycentric coordinate function.

Denote by $D_i := [d_0, \dots, d_i, \dots, d_n]$ the i-th (n − 1)-dimensional face

$$D_i := \{x \in D : t_i(x) = 0\}$$

For each point $x \in R^{n+1}$, $x = (x_0, \dots, x_n)$, let us put

$$|x| = \sum_{i=0}^{n} |x_i|$$

The purpose of our paper is to discuss some consequences of the following lemma which is equivalent to the Brouwer fixed point theorem.

Lemma 1. *Let $f : D \to [0, \infty)^{n+1}$, $f = (f_0, \dots, f_n)$, be a continuous map such that*

(1) $$f_i(D_i) = \{0\} \text{ for each } i = 0, \dots, n.$$

Then for each point $t \in T$, there is a point $x \in D$ such that

(2) $$f(x) = |f(x)| \cdot t.$$

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Proof. If there is a point $x \in D$ such that $f(x) = (0, \ldots, 0)$ then the lemma holds. Thus, without loss of generality, we may assume that

$$(3) \qquad f(x) \neq (0, \ldots, 0) \quad \text{for each} \quad x \in D$$

Define a continuous map $g : D \to D$,

$$(4) \qquad g(x) := \sum_{i=0}^{n} \frac{f_i(x)}{|f(x)|} \cdot d_i$$

According to the assumption (1) the map g has the following property

$$(5) \qquad g(D_i) \subset D_i \quad \text{for each} \quad i = 0, \ldots, n$$

To prove the lemma it suffices to show the map is "onto". The proof of this fact is easy when we use arguments from the degree theory (see Deimling [2]). Indeed, define a map $h : D \times [0, 1] \to D$,

$$(6) \qquad h(x, t) := (1 - t) \cdot x + t \cdot g(x)$$

From (5) it follows that for each $x \in D_i$ and $t \in [0, 1]$

$$(7) \qquad h(x, t) \in D_i .$$

From (6) and (7) we infer that for any $a \in \operatorname{Int} D$,

$$\deg(g, D, a) = \deg(Id, D, a) = 1$$

and this implies that $a \in g(D)$, for each $a \in \operatorname{Int} D$. But this is equivalent to $g(D) = D$.

2. For any point $x \in R^n$ and a set $A \subset R^n$ let $d(x, A)$ means the distance between the point x and the set A,

$$d(x, A) := \inf \{\|x - a\| : a \in A\}$$

From the lemma we get the following

Corollary. (Equilibrium Theorem). *Let be given sets $A_0, \ldots, A_n \subset R^n$ such that $D_i \subset A_i$ for each $i = 0, \ldots, n$. Then there exists a point $x \in D$ such that*

$$d(x, A_0) = \ldots = d(x, A_n)$$

Proof. Let us define $f_i : D \to [0, \infty)$

$$f_i(x) := d(x, A_i), \qquad i = 0, \ldots, n$$

Each of the maps f_i satisfies the assumption (1) and according to the Lemma 1 for the point $t = \left(\frac{1}{n+1}, \ldots, \frac{1}{n+1}\right)$ there is a point $x \in D$ such that $f(x) = |f(x)| \cdot t$, but this implies that

$$d(x, A_0) = \ldots = d(x, A_n) .$$

3. Let $\mu(A)$ means the n-dimensional Lebesgue measure of the set $A \subset R^n$. For any point $x \in D$ let us denote

$$D_i(x) := [d_0, \ldots, d_{i-1}, x, d_{i+1}, \ldots, d_n]$$

the convex hull of the set $\{d_0, \ldots d_{i-1}, x, d_{i+1}, \ldots, d_n\}$.

Corollary. (Sandwich Theorem). *Let $A \subset D$ be a measurable set. Then for any point $t \in T$ there exists a point $x \in D$ such that for each $i = 0, \ldots, n$*

$$\mu[A \cap D_i(x)] = t_i \cdot \mu(A) \tag{8}$$

Proof. Define a continuous map $f : D \to [0, \infty)^{n+1}$, $f = (f_0, \ldots, f_n)$,

$$f_i(x) := \mu[A \cap D_i(x)] \qquad i = 0, \ldots, n$$

It is clear that for each $x \in D$

$$|f(x)| = \mu(A) \tag{9}$$

According to the lemma 1 for each point $t \in T$ there is a point $x \in D$ such that $f(x) = |f(x)| \cdot t$. But from (9) we get that for each $i = 0, \ldots, n$, $f_i(x) = \mu(A) \cdot t_i$.

For a given set $A \subset R^n$ and a point $x \in R^n$ let

$$A - x := \{a - x : a \in A\}$$

means a translation of the set A.

Assume that $P := [p_0, \ldots, p_n]$ is an n-dimensional simplex such that $0 \in \operatorname{Int} P$. Let for each $i = 0, \ldots, M_i$ be the cone consisting of the union of all the rays joining 0 to the points of $(n - 1)$-dimensional face $P_i := [p_0, \ldots, \hat{p}_i, \ldots, p_n]$.

Corollary. (Kuratowski-Steinhaus Theorem). *Let $A \subset R^n$ be a bounded Lebesgue measurable set. Then for each point $t \in T$ there exist a point $x \in R^n$ such that for each $i = 0, \ldots, n$*

$$\mu[(A - x) \cap M_i] = \mu(A) \cdot t_i$$

Proof. Since the set A is bounded there exist a number $s > 0$ such that for the simplex $D := [d_0, \ldots, d_n]$, where $d_i = s \cdot p_i$ for each $i = 0, \ldots, n$, the following conditions hold

$$A \subset D \tag{10}$$

and for each $i = 0, \ldots, n$ and for each point $x \in D_i$

$$(A - x) \cap M_i = \emptyset \tag{11}$$

Define a continuous map $f : D \to [0, \infty)^{n+1}$, $f = (f_0, \ldots, f_n)$,

$$f_i(x) := \mu[(A - x) \cap M_i] \quad \text{for each} \quad i = 0, \ldots, n \tag{12}$$

From (10) and (11) it follows that for each $x \in D$

$$|f(x)| = \mu(A) \tag{13}$$

and for each $i = 0, \ldots, n$

$$f_i(D_i) = \{0\}. \tag{14}$$

Then for a given point $t \in T$ we obtain a point $x \in D$ such that

$$f(x) = \mu(A) \cdot t$$

And this means that for each $i = 0, \ldots, n$

$$\mu[(A - x) \cap M_i] = \mu(A) \cdot t_i .$$

4. In this part we shall consider some result related to the Urbanik paper [4].

Lemma 2. *Let* $g : [0, 1] \times [0, 1] \to [0, \infty)$ *be a continuous function with the following properties*

$$g(u, u) = 0 \tag{15}$$

$$g(u, v) \quad \textit{and} \quad g(v, w) = 0 \quad \textit{implies} \quad g(u, w) = 0 \tag{16}$$

$$g(0, 1) > 0 . \tag{17}$$

Then for each natural number $n > 0$ *there exist a real number* $d > 0$ *and a sequence*

$$0 = u_0 < \ldots < u_n < u_{n+1} = 1 \tag{18}$$

such that for each $i = 0, \ldots, n$

$$g(u_i, u_{i+1}) = d . \tag{19}$$

Proof. Let us define a continuous functions $u_i : D \to [0, 1]$ for $i = 0, \ldots, n+1$.

$$u_0(x) = 0 , \qquad u_i(x) = t_0(x) + \ldots + t_{i-1}(x) \tag{20}$$

and functions $f_i : D \to [0, \infty)$ for $i = 0, \ldots, n$

$$f_i(x) = g[u_i(x), u_{i+1}(x)] . \tag{21}$$

Observe that if $x \in D_i$ then $t_i(x) = 0$ and in consequence $u_i(x) = u_{i+1}(x)$ and now, from (15) we infer that $f_i(x) = 0$.
From the Lemma 1 it follows that there is a point $x \in D$ such that

$$f_0(x) = \ldots = f_n(x) . \tag{22}$$

Let us put for each $i = 0, \ldots, n$

$$u_i = u_i(x) \quad \text{and} \quad d = f_i(x) . \tag{23}$$

From (22) and (23) we infer that for each $i = 0, \ldots, n$

$$d = g(u_i, u_{i+1}) . \tag{24}$$

We show that $d > 0$. Suppose that $d = 0$. Then according to (16) we get

(25) $$g(u_0, u_1) = \ldots = g(u_n, u_{n+1}) = 0\,.$$

And this implies that $g(0, 1) = 0$, a contradiction to (17).

Corollary. (Urbanik) *Let $f : [0, 1] \to X$ be a continuous map into a metric space (X, d) such that $f(0) \neq f(1)$.*
Then for each natural number $n > 0$ there exist a real number $d > 0$ and a sequence

$$0 = u_0 < u_1 < \ldots < u_n < u_{n+1} = 1$$

such that for each $i = 0, \ldots, n$

$$d = d[f(u_i), f(u_{i+1})]\,.$$

Proof. Indeed, the function

(26) $$g(u, v) := d[f(u), f(v)]$$

satisfies the conditions (15)—(17) of the Lemma 2.

Corollary. *Let $f : S \to [0, \infty)$ be a continuous function defined on a triangle $S := \triangle ABC$ such that*

(27) $$f(x) = 0 \quad \textit{iff} \quad x \in \textit{side } AB\,.$$

Then for each natural number $n > 1$ there exists a sequence of points belonging to the side AB,

$$A = P_0 < P_1 < \ldots < P_n < P_{n+1} = B$$

such that

$$f(Q_0) = \ldots = f(Q_n)$$

where the points $Q_0, \ldots, Q_n \in S$ are vertices of the triangles $\triangle P_iQ_iP_{i+1}$, $i = 0, \ldots, n$ which are similar to the triangle S.

Proof. Consider a coordinate system such that the side AB is contained in the diagonal and $A = (0, 0)$ and the product $[0, 1] \times [0, 1]$ is equal to the parallelogram ABCD. Now, extend the function f to a continuous function g defining

$$g(u, v) = f(u, v) \quad \text{if} \quad u \leqq v\,, \quad \text{and} \quad g(u, v) = f(v, u) \quad \text{if} \quad v \leqq u\,.$$

According to the Lemma 2 there exist a real number $d > 0$ and a sequence $0 = u_0 < u_1 < \ldots < u_n < u_{n+1} = 1$ such that $d = g(u_i, u_{i+1})$ for each $i = 0, \ldots, n$. Now , the Corollary becomes obvious when put $Q_i := (u_i, u_{i+1})$ for $i = 0, \ldots, n$.

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