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A Note on Singular Points of Convex Functions in Banach Spaces

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The magnitude of sets $A^n(f)$ of points at which the subdifferential of a continuous convex function f defined on a Banach space with separable dual space contains a ball of finite codimension n is characterised.

Introduction

Let X be a Banach space and let f be a continuous convex function (or, more generally, a proper convex function) on X. For a nonnegative integer n, we denote by $A^n(f)$ the set of all points $x \in X$ at which the subdifferential $\partial f(x)$ contains a ball of codimension n (i.e. a ball in a closed affine subset of codimension n). We investigate how big the set $A^n(f)$ can be. If X is finite-dimensional, then a satisfactory characterisation of the magnitude of sets $A^n(f)$ is given in [Z] (the case of a continuous convex f) and in [V] (the case of a proper convex f).

The case when X^* is separable was considered also in [V]. In this case $A^0(f)$ is always countable and each set of the form $A^n(f)$, $n \geqq 1$, can be covered by countably many of special pieces of some n-dimensional Lipschitz surfaces in X, which are called δ -convex fragments.

In the present note we observe that the proof in [V] implicitely contains the fact that these δ - convex fragments have an additional property (they are UDC_n – fragments, cf. Definition 2 below).

The second observation is that a slightly modified construction from [Z] gives that if $E \subset X$ can be covered by countably many of UDC_n – fragments, then there exists a continuous convex function f on X such that $E \subset A^n(f)$. Thus we

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obtain a characterization of the magnitude of sets $A^n(f)$, but it is not too nice, since the notion of a UDC_n – fragment is rather complicated and there is a natural open question (cf. Problem below) whether it can be simplified.

A quite satisfactory characterisation we have for $n = 0$ and $n = 1$ only.

Finally we consider the set $A^0_*(f)$ of points x at which f "has a big singularity in all directions", more precisely, $d_v f(x) + d_{-v} f(x) > \epsilon$ for each v, $\|v\| = 1$, where $\epsilon > 0$ does not depend on v. It is easy to see that always $A^0(f) \subset A^0_*(f)$. But the opposite inclusion generally does not hold. Moreover, it is shown (Example 1) that $A^0_*(f)$ can be uncountable in l_2.

In the following we shall use the following notations and definitions.

Notation. *The open ball with center x and radius r is denoted by $B(x, r)$. The one-sided derivative of a function f on a normed linear space is defined as $d_v f(x) = \lim_{t \to 0+} (f(x + tv) - f(x))t^{-1}$. For the notion of a proper convex function see e.g. [P].*

Definition 1. *(cf. [VZ], p. 45, Problem 10) Let X, Y be linear spaces, $A \subset X$ be an open convex set and $\emptyset \neq M \subset A$. We shall say that $F: M \to Y$ is delta--convex on M w.r.t. A if ther exists a continuous convex function f (so called control function) on A such that for each $y^* \in Y^*$, $\|y^*\| = 1$, there exists a continuous convex function g_{y^*} on A such that $y^* \circ F = g_{y^*} - f$ on M.*

Note 1. *If $M = A$, we obtain the notion of a delta-convex mapping on A which generalizes in a natural way the well-known notion of a δ-convex function. The investigation of delta-convex mappings was started in [VZ] (cf. also [KM], where 2 from 10 problems contained in [VZ] are solved). It is still unknown (cf. Problem 10 from [VZ]) whether or not each $F: M \to Y$ which is delta-convex w.r.t. A can be always extended to a delta-convex mapping on A.*

Note 2. *A slightly different definition of a delta-convex mapping is used in [V]. Namely, the control function is demanded to be Lipschitz. This difference is not essential, since each continuous convex function is locally Lipschitz.*

Definition 2. *Let E be a subset of a Banach space X and $n < \dim X$ be a positive integer. Following [V] (p. 558) we shall say that E is a δ-convex fragment of dimension n ($E \in DC_n$) if there exists a closed subspace Z of X and its topological complement W of dimension n, $M \subset W$ and a Lipschitz mapping $\varphi: M \to Z$ which is delta-convex on M w.r.t. W with a Lipschitz control function f such that*

$$E = \{w + \varphi(w) : w \in M\}.$$

We shall say that E is a uniformly Lipschitz δ-convex fragment of dimension n ($E \in UDC_n$) if $E \in DC_n$ and, moreover, φ and f can be chosen in such way that all functions g_{y^*} from Definition 1 can be K-Lipschitz for some K (independent on y^*).

Fragments with $M = W$ will be called surfaces (curves for $N = 1$).

To prove our main result, we shall need the following characterization of the sets $A^n(f)$.

Lemma. *Let f be a continuous convex function defined on an open convex subset C of a Banach space X. Then for each nonnegative integer n the following conditions are equivalent:*
(i) $x \in A^n(f)$,
(ii) *There exists a closed subspace* $Z \subset X$ *of codimension n,* $y \in X^*$ *and* $\epsilon > 0$ *such that*

$$d_z f(x) \geqq (z, \ y) + \varepsilon \text{ for each } z \in Z, \|z\| = 1.$$

Proof. (a) Suppose that (i) holds. Then there exists a closed subspace $W \subset X^*$ of codimension n, $y \in X^*$ and $r > 0$ such that

$$B(y, \ r) \cap (y + W) \subset \partial f(x).$$

Chose a n-dimensional $V \subset X^*$ such that $V \oplus W = X^*$ and let $\pi_w : X^* \to W$ be the projection in the direction of V. Put $\varepsilon = \frac{1}{2} r \|\pi w\|^{-1}$ and $Z = {}^{\perp}V$. It is well known that codim $(Z) = n$. Choose $z \in Z$, $\|z\| = 1$. We know (cf. e.g. [P]) that

$$d_z f(x) = \sup \{(z, \ s) : s \in \partial f(x)\}. \tag{1}$$

Find $u \in X^*$, $\|u\| = 1$ such that $(z, \ u) = 1$ and denote $w = \pi w(u)$. Then obviously $(z, \ w) = 1$, $\|w\| \leqq \|\pi w\|$ and $y + \frac{r}{2\|w\|} w \in \partial f(x)$. Therefore by (1)

$$d_z f(x) \geqq (z, \ y) + \left(z, \frac{r}{2\|w\|} w\right) = (z, \ y) + \frac{r}{2\|w\|} \geqq (z, \ y) + \varepsilon$$

(b) Now suppose that (ii) holds. We can suppose without any loss of generality that $y = 0$ (if $y \neq 0$, we can consider the convex function $\tilde{f}(x) = f(x) - (x, \ y)$). Choose an n-dimensional $T \subset X$ such that $Z \oplus T = X$ and put $E = T^{\perp}$, $F = Z^{\perp}$. It is well-known that $E \oplus F = X^*$ and $\dim F = n$. Let $\pi_E : X^* \to E$ be the projection in the direction of F. We shall show that

$$\pi_E(\partial f(x)) \supset E \cap B(0, \ \varepsilon).$$

In fact, let $p \in E \cap B(0, \varepsilon)$. Since $y = 0$, (ii) implies that

$$(z, \ p) \leqq d_z f(x) \leqq f(x + z) - f(z) \quad \text{for each } z \in Z.$$

Therefore by the well-known version of the Hahn-Banach theorem (cf. e.g. [RW]) there exists $q \in \partial f(x)$ such that $p/Z = q/Z$ and consequently $p = \pi_E(q)$. Now let n_0 be the minimal nonnegative integer for which there exists a closed subspace

W of codimension n_0 such that a linear projection of $\partial f(x)$ on W has a nonempty interior in W. We have proved that $n \geqq n_0$. If $n_0 = 0$, then $(\partial f(x))^0 \neq \emptyset$, i.e. $x \in A^0(f)$. Thus suppose $n \geqq n_0 \geqq 1$ and choose a closed subspace W of codimension n_0, a linear projection $\pi_W : X^* \to W$ and a (relatively) open ball B in W such that $B \subset \pi_W(\partial f(x))$.

At first we shall show that $(\pi_W)^{-1}(w) \cap \partial f(x)$ is a singleton for each $w \in B$. Suppose on the contrary that there are $u^1 \neq u^2$ from $\partial f(x)$ and $y \in B$ for which $\pi_W(u^1) = \pi_W(u^2) = y$. Put $u = u^2 - u^1$, $Z = \pi_W^{-1}(\{0\})$ and choose a $((n_0 - 1)$-dimensional) subspace U such that $U \oplus Lin\{u\} = Z$. Further let $L = W \oplus Lin\{u\}$ and $\pi_L : X^* \to L$ be the projection in the direction of U. Clearly $A : \pi_L(\partial f(x))$ is convex and bounded. Therefore

$$\alpha(w) := \sup\{t : w + tu \in A\}$$

is a bounded concave function on B and

$$\beta(w) := \inf\{t : w + tu \in A\}$$

is a bounded convex function on B. Therefore u and l are continuous on B. Since clearly $\alpha(y) > \beta(y)$, we easily obtain that A has a nonempty (relative) interior in L, which is a contradiction with the definition of n_0.

Thus we know that $\pi_W^{-1}(w) \cap \partial f(x)$ is a singleton, say $\{\phi(w)\}$, for each $w \in B$. Let $\{u_1, \dots, u_{n_0}\}$ be a basis of Z. Considering for each $i \in \{1, \dots, n_0\}$ and $u := u_i$

$$U_i,\ L_i,\ A_i,\ \alpha_i,\ \beta_i \text{ defined as above,}$$

we obtain that $\alpha_i(w) = \beta_i(w)$ are continuous and affine on B and therefore also $\phi : B \to X^*$ is continuous affine on B and consequently has a unique continuous affine extension $\tilde{\phi} : W \to X^*$. It is easy to prove that $\tilde{\phi}(W)$ is a closed affine subspace of codimension n_0, $\partial f(x) \subset \tilde{\phi}(W)$ and $\phi(B) \subset \partial f(x)$ is open in $\tilde{\phi}(W)$, which proves (i).

Theorem. *Let X be a Banach space with a separable dual space X^* and $T \subset X$ be a set. Then the following assertions are equivalent:*

(i) *There exists a continuous convex function F on X such that $T \subset A^n(F)$.*
(ii) *There exists a proper convex function F on X such that $T \subset A^n(F)$.*
(iii) *T can be covered by countably many of uniformly Lipschitz δ-convex fragments.*

Proof. The implication $(i) \Rightarrow (ii)$ is trivial. The proof of the implication $(ii) \Rightarrow (iii)$ is implicitely contained in [V]. In fact, it is sufficient to observe that each function H_{y^*} constructed in [V] (p. 564) is Lipschitz with the constant $\frac{2q}{r}(m + r)$.

To prove the implication $(iii) \Rightarrow (i)$ consider at first a fragment $E \in UDC_n$ which is determined by W, Z, $\varphi : M \to Z$ and a control function f as in Definition 2. We can suppose that φ, f and all g_{z^*} are K-Lipschitz. Remember that by definitions

(2) $\quad z^*(\varphi(w)) = g_{z^*}(w) - f(w)$ for $z^* \in Z^*$, $\|z^*\| = 1$ and $w \in M$.

Now define the function c on $W \times Z$ (equiped with the maximum norm) by the formula

$$c(w, z) = \sup \{g_{z^*}(w) - z^*(z) : \|z^*\| = 1\}.$$

All functions $g_{z^*}(w) - z^*(z)$ are obviously convex and $(K + 1)$-Lipschitz on $W \times Z$. On account of (2) we have

(3) $\quad c(w, z) = \sup \{z^*(\varphi(w)) + f(w) - z^*(z) : \|z^*\| = 1\}$ for $w \in M$ and $z \in Z$

and consequently

$$c(w, \varphi(w)) = f(w) \text{ for } w \in M.$$

Consequently c is a finite $(K + 1)$-Lipschitz convex function on $W \times Z$. Further (3) implies that

$$c(w, \varphi(w) + h) = f(w) + \|h\| \text{ for each } w \in M \text{ and } h \in Z.$$

Identifying X and $W \times Z$, we obtain a Lipschitz convex function c on X such that $d_h c(x) = 1$ for each $x \in E$ and $h \in Z$, $\|h\| = 1$. Therefore Lemma gives $E \subset A^n(c)$.

Now suppose that $T \subset \bigcup_{k=1}^{\infty} E_k \in UDC_n$. For each natural k find a Lipschitz convex function c_k on X such that $E_k \subset A^n(c_k)$ and then a sequence $\{a_k\}$, $a_k > 0$ such that $F(x) := \sum_{k=1}^{\infty} a_k c_k(x)$ is a convex Lipschitz function on X. It is easy to prove that $T \subset A^n(F)$.

Note 3. *Since the nature of UDC_n-fragment is not sufficiently known, we cannot be satisfied with the characterization of the magnitude of the sets $A^0(f)$ for $n > 1$. The case $n = 0$ is easy (each countable set is a subset of some $A^0(f)$) and for $n = 1$ our Theorem and results from [V] give that the following assertions are equivalent:*

(i) *There is a continuous convex function F on X such that $T \subset A^1(F)$.*

(ii) *T can be covered by countably many curves with finite convexity (i.e. LFC--curves in the terminology of [V]).*

(iii) *T can be covered by countably many of δ-convex curves.*

The case $n > 1$ is unclear, since the following problem (analogical to Problem 10 from [VZ], cf. Note 1 above) is open.

Problem. *Is it true that each uniformly Lipschitz δ-convex fragment of dimension $n > 1$ is a subset of a (uniformly Lipschitz) δ-convex surface of dimension n?*

Note also that I do not know, whether each δ-convex fragment of dimension $n > 1$ can be covered by countably many of uniformly Lipschitz δ-convex fragments of dimension n. The following example which shows that $A_*^0(f)$ can be uncountable in l_2 was suggested to me by P. Holický, J. Tišer and L. Veselý.

Example 1.
Let

$$C = \{(x_n) \in l_2 : |x_n| \leqq 1/n\} \text{ and } A = \{(x_n) \in l_2 : |x_n| = 1/n\}.$$

Clearly C is a compact convex subset of l_2 and $A \subset C$ is uncountable perfect. Let

$$f(x) = dist(x, C) \text{ be the distance function determined by the set } C.$$

It is well known that f is a convex 1-Lipschitz function and obviously

$$d_v f(x) \geqq 0 \text{ for each } x \in C.$$

Now let a vector $v \in l_2$, $\|v\| = 1$ and $x \in A$ be given. Consider the sets

$$I^1 = \{n : x_n = 1/n,\ v_n \geqq 0\},\ I^2 = \{n : x_n = 1/n,\ v_n < 0\},$$
$$I^3 = \{n : x_n = -1/n,\ v_n \geqq 0\},\ I^4 = \{n : x_n = -1/n,\ v_n < 0\}.$$

Since $N = \bigcup_{k=1}^4 I^k$, we can choose $k \in \{1, 2, 3, 4\}$ such that $\sum\{(v_n)^2 : n \in Y^k\} \geqq 1/4$. We claim that $d_v f(x) \geqq 1/2$ if $k \in \{1, 4\}$ and $d_{-v} f(x) \geqq 1/2$ if $k \in \{2, 3\}$. Let, for example $k = 3$. Now choose $t > 0$ and consider the size of $f(x - vt) = dist(x - vt, C)$. If $c \in C$ and $n \in I_3$ we have

$$(x - vt - c)_n = -1/n - v_n t - c_n \leqq -v_n t$$
$$\text{and therefore } |(x - vt - c)_n| \geqq tv_n.$$

Consequently

$$\|x - vt - c\| \geqq \sqrt{\sum\{t^2(v_n)^2 : n \in I_3\}} \geqq t/2$$

and therefore

$$\frac{f(x - vt) - f(x)}{t} \geqq \frac{1}{2} \text{ for each } t > 0.$$

The cases $k = 1, 2, 4$ are quite similar. Thus we have proved that

$$d_v f(x) + d_{-v} f(x) \geqq 1/2 \text{ whenever } x \in A \text{ and } \|v\| = 1.$$

Example 1 implies that $A_*^0(f)$ does not coincide with $A^0(f)$ in l_2. The following simpler example illustrating this phenomenon was shown me by L. Veselý.

Example 2.

Let

$$C = \{x = \{x_i\} \in l_2 : x_i \geqq 0, \|x\| \leqq 1\}.$$

Then the support function

$$f(x) := \sigma_C(x) = \sup\{x, y) : y \in C\}$$

is a continuous convex function on l_2 with $\partial f(0) = C$. Consequently $0 \notin A^0(f)$. On the other hand, $0 \in A_*^0(f)$. In fact, for each $v = \{v_i\} \in l_2$, $\|v\| = 1$, the numbers $d_v f(0)$, $d_{-v} f(0)$ are clearly nonnegative, but one from them is at least $\frac{1}{2}$, since $v^+ := \{v_i^+\} \in C$, $v^- := \{v_i^-\} \in C$, $(v, v^+ - v^-) = 1$ and therefore one from the numbers

$$(v, v^+), (-v, v^-)$$

is at least $\frac{1}{2}$

References

[KM] Kopecká E. and Malý J., *Remarks on delta-convex functions*, Comment. Math. Univ. Carolinae **31** (1990), 501–510.

[P] Phelps R. R., *Convex functions, monotone operators and differentiability*, Springer-Verlag, 1989.

[RV] Roberts A. W. and Varberg D. E., *Convex functions*, New York and London, 1973.

[V] Veselý L., *On the multiplicity points of monotone operators on separable Banach spaces*, *Comment.* Math. Univ. Carolinae **27** (1986), 551–570.

[VZ] Veselý L. and Zajíček L., *Delta-convex mappings between Banach spaces and applications*, Dissertationes Mathematicae **CCLXXXIX** (1989), 1–48.

[Z] Zajíček L., *On the differentiation of convex functions in finite and infinite dimensional spaces*, Czechoslovak Math. J. **29** (1979), 340–348.