Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 34 (1993), No. 2, 67--70

Persistent URL: http://dml.cz/dmlcz/701995

On Subdifferentials of Convex Functions

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Received 14 April 1993

Let X be a real normed linear space, X^* and X^{**} its dual and bidual, respectively, $\langle , \rangle$ the pairing between X and X^*, $S_1(0)$ and $S_1^*(0)$ the unit sphere in X and X^*, respectively. By R we denote the set of all real numbers, while $\hat{x}$ denotes the image of an element $x \in X$ under the canonical mapping in X^{**}. If E is a subspace of X, denote by $E^\perp$ its annihilator in X^*. Let F and G be topological spaces, 2^G the family of all subsets of G, $T: F \to 2^G$ a mapping, $D(T) = \{u \in F: T(u) \neq \emptyset\}$ its domain, $G(T) = \{(u, v) \in F \times G: v \in T(u)$ for some $u \in D(T)\}$ its graph in the space $F \times G$. We shall say that $T: F \to 2^G$ is upper semicontinuous at $u_o \in F$, if for each open subset W of G such that $T(u_o) \subset W$ there exists an open neighborhood U of u_o such that $T(u) \subset W$ for every $u \in U$.

Suppose now that X is a normed linear space. By the symbols $\sigma(X, X^*)$ and $\sigma(X^*, X)$, we mean the weak and the weak* topology on X and X^*, respectively. Recall that $T: X \to 2^X$ is said to be

(i) monotone, if for every $u, v \in D(T)$ and every $u^* \in T(u)$, $v^* \in T(v)$ there is $\langle v^* - u^*, v - u \rangle \geqq 0$;

(ii) maximal monotone, if T is monotone and for a given element $(u_o, u_o^*) \in X \times X^*$ such that $\langle v^* - u_o^*, v - u_o \rangle \geqq 0$ for every $(v, v^*) \in G(T)$, we have that $(u_o, u_o^*) \in G(T)$.

Let $M \subset X$ be an open nonvoid convex subset of a normed linear space $X, f: M \to R$ a convex continuous function. The multivalued mapping $M \ni u \to \partial f(u)$ defined by $\partial f(u) = \{u^* \in X^*, \langle u^*, v - u \rangle \leqq f(v) - f(u)$ for every $v \in M\}$ is called the subdifferential mapping (or subdifferential) of f on M. Note that $u^* \in \partial f(u_o)$, where $u_o \in M$, if and only if the graph of the affine function $h(v) = f(u_o) + \langle u^*, v - u_o \rangle$ is a supporting hyperplane to the epigraph of f at the point $(u_o, f(u_o))$.

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Let us collect the main properties of the subdifferential mapping ∂f (see [9], [13]):

(a) For every $u \in M$, the set $\partial f(u)$ is nonvoid convex and weak* compact;

(b) If $u_o \in M$, then $\partial f(u_o)$ is a single point if and only if f is Gâteaux differentiable at u_o;

(c) f is Fréchet differentiable at $u_o \in M$ if and only if $\partial f(u_o)$ is singlevalued and $M \ni u \to \partial f(u)$ is norm to norm upper semicontinuous at u_o;

(d) f is Gâteaux (Fréchet) differentiable at $u_o \in M$ if and only if there exists a selection φ of $M \ni u \to \partial f(u)$ such that φ is norm to weak* (norm to norm) continuous at u_o;

(e) $M \ni u \to \partial f(u)$ is norm to weak* upper semicontinuous, maximal monotone and locally bounded on M;

(f) the so-called duality mapping $J: X \to 2^{X^*}$ defined by $J(u) = \{u^* \in X^*: \langle u^*, u\rangle = \|u\|^2, \|u^*\| = \|u\|\}$ is the example of the subdifferential mapping ∂f, where $f(u) = \frac{1}{2}\|u\|^2$. The support mapping $S_1(0) \ni u \to u_u^* \in \{u^* \in S_1^*(0): \langle u^*, u\rangle = 1\}$ is a selection of $J|_{S_1(0)}$. If X is smooth, then the support mapping coincides with $J|_{S_1(0)}$.

Theorem 1 ([11]). *Let X be a dual Banach space (i.e. $X = Z^*$ for some normed linear space Z), $M \subset X^*$ a convex open subset, $\hat{u}_0 \in M$, where $\hat{u}_o$ is a canonical image of $u_o \in Z$ in X^*. Let f: $M \to R$ be a weak* lower semicontinuous convex functional having the Gâteaux derivative $F'(\hat{u}_o)$ at $\hat{u}_o$. Then* (i) *$f'(\hat{u}_o) \in X$, i.e. $f'(\hat{u}_o)$ is a weak* continuous linear functional on X^*,* (ii) *if $(u_n^*) \subset M$, $u_n^* \to \hat{u}_o$ in the norm of X^* and $\hat{x}_n \in \partial f(u_n^*)$ for some sequence $(x_n) \subset X$, then $x_n \to f'(\hat{u}_o)$ weakly in X.*

Recall that Asplund [1] proved the following assertion: Let X be a Banach space, f: $X \to (-\infty, +\infty)$ a lower semicontinuous function such that $f \not\equiv +\infty$. If the dual function f^* defined on X^* is Fréchet differentiable at some point $u^* \in X^*$, then $(f^*)'(u^*) \in \hat{X}$.

Theorem 2 (cf. [12]). *Let X be a normed linear space, f a convex continuous function on X, v_o, w_o^* given elements of X and X^*, respectively. Assume that there exists a closed linear subspace E of X such that $\{u \in E: g(u) \leq c\}$ is nonempty and relatively weakly compact in E for some $c \in R$, where g: $E \to R$ is defined by $g(u) = f(u + v_o) - \langle w_o^*, u\rangle$ for every $u \in E$. Then:* (i) *there exists a point $u_o \in E$ such that $\partial f(u_o + v_o) \cap (w_o^* + E^\perp) \neq \emptyset$;* (ii) *if f is Gâteaux differentiable at the point $(u_o + v_o)$, then the intersection in* (i) *consists of exactly one point.*

Corollary 1. *Let X be a normed linear space, f: $X \to R$ a continuous convex function. Assume that there exists a reflexive subspace E of X such that $f(u)$. $\|u\|^{-1} \to +\infty$ as $\|u\| \to +\infty$. Then:* (i) *if v_o, w_o^* are arbitrary points of X and X^*, respectively, then there exists a point $u_o \in E$ such that $\partial f(u_o + v_o) \cap \cap (w_o^* + E^\perp) \neq \emptyset$;* (ii) *if f is Gâteaux differentiable at $(u_o + v_o)$, then the above intersection consists of exactly one point.*

Corollary 1 extends the results of Beurling and Livingston [3], Browder [4] and Asplund [2].

Theorem 3. *Let X be a Banach space, J and J^* the duality mapping on X and X^*, respectively. Then:* (i) *if X is nonreflexive and X^* is smooth, then the graph $G(J^*)$ of J^* is not closed in $(X^*, \sigma(X^*, X)) \times (X^{**}, \sigma(X^{**}, X^*))$;* (ii) *if X^* is smooth, then J^* is weak* continuous on the range $R(J)$ of J at $u_o^* \in R(J)$ if and only if J^{-1} is $\sigma(X^*, X) - \sigma(X, X^*)$ continuous at u_o^*.*

The next result was initiated by [7, 8]. We denote again by J and J^* the duality mapping on X and X^*, respectively.

Theorem 4. *Let X be a dual Banach space such that the weak* and strong convergence of sequences coincide on $S_1^*(0)$ of X^*, $u_o \in S_1(0)$. Assume that the norm of X^* is Gâteaux differentiable at the points of the set $J(u_o)$. Then for every n $(n = 1, 2, \ldots)$ there exist the points $u_n^* \in X^*$ and $v_n^* \in J(u_o)$ and a point $u_o^* \in S_1^*(0)$ such that $\|u_n^* - v_n^*\| = \text{dist}\,(u_n^*, J(u_o)) \leqq \frac{1}{n}$, $u_n^* \to u_o^*$ in the norm of X^*, $\langle u_o^*, u_o \rangle =$ $= 1$, $J^*(v_n^*) \to \hat{u}_o$ weakly* in X^* and $\hat{u}_o = J^*(u_o^*)$.*

Using the higher dual technique, Giles and Gregory and Sims [10] have proved the following result: Let X be a Banach space which can be equivalently renormed so that there exists a constant k $(0 < k < 1)$ such that for every $x \in S_1(0)$ and $x^* \in J(x)$ and $\hat{x}^* + x^\perp \in J^{**}(\hat{x})$, where $x^\perp \in X^\perp$ and J^{**} is the duality map on X^{**}, there is $\|x^\perp\| \leqq k$, then X is an Asplund space. In particular, if X can be equivalently renormed such that the weak* and weak topologies coincide on $J(S_1(0))$, then X is an Asplund space. It is observed that the proof of the above mentioned result shows that a Banach space X, whose dual X^* satisfies the condition of the above assertion or its consequence, is reflexive. The other proof of a similar assertion depends on the Eberlein-Šmulian and the Goldstine theorems.

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