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We are comparing two subclasses of first category sets in Hilbert spaces, one of $\sigma-\alpha$-cone porous and the other of $\sigma-\alpha$-angle porous sets. Our Theorem gives a positive result whence Example shows that these two notions do not coincide.

In this note we present two elementary observations (see Lemma and Example below) concerning the notions of σ-cone and σ-angle porosity. These and some other related notions of small sets in normed linear spaces appeared e.g. in [PZ], [Z]. They give subclasses of first category sets and they are used in the mentioned papers to describe the size of sets of singular points, such as points of non-differentiability of continuous convex functions or points of discontinuity of monotone multivalued operators. We recall the definitions and then we formulate two results which answer some questions posed by L. Zajíček at a Prague seminar and verify the hypothesis from Note 2 of [Z].

1. Notions

For simplicity we suppose that X is a Hilbert space if nothing else is mentioned. We use the notation $B(x, r)$ for the open r-ball centered at $x \in X$ and $S(x, r)$ for its boundary.

Under an *angle* (of size $\alpha \in (0, \pi)$) we understand a set of the form $A(x, v, \alpha) = x + A(0, v, \alpha)$, where $A(0, v, \alpha)$ is described, for given $x \in X$, $v \in S(0, 1)$, and $\alpha \in (0, \pi)$, by

$$A(0, v, \alpha) = \bigcup_{\lambda>0} \lambda B\left(v, \sin \frac{\alpha}{2}\right) = \left\{y \in X;\ \langle y, v\rangle > \|y\| \ \cos \frac{\alpha}{2}\right\}.$$

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Remark. In general normed linear spaces the third expression can be given the sense supposing that $v \in S(0, 1)$ in X^*. Nevertheless the set of all angles given by the second and the third expressions of the above definition differ. The third expression is used in the definition of a cone which is introduced and applied in [PZ] and [Z]. Formulating the next results in more general normed linear space one has to respect this difference.

We say that a set $S \subset X$ is *α-angle porous* ($A \in \mathfrak{A}(\alpha)$) if

$$(1) \qquad \forall_{x \in S} \ \forall_{r>0} \ \exists_{y \in B(x,r)} \ \exists_{v \in S(0,1)} \{S \cap A(y, v, \alpha) = \emptyset\}.$$

If $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathfrak{A}(\alpha)$, then A is called *$\sigma - \alpha$-angle porous* and we write $A \in \mathscr{A}_\sigma(\alpha)$.

The "local versions" of angle porosity are the following notions of cone porosity. We say that $S \subset X$ is *α-cone porous* ($A \in \mathfrak{C}(\alpha)$) if

$$(2) \qquad \forall_{x \in S} \ \exists_{R_x>0} \ \forall_{r>0} \ \exists_{y \in B(x,r)} \ \exists_{v \in S(0,1)} \{S \cap A(y, v, \alpha) \cap B(x, R_x) = \emptyset\}.$$

If $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathfrak{C}(\alpha)$, then A is called *$\sigma - \alpha$-cone porous* ($A \in \mathfrak{C}_\sigma(\alpha)$).

Remark. The property that A is c-cone (or c-angle) porous introduced in [Z] and α-cone (or α-angle) porosity defined above are equivalent for the choice $c = \cos \frac{\alpha}{2}$.

Notice that the notion of $\sigma - \alpha$-cone porosity is hereditary to subsets and the following proposition claims that it is local.

Proposition 1. *Let $S \subset X$ and $\mathfrak{U}$ be an open cover of S such that $S \cap U$ is $\sigma - \alpha$-cone porous for each $U \in \mathfrak{U}$. Then S is $\sigma - \alpha$-cone porous.*

Proof. There is an open cover $\mathfrak{V}$ of S which refines $\mathfrak{U}$ and such that $\mathfrak{V} = \bigcup_{n=1}^{\infty} \mathfrak{V}_n$, where $\mathfrak{V}_n$ is (metrically) discrete for every $n \in \mathbb{N}$ ([S]). The sets $S_n = S \cap \bigcup \mathfrak{V}_n$ are obviously $\sigma - \alpha$-cone porous and therefore S is $\sigma - \alpha$-cone porous, too.

Similarly we get.

Proposition 2. *Let $S \subset X$ be a separable $\sigma - \alpha$-cone porous set. Then S is $\sigma - \alpha$-angle porous.*

Proof. It suffices to verify the assertion for S separable and α-cone porous. Let $R_x > 0$, $x \in S$, be some choice of R_x from (2). Put $S_n = \{x \in S;\ R_x > \frac{1}{n}\}$ for $n \in \mathbb{N}$. Choose $x_m^{(n)} \in S_n$, $m \in \mathbb{N}$, such that $S_n \subset \bigcup_{m=1}^{\infty} B(x_m^{(n)}, \frac{1}{2n})$. The sets $S_{nm} = S_n \cap B(x_m^{(n)}, \frac{1}{2n})$ are α-angle porous.

2. Results

We show that the following equivalence between cone and angle porosity holds true.

Theorem. *The set $S \subset X$ is $\sigma-\alpha$-cone porous for some $\alpha \in (0, \pi)$, if, and only if, S is $\sigma-\alpha$-angle porous for some $\alpha \in (0, \pi)$.*

Proof. The "if" part is obvious and the "only if" follows from the following more quantitative assertion.

Lemma. *Let $S \subset X$ be $\sigma-\alpha$-cone porous for some $\alpha \in (0, \pi)$. Then $S \in \mathfrak{A}_\sigma(\frac{\alpha}{2} - \varepsilon)$ for every $\varepsilon \in (0, \frac{\alpha}{2})$.*

We postpone the proof of Lemma to the next section 3.

We point out the following special case of Lemma.

Corollary. *For $S \subset X$ the implication*

$$S \in \bigcap_{\alpha \in (0,\pi)} \mathfrak{C}_\sigma(\alpha) \Rightarrow S \in \bigcap_{\beta \in (0,\frac{\pi}{2})} \mathfrak{A}_\sigma(\beta)$$

hold true.

Remark. The sets from $\bigcap_{\alpha \in (0,\pi)} \mathfrak{C}_\sigma(\alpha)$ are called cone-small and the sets from $\bigcap_{\alpha \in (0,\pi)} \mathfrak{A}_\sigma(\alpha)$ angle-small in [Z]. The following example shows that Corollary can not be strengthened to the statement that cone-small sets are angle-small.

Example. *Let X be nonseparable and $\{e_\iota;\ \iota \in I\}$ be an orthonormal basis of X. Then, for sufficiently small $r > 0$, the set $S(r) = \bigcup_{\iota \in I} S(e_\iota, r)$ is in $\bigcap_{\alpha \in (0,\pi)} \mathfrak{C}(\alpha)$ but there are $\beta_r \in (0, \pi)$ such that $\lim_{r \to 0+} \beta_r = \beta_0 \in (0, \pi)$ and $S(r)$ is not in $\mathfrak{A}_\sigma(\beta_r)$.*

Moreover, we can find β_r and β_0 such that $\beta_0 = \frac{3}{4}\pi + \varphi_0$, where φ_0 solves the equation

$$\varphi_0 + \frac{\pi}{4} = \arctan\left(\sqrt{1 + 2\cot^2 \varphi_0}\right). \tag{3}$$

Remark. The set $S(r)$ from Example is obviously in $\bigcap_{\alpha > 0} \mathfrak{C}(\alpha)$ if $r \in (0, \frac{\sqrt{2}}{2})$. Thus it is cone-small in the terminology mentioned above. In fact, $S(r)$ is even "cone supported by halfspaces at every $x \in S(r)$" in an obvious sense. This is a strengthening of the notion of cone-supported sets in [Z]. We suppose that the example is far from being optimal in the sense that the bounds obtained for β_r are probably not the smallest possible for cone-small sets.

We indicate the proof of the remaining property of $S(r)$ from Example in the last section 4.

The assertions of Lemma and Example lead to the following natural.

Question. *What is the maximal $\beta(\alpha) \in (0, \pi)$ such that any $S \in \mathfrak{C}(\alpha)$ is in $\mathfrak{A}_\sigma(\beta)$ for every $\beta \in (0, \beta(\alpha))$? In particular, what is the* $\lim_{\alpha \nearrow \pi} \beta(\alpha)$?

Remark. We know, due to our above observations, that $\beta(\alpha) \geqq \frac{\alpha}{2}$ and $\beta(\alpha) \leqq \frac{3}{4}\pi + \varphi_0$ with φ_0 fulfilling the above equation (3).

3. Proof of Lemma

3.1. We may and shall suppose that $S \in \mathfrak{C}(\alpha)$. For every $x \in X$ we choose an $R_x > 0$ according to (2).

3.2. Let us define $S_p = \{x \in S;\ R_x \geqq \frac{1}{p}\}$ for $p \in \mathbb{N}$. It follows that, for every set U of diameter less than $\frac{1}{p}$, the set $S_p \cap U$ is in $\mathfrak{A}(\alpha)$.

3.3. Since X is metric there is a cover $\mathfrak{B}_p$ of X by open sets of diameter less than $\frac{1}{p}$ such that $\mathfrak{B}_p = \bigcup\{\mathfrak{B}_{p,q};\ q \in \mathbb{N}\}$, where every $\mathfrak{B}_{p,q}$ is metrically discrete, i.e. there are $\delta(q) > 0$ with distance of B_1 and B_2 at least $\delta(q)$ for every two distinct B_1, B_2 from $\mathfrak{B}_{p,q}([S])$. We put $S_{p,q} = S_p \cap \bigcup \mathfrak{B}_{p,q}$ and notice that $S_{p,q} \cap B \in \mathfrak{A}(\alpha)$ for $B \in \mathfrak{B}_p$ due to 3.2.

3.4. Later on we consider the set $S_{p,q}$ for $p, q \in \mathbb{N}$ fixed. Let us fix an arbitrary $v \in X$ with $\|v\| = 1$. We split X into subsets $X_s = \{x \in X;\ \langle v, x\rangle \in [s\varepsilon, (s+1)\,\varepsilon)\}$ for $s \in \mathbb{Z}$. We shall show that for $\varepsilon > 0$ small enough the sets $S_{p,q,s} = S_{p,q} \cap X_s$ are in $\mathfrak{A}(\beta)$ for $\beta \in (0, \frac{\alpha}{2})$.

3.5. Let us fix $\beta \in (0, \frac{\alpha}{2})$, $v \in X$, $p, q \in \mathbb{N}$ and $s \in \mathbb{Z}$, as in 3.4. Let x be an arbitrary element of $S_{p,q,s}$, $B_x \in \mathfrak{B}_{p,q}$ be the only element with $x \in B_x$. We know that for every $r > 0$ there is $y \in B(x; r)$ and $w \in S(0,1)$ such that $S_{p,q,s} \cap$ $\cap\ A(y, w, \alpha) \cap B_x = \emptyset$. The angles $A(y, v, \pi - \gamma)$ and $A(y, -v, \pi - \gamma)$ have empty intersections with $X_s \cap [\bigcup \mathfrak{B}_{p,q} \setminus B_x]$ for $\gamma \in (2 \arcsin \frac{\varepsilon}{\delta_q}, \pi)$. Hence, putting $\varepsilon > 0$ sufficiently small, the $\gamma > 0$ can be chosen as small as needed. Both the intersections $A(y, v, \pi - \gamma) \cap A(y, w, \alpha)$ and $A(y, -v, \pi - \gamma) \cap A(y, w, \alpha)$ do not intersect $S_{p,q,s}$, but one of them contains $A(y, u, \frac{\alpha}{2} - \gamma)$ for some suitably chosen $u = \mu v + \nu w$, $\|u\| = 1$. Thus $S_{p,q,s}$ is in $\mathfrak{A}(\frac{\alpha}{2} - \gamma)$ and for sufficiently small ε, and thus γ, we get $\frac{\alpha}{2} - \gamma \geqq \beta$.

Remark. The above theorem holds true for all normed linear spaces if we put $A(0, w, \alpha) = \bigcup \lambda B(w, \sin \frac{\alpha}{2})$ for $w \in S(0,1)$ in the definition of the angle. To prove this we may proceed similarly as in the proof of Lemma.

4. Proof of properties of $S(r)$ from Example

4.1. We already mentioned that $S(r)$ is in $\mathfrak{C}(\alpha)$ for $\alpha \in (0, \pi)$ and $r \in (0, \frac{\sqrt{2}}{2})$.

4.2. Instead of showing that $S(r) \notin \mathfrak{A}_\sigma(\beta)$ it is sufficient to show $W(r) \notin \mathfrak{A}_\sigma(\beta)$ for a subset $W(r)$ of $S(r)$ because the $\sigma - \beta$-porosity is hereditary. We find useful to define the subset $W(r)$ of $S(r)$ in the following way. Let $\varphi \in (0, \frac{\pi}{2})$. Consider the intersection $H(r, \varphi, \iota) = S(e_\iota, r) \cap \partial A(e_\iota, -e_\iota, 2\varphi)$, where ∂A stands for the boundary of A, and denote the center of $H(r, \varphi, \iota)$ by $h(r, \varphi, \iota)$. There is an uncountable family $\{f_x(r, \varphi, \iota) \in H(r, \varphi, \iota);\ x \in J\}$ such that $f_x(r, \varphi, \iota) - h(r, \varphi, \iota)$, $x \in J$, are mutually orthogonal due to the nonseparability of X. For $\delta > 0$, we put $W(r, \varphi, \delta, \iota) = S(e_\iota, r) \cap \bigcup B(f_x(r, \varphi, \iota), \delta)$. The subset we are interested in is $W(r) = \bigcup_{\iota \in I} W(r, \varphi(r), \delta(r), \iota)$ for a suitable choice of $\varphi(r)$, $\delta(r)$ and r sufficiently small.

4.3. Here we explain what the suitable choice of φ and δ means.

Claim 1. *There is an $\varphi_0 \in (0, \frac{\pi}{4})$ and, for sufficiently small $r > 0$, there are $\delta(r) \in (0, \infty)$ with $\lim_{r \to 0+} \delta(r) = 0$ and $\gamma(r) > 0$ with $\lim_{r \to 0+} \gamma(r) = 0$ such that for every $t \in S(e_\iota, r) \cap B(f_x(r, \varphi_0, \iota), \delta(r))$*

$$\text{(a)} \qquad \bigcup_{x' \neq x} B(f_{x'}(r, \varphi_0, \iota), \delta(r)) \subset A\left(t, \frac{e_\iota - t}{\|e_\iota - t\|}, \frac{\pi}{2} + 2\varphi_0 + \gamma(r)\right);$$

$$\text{(b)} \qquad S(r) \setminus S(e_\iota, r) \subset A\left(t, \frac{t - e_\iota}{\|t - e_\iota\|}, \frac{\pi}{2} + 2\varphi_0 + \gamma(r)\right).$$

Proof. We write f_x instead of $f_x(r, \varphi, \iota)$ during this proof. Consider first the degenerate situation with $B(f_x, \delta(r))$ replaced by $\{f_x\}$.

We have that $f_{x'} \in \partial A(f_x, \frac{e_\iota - f_x}{\|e_\iota - f_x\|}, 2\varphi^*)$, where $\varphi^* \in (0, \frac{\pi}{2})$ is the angle between the lines given by pairs $f_x, f_{x'}$ and e_ι, f_x, respectively.

Also we get

$$\forall_{\varphi \in (0, \frac{\pi}{4})} \{e_{\iota'};\ \iota' \neq \iota\} \subset \overline{A\left(e_\iota, \frac{f_x - e_\iota}{\|f_x - e_\iota\|}, \frac{\pi}{2} + 2\varphi\right)}.$$

By elementary computations we obtain $\varphi^* = \arctan(\sqrt{1 + 2\cot^2 \varphi})$ and there is a solution $\varphi_0 \in (0, \frac{\pi}{2})$ of the equation $2\varphi^* = \frac{\pi}{2} + 2\varphi$ which is equivalent to (3).

Thus, by continuity arguments, we get the result of Claim 1.

Finally, we do some choice of $\delta(r)$ and $\gamma(r)$ along the lines of Claim 1 and put $W(r) = \bigcup_{\iota \in I} W(r, \iota)$. Here and later on we write (r, ι) instead of $(r, \varphi(r), \delta(r), \iota)$, or (r, φ, ι), respectively.

4.4. As a corollary of Claim 1 the following assertion can be derived.

Claim 1'. *Let $\varphi_0 \in (0, \frac{\pi}{4})$ be the only solution of (3), and $\delta(r)$ and $\gamma(r)$ be as in Claim 1. Let $t \in S(e_\iota, r) \cap B(f_x(r, \iota), \delta(r))$. Then*

(a) if $v \in S(0, 1)$ and $\beta \in (0, \pi)$ be such that $A(t, v, \beta) \subset A(t, \frac{e_\iota - t}{\|e_\iota - t\|}, \pi)$, then the relation

$$\bigcup_{x' \neq x} B(f_{x'}(r, \iota), \delta(r)) \nsubseteq A(t, v, \beta)$$

implies that

$$\beta < \varphi_0 + \frac{3}{4}\pi + \frac{\gamma(r)}{2};$$

(b) if $v \in S(0,1)$ and $\beta \in (0,\pi)$ be such that $A(t,v,\beta) \subset X \setminus B(e_\iota, r)$, then

$$S(r) \setminus S(e_\iota, r) \nsubseteq A(t, v, \beta)$$

implies that

$$\beta < \frac{3}{4}\pi + \varphi_0 + \frac{\gamma(r)}{2}.$$

4.5. Before we formulate another auxiliary assertion, we make an agreement. The subset V of $W(r, \iota)$ is called "big" if

(1) the closure of V contains an interior point of $S(r)$;

(2) $V \cap B(f_x(r, \iota), \delta(r)) \neq \emptyset$ for uncountably many $x \in J$.

Further, we assume that $W(r) = \bigcup_{m \in \mathbb{N}} W_m(r)$.

Claim 2. *There is an $m(r) \in \mathbb{N}$ such that*

$$I_{m(r)}(r) = \{\iota \in I;\ W_{m(r)}(r) \cap W(r, \iota) \textit{ is "big"}\}$$

is uncountable.

Proof. Suppose by contradiction that $I_m(r)$ is countable for all $m \in \mathbb{N}$. Obviously, $I \setminus \bigcup_{m \in \mathbb{N}} I(m) \neq \emptyset$ and we choose $\iota \in I \setminus \bigcup_{m \in \mathbb{N}} I_m(r)$. Put

$$\mathbb{N}(r) = \{m \in \mathbb{N};\ W_m(r) \cap B(f_x(r, \iota), \delta(r)) \neq \emptyset \text{ for at most countably many } x \in J\}.$$

Consider the relatively open and "big" subset $W(r, \iota) \setminus \overline{\bigcup_{m \in \mathbb{N}(r)} W_m(r)}$ of $W(r, \iota)$. It is a Baire space and therefore, by the Baire category theorem, there is an $m(r) \in \mathbb{N} \setminus \mathbb{N}(r)$ such that $\overline{W_{m(r)}(r)}$ contains an interior point of $W(r, \iota) \setminus \overline{\bigcup_{m \in \mathbb{N}(r)} W_m(r)}$ and thus of $W(r, \iota)$ and $S(r)$ as well. Therefore $W_{m(r)}(r) \cap W(r, \iota)$ is "big" and $\iota \in I_{m(r)}(r)$, which is a contradiction.

4.6. Let $W(r) = \bigcup_{m \in \mathbb{N}} W_m(r)$ and $m(r) \in \mathbb{N}$ be as in the above Claim 2. Thus we may choose two distinct elements ι and ι' of $I_{m(r)}(r)$. Let t be an interior point of $\overline{W_{m(r)}(r)} \cap W(r, \iota)$ in $S(r)$.

Suppose that $W_{m(r)}(r) \in \mathfrak{A}(\beta(r))$ for some $\beta(r) \in (0, \pi)$. Therefore

$$\forall_{r > 0} \ \exists_{y_\tau \in B(t, \tau)} \ \exists_{v_\tau \in S(0,1)} \{W_{m(r)}(r) \cap A(y_\tau, v_\tau, \beta(r)) = \emptyset\}.$$

If τ is small enough the angle $A(y_\tau, v_\tau, \beta(r))$ is contained either in $A(t, \frac{e_\iota - t}{\|e_\iota - t\|}, \pi)$, or in the complement of the angle $A(t, \frac{e_\iota - t}{\|e_\iota - t\|}, \pi - \gamma(r))$. (Here and in what follows we use that t is an interior point of $\overline{W_{m(r)}(r)}$ in $S(r)$.)

If $A(y_\tau, v_\tau, \beta(r)) \subset A(t, \frac{e_\iota - t}{\|e_\iota - t\|}, \pi)$, then 4.4(a) applies and, by continuity of the scalar product and smoothness of $S(e_\iota, r)$, we get $\beta(r) < \frac{3}{4}\pi + \varphi_0 + \gamma(r)$ for τ small enough.

In the other case we use 4.4(b) and, using similar arguments as above, we get $\beta(r) < \frac{3}{4}\pi + \varphi_0 + 2\gamma(r)$ for τ sufficiently small again.

So $S(r) \notin \mathscr{A}_\sigma(\beta_r)$ for $\beta_r = \frac{3}{4}\pi + \varphi_0 + 2\gamma(r)$ and the assertion of Example is proved.

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