Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 33 (1992), No. 2, 41--44

Persistent URL: `http://dml.cz/dmlcz/701975`

Stability in Vector Valued l^∞-spaces

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Received 10 May 1992

A convex subset Q of topological vector space is called *stable* if the midpoint map $Q \times Q \ni (x, y) \to \frac{1}{2}(x + y) \in Q$ is open with respect to the inherited topology in Q. The purpose of the paper is to discuss stabiliy of the unit balls of spaces $l^\infty(E)$ and $\mathfrak{L}(l^1, E)$ (E is a Banach space).

For a Banach space E by $B(E)$ we denote the unit ball of E. By $l^\infty(E)$ we denote the Banach space of E-valued bounded sequences equipped with the norm $\|(x_n)\| = \sup_{n\in N}\|x_n\|$.

A convex set Q of a real Hausdorff topological vector space is called *stable* if the midpoint map $Q \times Q \in (x, y) \to \frac{1}{2}(x + y) \in Q$ is open with respect to the inherited topology in Q. Stable compact sets have been studied by Vesterstrøm [13], Lima [8], O'Brien [10]. But the interesting properties of the stable sets were presented by Clausing and Papadopoulou [2, 11, 12]. In particular a full description of stable convex subsets of finite dimensional topological spaces can be found in [11]. It also known description of stable unit balls in Orlicz spaces [6, 3, 14]. Note that stability is also a useful tool in studying extreme operators between Banach spaces. Stability arguments can be applied to the description of extreme points of the unit ball of the space of vector valued continuous maps $C(K, E)$, K being a compact Hausdorff space, namely, applying the Michael selection theorem [9],

$$f \in \operatorname{Ext} B(C(K, E)) \Leftrightarrow f(k) \in \operatorname{Ext} B(E) \quad \text{for every} \quad k \in K$$

provided $B(E)$ is stable.

The aim of this paper is to continue this investigation providing some additional facts and examples.

Let

$$Q_n = \operatorname{conv}\{(0, 1, 0), (0, -1, 0), (1, 0, 1), (1, 0, -1), (-1, 0, 1), (-1, 0, -1), \\ (1, 1/n, 0), (1, -1/n, 0), (-1, 1/n, 0), (-1, -1/n, 0)\}.$$

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Obviously Q_n in a stable convex symmetric compact subset of $\mathbb{R}^3$ and can be considered as a unit ball of some three dimensional Banach space. The norm on this space (generated by the Minkowski functional) we denote by $\|\cdot\|_n$. Therefore we have $B(\mathbb{R}^3, \|\cdot\|_n) = Q_n$.

Let $E_1 = \bigoplus_{n=1}^{\infty} (\mathbb{R}^3, \|\cdot\|_n)_1$ be the l^1 – sum of $(\mathbb{R}^3, \|\cdot\|_n)$. Namely E_1 consist of all real sequences whose norm $\|\cdot\|$ is finite, where

$$\|(x_i)\| = \sum_{n=1}^{\infty} \|(x_{3n-2}, x_{3n-1}, x_{3n})\|_n$$

The unit ball of E_1 is stable.

Indeed, fix $\varepsilon > 0$ and $x = (x_i)$, $y = (y_i)$, $z = (z_i) \in B(E_1)$ such that $x = (y + z)/2$. We need to find $\delta > 0$ such that for every $x' \in B(E_1)$ with $\|x - x'\| < \delta$ there exist $y', z' \in B(E_1)$ with $\|y - y'\| < \varepsilon$, $\|z - z'\| < \varepsilon$ such that $x' = (y' + z')/2$. Let N_0 be such that

$$\sum_{n=N_0+1}^{\infty} \|(x_{3n-2}, x_{3n-1}, x_{3n})\|_n < \varepsilon/4\,,$$

$$\sum_{n=N_0+1}^{\infty} \|(y_{3n-2}, y_{3n-1}, y_{3n})\|_n < \varepsilon/4\,,$$

$$\sum_{n=N_0+1}^{\infty} \|(z_{3n-2}, z_{3n-1}, z_{3n})\|_n < \varepsilon/4\,.$$

By stability of Q_n, let $\delta_n > 0$ such that for every

$(x'_{3n-2}, x'_{3n-1}, x'_{3n}) \in Q_n$ with

$$\|(x'_{3n-2}, x'_{3n-1}, x'_{3n}) - (x_{3n-2}, x_{3n-1}, x_{3n})\|_n < \delta_n$$

there exist $(y'_{3n-2}, y'_{3n-1}, y'_{3n})$, $(z'_{3n-1}, z'_{3n-2}, z'_{3n}) \in Q_n$ with

$$\|(y'_{3n-2}, y'_{3n-1}, y'_{3n}) - (y_{3n-2}, y_{3n-1}, y_{3n})\|_n < \varepsilon/_{2^{n+1}}\,,$$

$$\|(z'_{3n-2}, z'_{3n-1}, z'_{3n}) - (z_{3n-2}, z_{3n-1}, z_{3n})\|_n < \varepsilon/_{2^{n+1}}$$

such that

$$(x'_{3n-2}, x'_{3n-1}, x'_{3n}) = \tfrac{1}{2}[(y'_{3n-2}, y'_{3n-1}, y'_{3n}) + (z'_{3n-2}, z'_{3n-1}, z'_{3n})]\,.$$

Put $\delta = \min_{n \leq N_0} \delta_n$. Take $x' \in E_1$ with $\|x' - x\| < \delta$. Then from the above we choose y'_n, z'_n for $n \leqq 3N_0$ and we put $y'_n = z'_n = x'_n$ for $n > 3N_0$. We have $\|y' - y\| =$

$$\sum_{n=1}^{N_0} \|(y'_{3n-2}, y'_{3n-1}, y'_{3n}) - (y_{3n-2}, y_{3n-1}, y_{3n})\|_n +$$

$$+ \sum_{n=N_0+1}^{\infty} \|(y'_{3n-2}, y'_{3n-1}, y'_{3n}) - (y_{3n-2}, y_{3n-1}, y_{3n})\|_n \leqq$$

$$\leqq \sum_{n=1}^{N_0} \varepsilon/2^{n+1} + \varepsilon/4 + \varepsilon/4 < \varepsilon\,.$$

Similar by we get $\|z' - z\| < \varepsilon$, what ends the proof that $B(E_1)$ is stable.

Now we show that $B(l^\infty(E_1))$ is not stable though $B(E_1)$ is stable. Indeed, let

$f, g_1, g_2 \in B(l^\infty(E_1))$ be defined by

$$f(n) = e_{3n-2} \quad \text{and} \quad g_i(n) = e_{3n-2} + (-1)^i e_{3n} .$$

(we denote $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, \ldots), \ldots$).

We have $f = (g_1 + g_2)/2$. Now we consider a sequence $f_k \in B(l^\infty(E_1))$ defined by $f_k(n) = e_{3n-2} + \min(1/k, 1/n)\, e_{3n-1}$. Obviously $\|f_k - f\| = 1/k$. Suppose that $f_k = (h_1 + h_2)/2$, $h_1, h_2 \in B(l^\infty(E_1))$. Because $e_{3n-2} \pm 1/ne_{3n-1} \in \operatorname{ext} B(l^\infty(E_1))$ we have $h_1(n) = h_2(n) = f_k(n)$ for $n \geqq k$. Therefore $\|g_i - h_j\| = 1$ $(i, j = 1, 2)$. This shows that $B(l^\infty(E_1))$ is not stable.

Now me make a small modification of the above example. Let Q'_n be a symmetric compact strictly convex subset of $\mathbb{R}^3$ such that $\operatorname{ext} Q_n \subseteq \operatorname{ext} Q'_n$. We denote by $\|\cdot\|'_n$ the corresponding to Q'_n norm. And, for $p \in (1, \infty)$, let $E_p = \bigoplus_{n=1}^{\infty} (\mathbb{R}^3, \|\cdot\|'_n)_p$ be the l^p – sum of $(\mathbb{R}^3, \|\cdot\|'_n)$. The space E_p is equipped with the norm

$$\|(x_i)\| = \sum_{n=1}^{\infty} (\|(x_{3n-2}, x_{3n-1}, x_{3n})\|'^p_n)^{1/p} .$$

The space E_p is strictly convex with $B(E_p)$ stable but $B(l^\infty(E_p))$ is not stable. To get it we use the same arguments as before.

Let E, F be Banach spaces. By $\mathfrak{L}(E, F)$ we denote the linear space of all bounded linear operators from E into F. Note that the following spaces are isometrically isomorphic:

$$l^\infty(E) = \mathfrak{L}(l^1, E)$$

(and $\mathfrak{L}(E, l^\infty) = l^\infty(E^*) = \mathfrak{L}(l^1, E^*)$, E^* denotes the dual space of E).

Now we present some positive results.

J. A. Clarkson [1] has introduced a very important class of Banach spaces. Recall that E is *uniformly convex* if for every $\varepsilon \in (0, 2)$ there exists $\delta(\varepsilon) > 0$ such that $\|x - y\| > \varepsilon$ implies that $\|(x + y)/2\| < 1 - \delta(\varepsilon)$.

Theorem. *Let E be uniformly convex Banach space. Then $B(\mathfrak{L}(l^1, E)\,[B(l^\infty(E))]$ is stable.*

Proof. Let E be uniformly convex.

And let $T, R, S \in B(\mathfrak{L}(l^1, E))$ be such that $T = (R + S)/2$. Fix $\varepsilon > 0$. Let $\delta(\varepsilon) > 0$ be corresponding to ε with respect to the strict convexity of E. In particular the inequality $\|(Re_i + Se_i)/2\| \geqq 1 - \delta(\varepsilon)$ implies $\|Re_i - Se_i\| \leqq \varepsilon$.

Put $\delta_0 = \min(\varepsilon/2, \varepsilon\delta(\varepsilon)/4)$ and put $\lambda = \varepsilon/4$. Let $T' \in B(\mathfrak{L}(l^1, E))$ be such that $\|T' - T\| < \delta_0$. We need to find $R', S' \in B(\mathfrak{L}(l^1, E))$ with $\|R' - R\| < \varepsilon$, $\|T' - T\| < \varepsilon$ such that $T' = (R' + S')/2$.

If $\|Te_i\| \geqq 1 - \delta(\varepsilon)$ we put $R'e_i = S'e_i = T'e_i$. Then, be the strict convexity, we have

$$\|R'e_i - Re_i\| = \|T'e_i - Te_i\| + \|Te_i - Re_i\| \leqq \delta_0 + \varepsilon/2 < \varepsilon .$$

Similarly, $\|T'e_i - Te_i\| < \varepsilon$.

Now, if $\|Te_i\| < 1 - \delta(\varepsilon)$ we put

$$R'e_i = (T' - T)\, e_i + (1 - \lambda)\, Re_i + \lambda Te_i \quad \text{and}$$
$$S'e_i = (T' - T)\, e_i + (1 - \lambda)\, Se_i + \lambda Te_i\,.$$

We have

$$\|R'e_i\| = \|T' - T\| + (1 - \lambda) + \lambda(1 - \delta(\varepsilon)) < \delta_0 + 1 - \varepsilon\delta(\varepsilon)/4 \leqq 1$$
$$\text{and} \quad \|S'e_i\| \leqq 1\,.$$

And

$$\|(R' - R)\, e_i\| = \|T' - T\| + \lambda\|T - R\| < \delta_0 + 2\lambda \leqq \varepsilon$$
$$\text{and} \quad \|(S' - S)\, e_i\| \leqq \varepsilon\,,$$

what ends the proof. □

Corollary. *Let* $1 < p < \infty$. *Then* $B(\mathfrak{L}(l^1, l^p)\, [B(l^\infty(l^p))]$ *is stable.* □

Point out that $B(\mathfrak{L}(l^p, l^p))$ $(1 < p < \infty, p \neq 2)$ is not stable because ext $B(\mathfrak{L}(l^p, l^p)$ is not closed (see [4, 5]). Moreover in [4] is shown that $B(\mathfrak{L}(l_n^2, l_m^2))$, $n, m < \infty$, is stable. The problem if $B(\mathfrak{L}(H, H))$ and $B(\mathfrak{K}(H, H))$ are stable for infinite dimensional Hilbert space H is still open ($\mathfrak{K}(H, H)$ denotes the set of all compact operators on a Hilbert space H equipped with the operator norm). It is more so interesting as

$$f \in \operatorname{ext} B(C(K, \mathfrak{L}(H, H))) \Leftrightarrow f(k) \in \operatorname{ext} B(\mathfrak{L}(H, H)) \quad \text{for every} \quad k \in K$$

(see [7]).

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