Ján Haluška

On the generalized continuity of the semivariation in locally convex spaces

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 32 (1991), No. 2, 23--28

Persistent URL: http://dml.cz/dmlcz/701964

On the Generalized Continuity of the Semivariation in Locally Convex Spaces

J. HALUŠKA*)

Czechoslovakia

Received 11 March 1991

If Condition (GB) introduced in [7] and [8], is fulfilled, then the everywhere convergence of the net of measurable functions implies the convergence of these functions with respect to the semivariation on a set of the finite variation of the measure $m : \Sigma \to L(X, Y)$, where Σ is a σ-algebra of subsets of the set $T \neq 0$, X, Y are both locally convex spaces. The generalized strong continuity of the semivariation of the measure, introduced in this paper, implies Condition (GB).

Introduction

The notion of the continuity of the semivariation of the measure is needed in many occasions in the integration theory with respect to the operator valued measure, cf. [4], e.g. convergence theorems are based on this notion in the case of operator valued measure countable additive in the strong operator topology, cf. also [2] and [5]. Our notion of the generalized continuity of the semivariation of the measure enables us to develop a concept of an integral with respect to the $L(X, Y)$-valued measure based on the net convergence of simple functions, where both X, Y are locally convex topological spaces. Of course, the using of nets instead sequences leads to the generalization of the notion of the continuity of the measure which is sufficiently "fine". More precisely, we use the notion of the inner semivariation for this generalization. This way we restrict the set of $L(X, Y)$-valued measures which can be taken for such type of integration. For instance, every atomic measure is generalized continuous. So, the set of measures with the generalized continuous semivariation is a nonempty set.

For terms concerning the nets cf. [9]. For Condition (GB) cf. [7], [8].

If Condition (GB) is fulfilled, then the everywhere convergence of the net of measurable functions implies the convergence of these functions with respect to the semivariation on the set of the finite variation. The generalized strong continuity

*) Mathematical Institute of Slovak Academy of Sciences, Grešákova 6, 040 01 Košice, Czechoslovakia.

of the semivariation implies Condition (GB). The classical Lebesgue measure does not satisfy Condition (GB).

1. Definition

Let $T \neq \emptyset$ be a set and let Σ be a σ-algebra of subsets of T. Let X, Y be two Hausdorff locally convex topological vector spaces. Let P and Q be two families of seminorms which define the topologies on X and Y, respectively, $\mathbb{N} = \{1, 2, \ldots\}$. Let $m : \Sigma \to L(X, Y)$ be an operator valued measure σ-additive in the strong operator topology, i.e. if $E \in \Sigma$, then $m(E)x$ is an Y-valued measure for every $x \in X$.

Definition 1.1.

(a) By the p, q-semivariation of the measure m, cf. [1], [11], we mean the family of set functions $\hat{m}_{p,q} : \Sigma \to [0, \infty]$, $p \in P$, $q \in Q$, defined as follows:

$$\hat{m}_{p,q}(E) = \sup q(\sum_{i=1}^{I} m(E_i)x_i),\ E \in \Sigma\ ,$$

where the supremum is taken over all finite disjoint partitions $\{E_i \in \Sigma : i = 1, 2, \ldots, I,\ E = \bigcup_{i=1}^{I} E_i,\ E_i \cap E_j = \emptyset,\ i \neq j\}$, of E, and all finite sets $\{x_i \in X\colon\ p(x_i) \leqq 1,\ i = 1, 2, \ldots, I\}$.

(b) By the p,q-variation of the measure m we mean the set function $v_{p,q}(m, .) : \Sigma \to [0, \infty]$ defined by the equality:

$$v_{p,q}(m, E) = \sup \sum_{i=1}^{I} q_p(m(E_i)),\ E \in \Sigma\ ,$$

where the supremum is taken over all finite disjoint partitions $\{E_i \in \Sigma : i = 1, 2, \ldots, I,\ E = \bigcup_{i=1}^{I} E_i,\ E_i \cap E_j = \emptyset,\ i \neq j\}$, of E and $q_p(m(E)) = \sup_{p(x) \leqq 1} q(m(E)x)$, $p \in P$, $q \in Q$.

(c) By the inner p,q-semivariation of the measure m we mean the set function $\hat{m}^*_{p,q} : 2^T \to [0, \infty]$, $q \in Q$, $p \in P$, defined as follows:

$$\hat{m}^*_{p,q}(F) = \sup_{E \subset F, E \in \Sigma} \hat{m}_{p,q}(E)\ .$$

Lemma 1.2. *The p,q-(semi)variation of the measure m is a monotone, σ-additive (σ-subadditive) set funtion, and*

$$v_{p,q}(m, \emptyset) = 0,\ (\hat{m}_{p,q}(\emptyset) = 0)\ ,$$

for every $p \in P$, $q \in Q$.

Proof. Trivial. □

Definition 1.3.

(a) The set $E \in \Sigma$ is said to be of the positive variation if there exist $q \in Q$, $p \in P$, such that $v_{p,q}(m, E) > 0$.

(b) We will say that $E \in \Sigma$ is a $\hat{m}$-null set if $\hat{m}_{p,q}(E) = 0$ for every $p \in P$, $q \in Q$.

(c) We will say that the set $E \in \Sigma$ is of the finite variation of the measure m if to every $q \in Q$ there exists a $p \in P$, such that $v_{p,q}(m, E) < \infty$. We will denote this relation shortly $Q \to {}_E P$, or, $q \mapsto {}_E p$, $q \in Q$, $p \in P$.

Remark 1.4. The relation $Q \to_E P$ in Definition 1.3(c) may be different for different sets of the finite variation of the measure m.

Definition 1.5. We say that the measure m satisfies Condition (GB) if for every $E \in \Sigma$ of the finite and positive variation, and every net of sets $E_i \in \Sigma$, $E_i \subset E$, $i \in I$, there exist real numbers $\delta(q, p, E) < 0$, $p \in P$, $q \in Q$, such that the following implication

$$\hat{m}_{p,q}(E_i) \geqq \delta(q, p, E),\ i \in I \Rightarrow \limsup_{i \in I} E_i \neq \emptyset$$

hold for every couple $(p, q) \in P \times Q$, such that $q \mapsto_E p$.

Definition 1.6. We say that the set $E \in \Sigma$ of the positive variation is an $\hat{m}$-atom if every subset a of E is either $\emptyset$ or $a \notin \Sigma$. We say that the measure m is atomic if each $E \in \Sigma$ can be expressed in the form $E = \bigcup_{i=1}^{\infty} a_i$, where a_i, $i \in \mathbb{N}$, are $\hat{m}$-atoms.

2. On the generalized continuity of the semivariation

Theorem 2.1. *If m is a continuous semivariation of the $L(X, Y)$-valued measure σ-auditive in the strong operator topology, where X, Y are Banach spaces, then the measure m satisfies Condition (GB) for sequences.*

Proof. Let $E \in \Sigma$, $0 < v(m, E) < \infty$, be given. Let $E_i \in \Sigma$, $E_i \subset E$, $i \in \mathbb{N}$, be a sequence of sets, such that there exists $\delta < 0$, such that $\hat{m}(E_i) \geqq \delta$ foe every $i \in \mathbb{N}$. We have:

$$\limsup_{i \in \mathbf{N}} E_i = \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} E_j = \bigcap_{i=1}^{\infty} H_i ,$$

where $H_i = \bigcup_{j=i}^{\infty} E_j \in \Sigma$, $i \in \mathbb{N}$, is a nonincreasing set sequence. Since $\hat{m}$ is continuous, there is:

$$(1) \qquad \hat{m}(\limsup_{i \in \mathbf{N}} E_i) = \hat{m}\Big(\bigcap_{i=1}^{\infty} H_i\Big) = \lim_{i \in \mathbf{N}} \hat{m}(H_i) .$$

Since $\hat{m}(E_i) \geqq \delta$ and $E_i \subset H_i$ for every $i \in \mathbb{N}$, there is $\hat{m}(H_i) \geqq \delta$ for every $i \in \mathbb{N}$. Thus (1) implies $\hat{m}(\limsup_{i \in \mathbf{N}} E_i) \geqq \delta$, i.e. $\limsup_{i \in \mathbf{N}} E_i \neq \emptyset$. □

Lemma 2.2. *Let Σ contain all singletons (= one-point subsets) of the set $E \in \Sigma$ and $\hat{m}_{p,q}(\{t\}) = 0$ for every $t \in E$ and every couple $(p, q) \in P \times Q$. Let E be of the positive and finite variation. Then Condition (GB) is not satisfied.*

Proof. Denote $E_{t_1,\dots,t_n} = E \setminus \{t_1, \dots, t_n\}$, $n \in \mathbb{N}$. Then the net $(E_{t_1,\dots,t_n} : t_1, \dots, t_n \in E,\ n \in \mathbb{N}$ is a nondecreasing net of subsets of the set E_i with $\bigcap_{t_1,\dots,t_n \in E} E_{t_1,\dots,t_n} = \emptyset$. Show that

$$\hat{m}_{p,q}(E) = \hat{m}_{p,q}{'}E_{t_1,\dots,t_n}) \tag{2}$$

for every $p \in P$, $q \in Q$, and every $t_1, \dots, t_n \in E$, $n \in \mathbb{N}$.

We have:

$$\hat{m}_{p,q}(E) = \hat{m}_{p,q}([E \setminus \{t_1, \dots, t_n\}] \cup \{t_1, \dots, t_n\}) \leqq \hat{m}_{p,q}(E \setminus \{t_1, \dots, t_n\}) + \\ + \sum_{i=1}^{n} \hat{m}_{p,q}(\{t_i\}) = \hat{m}_{p,q}(E \setminus \{t_1, \dots, t_n\}),$$

for every $q \in Q$, $p \in P$, and every finite set $\{t_1, \dots, t_n\}$, $n \in \mathbb{N}$, of points from E. The inverse inequality follows from the fact that the p, q-semivariation, $p \in P$, $q \in Q$, is a monotone set function.

So, we found the net $(E_{t_1,\dots,t_n} \in \Sigma,\ t_1, \dots, t_n \in E,\ n \in \mathbb{N})$, of sets with the empty intersection and such that

$$\hat{m}_{p,q}(E_{t_1,\dots,t_n}) = \hat{m}_{p,q}(E) = \delta(q, p, E) > 0$$

for some $p \in P$, $q \in Q$, such that $q \mapsto_E p$. So, Condition (GB) is not satisffied. □

Corollary 2.3. *The Lebesgue measure does not satisfy Condition (GB) on the real line (for arbitrary nets of sets).*

Corollary 2.4. *If the operator valued measure m satisfies Condition (GB), then for every $E \in \Sigma$ of the positive and finite variation there exists a singleton $\{t\}$, $t \in E$, such that $\hat{m}_{p,q}(\{t\}) > 0$ for some $p \in P_i$ $q \in Q$. Such singleton is clearly an $\hat{m}$-atom.*

Definition 2.6. We say that the semivariation of the measure $m : \Sigma \to L(X, Y)$ is GS-(= generalized strongly) continuous if for every set of the finite variation $E \in \Sigma$ and every monotone net of sets $E_i \subset T$, $E_i \subset E$, $i \in I$, the following equality

$$\lim_{i \in I} \hat{m}^*_{p,q}(E_i) = \hat{m}^*_{p,q}(\lim_{i \in I} E_i)$$

holds for every couple $(p, q) \in P \times Q$, such that $q \mapsto_E p$.

Theorem 2.7. *If the semivariation of the measure $m : \Sigma \to L(X, Y)$ is GS-continuous, then the measure m satisfies Condition (GB).*

Proof. Let $E \in \Sigma$ be a set of the positive and finite variation. Let $E_i \subset E$, $\hat{m}^*_{p,q}(E_i) \geqq \geqq \delta = \delta(q, p, E) > 0$ for every $i \in I$. We have:

$$\limsup_{i \in I} E_i = \bigcap_{i \in I} \bigcup_{j \geqq i} E_j = \bigcap_{i \in I} H_i,$$

where H_i, $i \in I$, is a nonincreasing net of sets. So, by assumption

$$\text{(3)} \qquad \lim_{i \in I} \hat{m}^*_{p,q}(H_i) = \hat{m}^*_{p,q}(\lim_{i \in I} H_i) = \hat{m}^*_{p,q}(\bigcap_{i \in I} H_i) = \hat{m}^*_{p,q}(\limsup_{i \in I} E_i) .$$

Since $\hat{m}^*_{p,q}(E_i) \geqq \delta$ and $H_i \supset E_i$ for every $i \in I$, we have $\hat{m}^*_{p,q}(H_i) \geqq \delta$, too. Then (3) implies $\hat{m}^*_{p,q}(\limsup_{i \in I} E_i) \geqq \delta$. Recall the definition of the inner p,q-semivariation:

$$\hat{m}^*_{p,q}(F) = \sup_{E \subset F, E \in \Sigma} \hat{m}_{p,q}(E), F \in 2^T .$$

So, there is a couple $(p, q) \in P \times Q$, $q \to_E p$, such that for every $\varepsilon > 0$, $\delta > \varepsilon > 0$, there exists a set $G \in \Sigma$, $G \subset \limsup_{i \in I} E_i$, such that

$$0 < \delta - \varepsilon \leqq \hat{m}^*_{p,q} (\limsup_{i \in I} E_i) - \varepsilon \leqq \hat{m}_{p,q}(G) ,$$

and therefore $G \neq \emptyset$, i.e. $\limsup_{i \in I} E_i \neq \emptyset$ and Condition (GB) is satisfied. □

Example 2.8. Let $m: \Sigma \to L(X, Y)$ be a (countable) atomic measure. The the semivariation of the measure m is GS-continuous.

Indeed, let $E \in \Sigma$ be a set of the finite and positive variation. Let E_i, $i \in I$, be an arbitrary decreasing net of sets. Recall that $E_i \searrow G$ $(\in 2^T)$ iff (1) $i \leqq j \Rightarrow E_i \supset E_j$, (2) $\bigcap_{i \in I} E_i = G$. It is clear that it is enough to consider the case $G = \emptyset$, because $E_i \searrow G \Leftrightarrow G \subset E_i$, $E_i \setminus G \searrow \emptyset$.

In the case $E_i \in \Sigma$, $i \in I$, there is

$$\text{(4)} \qquad \lim_{i \in I} \hat{m}^*_{p,q}(E_i) = \lim_{i \in I} \hat{m}_{p,q}(E_i) ,$$

and since the family of all atoms is a countable set, there is $\lim_{i \in I} E_i = \bigcap_{i \in I} E_i \in \Sigma$ and therefore

$$\text{(5)} \qquad \hat{m}^*_{p,q}(\lim_{i \in I} E_i) = \hat{m}_{p,q}(\lim_{i \in I} E_i) ,$$

for every $p \in P$, $q \in Q$.

Take an arbitrary set $E \in \Sigma$ of the positive and finite variation. Denote A the set of all $\hat{m}$-atoms. Denote $k(i, E) = (A \cap E) \setminus E_i$, $i \in I$. Clearly

$$i \leqq j,\ i, j \in I \Rightarrow k(i, E) \subset k(j, E)$$

and there exist atoms $a_n \in A$, $n \in \mathbb{N}$, such that $k(i, E) = \{a_1, a_2, \ldots, a_n, \ldots\}$.

By Lemma 1.2. we have

$$v_{p,q}(m, E) = v_{p,q}(m, E_i) + \sum_{a_n \in k(i,E)} v_{p,q}(m, a_n) ,$$

where $i \in I$, $p \in P$, $q \in Q$. Since

$$\hat{m}_{p,q}(E_i) \leqq v_{p,q}(m, E_i),\ i \in I,\ p \in P,\ q \in Q\ ,$$

there is

$$\hat{m}_{p,q}(E_i) \leqq v_{p,q}(m, E) - \sum_{a_n \in k(i,E)} v_{p,q}(m, a_n)\ ,$$

where $i \in I$, $p \in P$, $q \in Q$. Since $v_{p,q}(m, E \cap \cdot) : \Sigma \to [0, \infty)$, is a finite real measure for every $p \in P$, $q \in Q$, such that $q \mapsto_E p$, then for every $\varepsilon > 0$, $q \in Q$, $p \in P$, such that $q \mapsto_E p$, there exists an index $i_0 = i_0(\varepsilon, p, q, E) \in I$, such that

$$\hat{m}_{p,q}(m, E_i) < \varepsilon \tag{6}$$

holds for every $i \geqq i_0$, $i \in I$. Combining (4), (5), (6) and Definition 2.6 we see that the assertion is proved for the case when $E_i \in \Sigma$, $i \in I$, is a decreasing net of sets. The other cases of monotone nets of sets can be proved analogously.

Let now $G \subset T$ an arbitrary set. Then there is a set $F^* = A \cap G$ with the property

$$\hat{m}^*_{p,q}(G) = \sup_{F \subset G, F \in \Sigma} \hat{m}_{p,q}(F) = \hat{m}_{p,q}(F^*)\ ,$$

for every $p \in P$, $q \in Q$. The proof for the inner semivariation and the arbitrary net of subsets $E_i \subset T$, $i \in I$, we obtain now repeating the previous procedure of the proof concerning the set system Σ. □

References

[1] Bartle, R. G., A general bilinear vector integral, Studia Math. 15 (1956), 337—352.
[2] Bourbaki, N., Eléments de Mathématique, livre VI (Integration), Herman, Paris, 1952, 1956.
[3] Dobrakov I., On integration in Banach spaces, I, Czech. Math. J. 20 (1970), 511—536.
[4] Dobrakov, I., On integration in Banach spaces, II. Czech. Math. J. 20 (1970), 680—695.
[5] Dobrakov, I., On extention of submeasures. Math. Slovaca 34 (1984), 265—271.
[6] Goguadze, B. F., On Kolmogoroff integrals and some their applications, (in russian), Micniereba, Tbilisi, 1979.
[7] Haluška, J., On Goguadze-Luzin's L(X, Y)-measure condition in locally convex spaces. Proc. of the 2nd Winter school on measure theory (Liptovský Ján), Slovak Acad. of Sci., Bratislava, 1990, 28—31.
[8] Haluška, J., On the semivariation in locally convex spaces. Math. Slovaca (to appear).
[9] Kelley, J. L., General topology. Van Nostrand, New York, 1955.
[10] Luzin, N. N., Collected works. Integral and trigonometric series, (in russian). Izd. Akad. Nauk SSSR, Moscow, 1953, p. 58.
[11] Smith, W. and Tucker, D. H., Weak integral convergence theorem and operator measures. Pacif. J. of Math. 111 (1984), 243—256.