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In: Zdeněk Frolík and Vladimír Souček and Marián J. Fabián (eds.): Proceedings of the 14th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1987. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 14. pp. [329]--333.

Persistent URL: http://dml.cz/dmlcz/701904

ALMOST CONTINUOUS FUNCTIONS WITH CLOSED GRAPHS

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A function $f: X \longrightarrow Y$ is almost continuous if for every $x \in X$ and for every open set $V \subset Y$ containing $f(x)$, $\overline{f^{-1}(V)}$ is a neighbourhood of x. The main theorem of this paper states that if $f: X \longrightarrow Y$ is almost continuous with a closed graph (closed in $X \times Y$), X is a locally almost countably complete space and Y is a regular space, which possesses a complete sequence of open coverings of Y, then f is continuous.

1. Introduction

In the theory of functions with closed graphs the notion of the almost continuity is essentially used. We will use this notion to prove a general theorem on closed graphs where the underlying spaces are those complete in the sense of Frolík.

The notion of the almost continuity was introduced by Blumberg in 1922 and used by Pták, Husain and several other authors.

This paper completes the results of the papers [1] and [2]. The main theorem of [1] states that if $f: X \longrightarrow Y$ is almost continuous with a closed graph and X and Y are complete metric spaces, then f is continuous. In case X and Y are complete in the sense of Čech this theorem is proved in [2].

This paper is in final form and no version of it will be submitted for publication elsewhere.

2.

In what follows X, Y denote topological spaces. For a subset A of a topological space denote $\overline{A}$ and Int A the closure and the interior of A respectively.

The intersection of a family $\mathcal{U}$ of sets will be denoted by $\bigcap \mathcal{U}$. N denotes a set of all positive integers.

Now, let us recall some definitions and basic facts which will be used throughout this note.

Definition 1. (See [4]) Let $\{\mathcal{U}_n\}$ be a sequence of open families (an open family is a family consisting of open sets) in a space X. The sequence $\{\mathcal{U}_n\}$ is said to be countably complete if , for every centered sequence of sets $\{A_{n_k}\}$, where $A_{n_k} \in \mathcal{U}_{n_k}$, the set $\bigcap_{k=1}^{\infty} \overline{A}_{n_k} \neq \emptyset$. The sequence $\{\mathcal{U}_n\}$ is said to be strongly countably complete if the following condition is satisfied: If $\{F_n\}$ is a centered sequence of closed subsets of X and if every F_n is contained in some $A_n \in \mathcal{U}_n$, then the set $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Definition 2. (See [3]) A sequence $\{\mathcal{U}_n\}$ of open coverings of a space X is said to be complete if the following condition is satisfied: If $\mathcal{F}$ is a centered family of closed subsets of the space X such that for every n = 1, 2, ... some $F_n \in \mathcal{F}$ is contained in some $A_n \in \mathcal{U}_n$, then $\bigcap \mathcal{F} \neq \emptyset$.

Definition 3. (See [4]) A space X is said to be countably complete if there exists a countably complete sequence of open bases for X. X is said to be strongly countably complete if there exists a strongly countably complete sequence of open coverings of X.

It is known that a Tychonoff space is complete in the sense of Čech iff it has a complete sequence of open coverings.

It is easy to see that, every regular strongly countably complete space is a countably complete space. According to [3], example 3.1 there exists a completely regular countably compact space, which is not complete in the sense of Čech, that means there exists a completely regular countably complete space, which is not complete in the sense of Čech.

Definition 4. (See [4]) An open almost-base for a space X is a family $\mathcal{U}$ of open subsets of X such that every non-void open subset of X contains some non-void $A \in \mathcal{U}$.

A space X is said to be an almost countably complete space if there exists a countably complete sequence of open almost-bases for X. A space X is said to be locally almost countably complete if and only if every $x \in X$ has a neighbourhood which is an almost countably complete space.

Definition 5. (See [1]) The function $f: X \longrightarrow Y$ is almost continuous at $x \in X$ if and only if for each open $V \subset Y$ containing f(x), $x \in \operatorname{Int} \overline{f^{-1}(V)}$.

Definition 6. We say that the diameter of a subset M of a space X is less than a covering $\mathcal{U} = \{A_s : s \in S\}$ of this space ($\operatorname{diam} M < \mathcal{U}$) provided there exists an $s \in S$ such that $M \subset A_s$.

Theorem 1. Let X be a locally almost countably complete space, Y be a regular space. Suppose Y possesses a complete sequence of open coverings of Y. Let $f\colon X \longrightarrow Y$ be an almost continuous function with a closed graph. Then f is continuous.

Proof. Suppose f is not continuous at a point $p \in X$. Let U be an almost countably complete neighbourhood of the point p. Let $\{\mathcal{U}_n\}_{n=1}^{\infty}$ be a countably complete sequence of open in U almost bases for U and $\{\mathcal{V}_n\}_{n=1}^{\infty}$ be a complete sequence of open coverings of Y.

We will inductively define a sequence $\{p_i\}_{i=1}^{\infty}$ of points of X, a sequence $\{V_i\}_{i=1}^{\infty}$ of open subsets of Y, sequences $\{G_i\}_{i=1}^{\infty}$, $\{U_i\}_{i=1}^{\infty}$ of open subsets of X satisfying the following conditions:

(i) $p_i \in U_i$, $i = 1, 2, \dots$

(ii) $f(p_i) \in V_i$, $i = 1, 2, \dots$

(iii) $\overline{V_1} \cap \overline{V_2} = \emptyset$

(iv) if i and j are either both even or both odd and $i < j$ then $\overline{V_j} \subset V_i$

(v) $p_{i+2} \in G_i$, $i = 1, 2, \dots$

(vi) $G_i \subset U_{i+1}, U_{i+2} \subset G_i$, $G_{i+1} \subset G_i$, $i = 1, 2, \dots$

(vii) $U_i \subset f^{-1}(V_i)$, $U_{i+1} \subset U_i$, $i = 1, 2, \dots$

(viii) $\operatorname{diam}(V_1) < \mathcal{V}_1$, $\operatorname{diam}(V_i) < \mathcal{V}_{i-1}$, $\operatorname{diam}(V_i) < \mathcal{V}_i$, $i = 2, 3, \dots$

Put $p_1 = p$. There is an open set V containing $f(p_1)$ such that $f^{-1}(V)$ is not a neighbourhood of p_1. Let V_1 be an open set containing $f(p_1)$ such that $\operatorname{diam}(V_1) < \mathcal{V}_1$ and $\overline{V_1} \subset V$. By the almost continuity of f at p_1, $p_1 \in \operatorname{Int} \overline{f^{-1}(V_1)}$. Put $U_1 = \operatorname{Int} \overline{f^{-1}(V_1)} \cap \operatorname{Int} U$. Then $U_1 \subset \overline{f^{-1}(V_1)}$. There must be a point $p_2 \in U_1$ such that $f(p_2) \notin V$ (thus $f(p_2) \notin \overline{V_1}$). Let V_2 be an open set containing $f(p_2)$ such that $\overline{V_1} \cap \overline{V_2} = \emptyset$, $\operatorname{diam}(V_2) < \mathcal{V}_1$, $\operatorname{diam}(V_2) < \mathcal{V}_2$. The almost continuity of f at p_2 implies $p_2 \in \operatorname{Int} \overline{f^{-1}(V_2)}$. Put $U_2 = \operatorname{Int} \overline{f^{-1}(V_2)} \cap U_1$. Then $U_2 \subset U_1$ and $U_2 \subset \overline{f^{-1}(V_2)}$. There exists $G_1 \in \mathcal{U}_1$ such that $G_1 \neq \emptyset$, $G_1 \subset U_2$ and G_1 is open in X. The inclusions $G_1 \subset U_2$ and $U_2 \subset U_1$, $U_1 \subset \overline{f^{-1}(V_1)}$ imply there exists a point $p_3 \in G_1$ such that $f(p_3) \in V_1$.

Let $j \geq 2$. Suppose now we have defined V_i, U_i, p_i for all $i \leq j$ and G_i for all $i \leq (j-1)$ satisfying (i)-(viii). Since G_{j-1} is open in X and $\emptyset \neq G_{j-1} \subset U_j \subset U_{j-1}$ and $U_{j-1} \subset \overline{f^{-1}(V_{j-1})}$ there is a point $p_{j+1} \in G_{j-1}$ such that $f(p_{j+1}) \in V_{j-1}$. Let V_{j+1} be an open set containing $f(p_{j+1})$ such that $\overline{V_{j+1}} \subset V_{j-1}$ and $\operatorname{diam}(V_{j+1}) < \mathcal{V}_j$, $\operatorname{diam}(V_{j+1}) < \mathcal{V}_{j+1}$. The almost continuity of f at p_{j+1} implies $p_{j+1} \in \operatorname{Int} \overline{f^{-1}(V_{j+1})}$. Put $U_{j+1} = \operatorname{Int} \overline{f^{-1}(V_{j+1})} \cap G_{j-1}$. Then $U_{j+1} \subset G_{j-1}$, $U_{j+1} \subset \overline{f^{-1}(V_{j+1})}$. Since U_{j+1} is non-empty open set in X and $U_{j+1} \subset G_{j-1}$, there exists a non-empty open set $G_j \in \mathcal{U}_j$ such that $G_j \subset U_{j+1}$ and $G_j \subset G_{j-1}$. This completes the inductive definitions.

$\bigcap_{i=1}^{\infty} \overline{G}_i \neq \emptyset$. (U is almost countably complete, that means $\bigcap_{i=1}^{\infty} (\overline{G}_i \cap U) \neq \emptyset$). Let $x \in \bigcap_{i=1}^{\infty} \overline{G}_i$. Let H be a neighbourhood of x. Since $\bigcap_{i=1}^{\infty} \overline{G}_i \subset \bigcap_{i=1}^{\infty} \overline{f^{-1}(V_{2i-1})}$ there exists a net $\{x_{2i-1}\}_{i=1}^{\infty}$ such that $x_{2i-1} \in H$ and $f(x_{2i-1}) \in V_{2i-1}$ for $i = 1, 2, \ldots$

Let $\mathcal{O}$ be an open neighbourhood base at the point x. $\mathcal{O} \times N$ is directed set. (If $(U,n) \in \mathcal{O} \times N$ and $(V,m) \in \mathcal{O} \times N$ then $(U,n) \geq (V,m)$ if and only if $U \subset V$ and $n \geq m$). Define a net $\{x_{O,n}: O \in \mathcal{O}, n \in N\}$ as follows. Let $O \in \mathcal{O}$, $n \in N$. O is a neighbourhood of x, $x \in \overline{f^{-1}(V_{2n-1})}$. Let $x_{O,n}$ be such point in O for which $f(x_{O,n}) \in V_{2n-1}$. It is clear that the net $\{x_{O,n}: O \in \mathcal{O}, n \in N\}$ converges to x. (Let V be a neighbourhood of x. If $(B,m) \geq (V,1)$ then $x_{B,m} \in V$). There exists a cluster point of the net $\{f(x_{O,n}): O \in \mathcal{O}, n \in N\}$. Put $A_{O,n} = \{f(x_{B,m}): (B,m) \geq (O,n)\}$. The system $\{\overline{A}_{O,n}: O \in \mathcal{O}, n \in N\}$ satisfies the conditions of Definition 3. for complete sequence of open coverings of Y, that is $\bigcap \{\overline{A}_{O,n}: O \in \mathcal{O}, n \in N\} \neq \emptyset$. Let $y_1 \in \bigcap \{\overline{A}_{O,n}: O \in \mathcal{O}, n \in N\}$. By Theorem 2.2 in [6] y_1 is a cluster point of the net $\{f(x_{O,n}): O \in \mathcal{O}, n \in N\}$.

Since $\bigcap_{i=1}^{\infty} \overline{G}_i \subset \bigcap_{i=1}^{\infty} \overline{f^{-1}(V_{2i})}$, $x \in \bigcap_{i=1}^{\infty} \overline{f^{-1}(V_{2i})}$. We will define analogical the nets $\{y_{O,n}: O \in \mathcal{O}, n \in N\}$ and $\{f(y_{O,n}): O \in \mathcal{O}, n \in N\}$ such that $y_{O,n} \in O$ and $f(y_{O,n}) \in V_{2n}$. The net $\{y_{O,n}: O \in \mathcal{O}, n \in N\}$ converges to x and there exists a cluster point y_2 of the net $\{f(y_{O,n}): O \in \mathcal{O}, n \in N\}$ by similar argument as above.

Since $y_1 \in \overline{V_1}$ and $y_2 \in \overline{V_2}$, $y_1 \neq y_2$. But the points (x,y_1) and (x,y_2) are both limit points of the graph of f, contradicting the fact the graph of f is closed.

Corollary 1. (See [2]) Let X, Y be spaces complete in the sense of Čech. If $f: X \longrightarrow Y$ is an almost continuous function with a closed graph, then f is continuous.

Theorem 2. Let X be a first countable locally complete space, Y be a regular strongly countably complete space. If $f: X \longrightarrow Y$ is an almost continuous function with a closed graph, then f is continuous.

Proof. We will proceed as well as in the proof of Theorem 1. Since there exists a countable open neighbourhood base at the point x, the systems $\mathcal{O}$ and $\{\overline{A}_{O,n}: O \in \mathcal{O}, n \in N\}$ in the proof of Theorem 1. are countable.

Definition 7. A subset of a topological space is called almost open if it is in the interior of its closure, a function is called almost open if the image of every open subset is almost open.

Corollary 2. Let X, Y be topological spaces. Let Y be a locally almost countably complete space, X be a regular space. Suppose X possesses a complete sequence of open coverings of X. Let f: X⟶Y be a bijective almost open function with a closed graph. Then f is open.

Acknowledgement. The author thanks to Prof.T.Neubrunn for discussing the paper.

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