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ON SHAPE GROUPS AND ČECH HOMOLOGY GROUPS OF A COMPACT SPACE

Davide Carlo Demaria - Rosanna Garbaccio Bogin

Given a pretopological space $S=(X,P)$, we associate to any interior covering X of S a symmetrical pf-space S_X on the set X (see [2], [3]). Precisely, to obtain the pretopology of S_X, we take for each point x of X the principal filter $\overline{St(x,X)}$.

Then we associate to S the inverse system $\hat{S}=(S_X,\ p_{XX'},\ Cov(S))$, where $p_{XX'}:S_{X'}\to S_X$ is the identity in X and Cov(S) is the collection of all interior coverings of S.

For each dimension n, we associate to $\hat{S}$ an inverse system of prehomotopy groups $\Pi_n(S_X,\ a)$ and an inverse system of singular homology groups $H_n(S_X)$. Taking the inverse limits $\varprojlim \Pi_n(S_X,\ a)$ and $\varprojlim H_n(S_X)$, we obtain the shape groups $\check{\Pi}_n(S,\ a)$ and the Čech homology groups $\check{H}_n(S)$ of the pretopological space S.

In this way, if S is a topological space, instead to approximate it by means of polyhedra, we reduce the more the set of admissible functions into S, in such a way to obtain the set of continuous maps.

Here we prove that our shape groups and Čech homology groups of a connected compact topological space S are isomorphic to the classical ones.(1)

In [2] we proved that, if the covering $X=\{X_i\}(i\in J)$ is finite, then S_X belongs to the same homotopy type of a finite symmetrical pf-space (i.e. an undirected graph) $G'(X)$, that we obtain in the following way. The vertices $v_{i_1\dots i_n}$ of $G'(X)$ correspond to the maximal subsets $\{i_1,\dots,i_n\}$ of J such that $\bigcap_{r=1}^{n} X_{i_r}\neq\emptyset$, and there is the edge $v_{i_1\dots i_n}v_{j_1\dots j_m}$ iff $\{i_1,\dots,i_n\}\cap\{j_1,\dots,j_m\}\neq\emptyset$.

Here (§2, §3) we consider a suitable collection Cov'(S) of open coverings of S which is cofinal in Cov(S), and for any $X\in Cov'(S)$ we construct an open covering Z such that the nerve $N(X)$ of X is isomorphic to the complex $K_{G'(Z)}$ of the graph $G'(Z)$. This is possible if the covering X is independent and non singular. In fact, if X is independent, we obtain Z such that the graph $G_{N(X)}$ of the edges of $N(X)$ is

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This paper is in final form and no version of it will be submitted for publication elsewhere.

(1) Any compact topological space is supposed to be Hausdorff. Moreover we consider only infinite spaces, since any finite connected compact space is a singleton.

isomorphic to $G'(Z)$. Moreover, if X is also non singular, the complex $N(X)$ is complete and therefore isomorphic to the complex $K_{G_{N(X)}}$.

Afterwards, given $X=\{X_i\}(i\in I)$ and $X'=\{X'_h\}(h\in H)$ in $Cov'(S)$ such that $X\leq X'$ and $Z\leq Z'$ and a suitable function $\phi:H\to J$ such that $X'_h\subseteq X_{\phi(h)}$ for each $h\in H$, we show that the following diagram over pretopological spaces:

$$\begin{array}{ccc}
S_{Z'} & \xrightarrow{p_{ZZ'}} & S_Z \\
\downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\
G'(Z') & \xrightarrow{\tilde{\phi}} & G'(Z) \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
G_{N(X')} & \xrightarrow{\bar{\phi}} & G_{N(X)}
\end{array}$$

where $\bar{\phi}$ and $\tilde{\phi}$ are precontinuous maps induced by ϕ, is such that $\bar{\phi}f'=f\tilde{\phi}$ and $\tilde{\phi}p'\sim pp_{ZZ'}$.

Hence (§4) we obtain the following commutative diagrams:

$$\begin{array}{ccc}
\Pi_n(S_{Z'},a) & \xrightarrow{p^*_{ZZ'}} & \Pi_n(S_Z,a) \\
\downarrow{\scriptstyle h'^*} & & \downarrow{\scriptstyle h^*} \\
\Pi_n(|N(X')|,X'_1) & \xrightarrow{|\bar{\phi}|^*} & \Pi_n(|N(X)|,X_1)
\end{array}$$

$$\begin{array}{ccc}
H_n(S_{Z'}) & \xrightarrow{p^{ZZ'}_*} & H_n(S_Z) \\
\downarrow{\scriptstyle h'_*} & & \downarrow{\scriptstyle h_*} \\
H_n(N(X')) & \xrightarrow{\bar{\phi}_*} & H_n(N(X))
\end{array}$$

where h'^*, h^*, h'_*, h_* are isomorphisms.

Since also the collection $Cov''(S)$ of the coverings Z is cofinal in $Cov(S)$, we obtain:

$\varprojlim \Pi_n(S_Z,a) \simeq \varprojlim \Pi_n(|N(X)|,X_1)$;

$\varprojlim H_n(S_Z) \simeq \varprojlim H_n(N(X))$.

Finally we give some examples.

1. On some finite open coverings of S.

Let $X=\{X_1,\dots,X_p\}$ be a covering of a nonempty set S. For any positive integer $n\leq p$ and any n-tuple $(i_1,\dots,i_n)$ such that $1\leq i_1<i_2<\dots<i_n\leq p$, we put:

$X_{i_1\dots i_n} = X_{i_1}\cap\dots\cap X_{i_n}$;

$X_{i_1\dots\hat{i}_r\dots i_n} = X_{i_1}\cap\dots\cap X_{i_{r-1}}\cap X_{i_{r+1}}\cap\dots\cap X_{i_n}$.

1.1 *Definition* The covering X is independent if:

$X_{i_1\dots i_n}\neq\emptyset \Longrightarrow X_{i_1\dots i_n} \not\subseteq \cup\{X_j/j\notin\{i_1,\dots,i_n\}\}$ for any n-tuple $(i_1,\dots,i_n)$ with $1\leq n<p$.

1.2 *Definition* Let N be an integer such that $3\leq n\leq p$. $\{X_{i_1},\dots,X_{i_n}\}$ is a singularity of X with degree n and indices $i_1,\dots,i_n$, if the following conditions hold:

$X_{i_1\dots i_n} = \emptyset$;

$X_{i_1\dots\hat{i}_r\dots i_n} \neq \emptyset$ for $r=1,2,\dots,n$.

Then $\mathcal{X}$ is non singular, if there are no singularities of $\mathcal{X}$.

1.3 *Proposition* Let S be a connected compact topological space. Any open covering $\mathcal{X}=\{X_1,\dots,X_p\}$ of S has an independent open refinement $\mathcal{Y}=\{Y_1,\dots,Y_p\}$.

Proof: First we construct a finite set X of distinct points of S, taking a point $x_{i_1\dots i_n}$ in each $X_{i_1\dots i_n}\neq\emptyset$ for $n=1,\dots,p$. (This is possible since any nonempty open subset of S is infinite). Then we put:

$Y_i = X_i - X(\hat{i})$ where $X(\hat{i}) = \{x_{i_1\dots i_n}\in X / i\notin\{i_1,\dots,i_n\}\}$.

1.4 *Remark.* $x_{i_1\dots i_n}\in Y_{j_1\dots j_m}$ iff $\{j_1,\dots,j_m\}\subseteq\{i_1,\dots,i_n\}$; so $\mathcal{Y}$ is minimal. Moreover $Y_{i_1\dots i_n}\neq\emptyset$ iff $X_{i_1\dots i_n}\neq\emptyset$. The point x_i will be called characteristical point of Y_i, since Y_i is the only element of $\mathcal{Y}$ containing x_i.

1.5 *Proposition* Let S be a connected compact topological space. Any independent open covering $\mathcal{X}=\{X_1,\dots,X_p\}$ of S has an independent shrinking $\mathcal{Y}=\{Y_1,\dots,Y_p\}$ such that $Y_{i_1\dots i_n}\neq\emptyset$ iff $X_{i_1\dots i_n}\neq\emptyset$ for any n-tuple of indices.

Proof: Construct a finite set X of distinct points of S, taking a point $x_{i_1\dots i_n}$ in $X_{i_1\dots i_n} - \cup\{X_j / j\notin\{i_1,\dots,i_n\}\}$ whenever $X_{i_1\dots i_n}\neq\emptyset$, for $n=1,\dots,p$. Then consider the closed subset:

$Y^{(i)} = X(i)\cup(S-\bigcup_{j\neq i}X_j)$ where $X(i)=\{x_{i_1\dots i_n}\in X / i\in\{i_1,\dots,i_n\}\}$.

Finally take an open subset Y_i of S such that:

$Y^{(i)}\subseteq Y_i\subseteq\overline{Y_i}\subseteq X_i$.

1.6 *Remark.* $x_{i_1\dots i_n}\notin\overline{Y_j}$ for $j\notin\{i_1,\dots,i_n\}$, since $\overline{Y_j}\subseteq X_j$.

1.7 *Lemma* Let S be a connected compact topological space and $\mathcal{X}=\{X_1,\dots,X_p\}$ an independent open covering of S. If $\{X_1, X_2,\dots, X_n\}$ is the singularity of $\mathcal{X}$ relative to $(1,2,\dots,n)$, we can construct an independent open refinement $\mathcal{X}' =$ $= \{U', V', X'_2,\dots, X'_n\}$ of $\mathcal{X}$ such that:

(i) $U'\subseteq X_1$, $V'\subseteq X_1$, $X'_i\subseteq X_i$ for $i=2,\dots,p$;

(ii) $\{U', V', X'_2,\dots, X'_n\}$, $\{U', X'_2,\dots, X'_n\}$ and $\{V', X'_2,\dots, X'_n\}$ are not singularities;

(iii) given m indices $i_1,i_2,\dots,i_m$ such that $1<i_1<i_2<\dots<i_m\leq p$ and $i_r>n$ for some r, we have:

a) $\{U', V', X'_{i_1},\dots, X'_{i_m}\}$ is not a singularity;

b) if $\{U', X'_{i_1},\dots, X'_{i_m}\}$ or $\{V', X'_{i_1},\dots, X'_{i_m}\}$ is a singularity of $\mathcal{X}'$, then $\{X_1, X_{i_1},\dots, X_{i_m}\}$ is a singularity of $\mathcal{X}$.

Proof: Construct a shrinking $\mathcal{Y}=\{Y_1,\dots,Y_p\}$ of $\mathcal{X}$ with the process from Proposition 1.5, and put:

$Y_{\hat{i}} = Y_1\cap\dots\cap Y_{i-1}\cap Y_{i+1}\cap\dots\cap Y_n$ for $i=2,3,\dots,n$;

$U = Y_1\cap(S-\bigcup_{i>2}\overline{Y_{\hat{i}}})$;

$V = Y_1\cap(S-\overline{Y_{\hat{2}}})$;

$\mathcal{Y}' = \{U, V, Y_2,\dots, Y_p\}$.

Clearly $\mathcal{Y}'$ is an open covering of S and $\{U, V, Y_2,\dots, Y_n\}$, $\{U, Y_2,\dots, Y_n\}$, $\{V, Y_2,\dots, Y_n\}$ are not singularities of $\mathcal{Y}'$.

Now consider m indices $i_1,i_2,\dots,i_m$ such that $1<i_1<i_2<\dots<i_m\leq p$ and $i_r>n$ for some r, and distinguish two cases.

I) If $Y_1\cap Y_{i_1\dots i_m}=\emptyset$, then $\{U, V, Y_{i_1},\dots, Y_{i_m}\}$ is not a singularity of $\mathcal{Y}'$. Moreover, if $\{U, Y_{i_1},\dots, Y_{i_m}\}$ or $\{V, Y_{i_1},\dots, Y_{i_m}\}$ is a singularity of $\mathcal{Y}'$, then $\{Y_1, Y_{i_1},\dots, Y_{i_m}\}$ is a singularity of $\mathcal{Y}$, and hence $\{X_1, X_{i_1},\dots, X_{i_m}\}$ is a singularity of $\mathcal{X}$.

II) If $Y_1\cap Y_{i_1\dots i_m}\neq\emptyset$, put $I=\{2,3,\dots,n\}$ and distinguish three possibilities.

1) $I-\{i_1,\dots,i_m\} = \{2\}$.

Since $Y_1\cap Y_{i_1\dots i_m}\subseteq Y_{\hat{2}}\subseteq U\subseteq Y_1$, we obtain $Y_1\cap Y_{i_1\dots i_m} = U\cap Y_{i_1\dots i_m}\neq\emptyset$; therefore $\{U, Y_{i_1},\dots, Y_{i_m}\}$ is not a singularity of $\mathcal{Y}'$.

Moreover $V\cap Y_{i_1\dots i_m}\subseteq Y_{\hat{2}}\subseteq S-V$; hence both $\{V, Y_{i_1},\dots, Y_{i_m}\}$ and $\{U, V, Y_{i_1},\dots, Y_{i_m}\}$ are not singularities of $\mathcal{Y}'$.

2) $I-\{i_1,\dots,i_m\} = \{j\}$ with $j>2$.

Both $\{U, Y_{i_1},\dots, Y_{i_m}\}$ and $\{U, V, Y_{i_1},\dots, Y_{i_m}\}$ are not singularities of $\mathcal{Y}'$, since $U\cap Y_{i_1\dots i_m}\subseteq Y_{\hat{j}}\subseteq S-U$.

Moreover $\{V, Y_{i_1},\dots, Y_{i_m}\}$ is not a singularity of $\mathcal{Y}'$, because $V\cap Y_{i_1\dots i_m} = Y_1\cap Y_{i_1\dots i_m}\neq\emptyset$.

3) $I-\{i_1,\dots,i_m\}\supseteq\{h,k\}$ with $h<k$.

The point $z=x_{1i_1\dots i_m}$, we fixed to construct the shrinking $\mathcal{Y}$ of $\mathcal{X}$, is such that $z\notin\overline{Y_h}\cup\overline{Y_k}$. So $z\notin\overline{Y_{\hat{i}}}$ for $i=2,3,\dots,n$; hence $z\in U\cap V\cap Y_{i_1\dots i_m}$. Therefore $\{U, Y_{i_1},\dots, Y_{i_m}\}$, $\{V, Y_{i_1},\dots, Y_{i_m}\}$, $\{U, V, Y_{i_1},\dots, Y_{i_m}\}$ are not singularities of $\mathcal{Y}'$.

Finally, construct an independent open refinement $\mathcal{X}'=\{U', V', X_2',\dots, X_p'\}$ of $\mathcal{Y}'$ applying Proposition 1.3.

1.8 *Remark.* To construct $\mathcal{X}'$ we replace the element X_1 of $\mathcal{X}$ with two subsets U' and V' of X_1, that we can associate again to the index 1. Instead each element of $\mathcal{X}$ with index greater than 1 is replaced with one subset with the same index. From each singularity of $\mathcal{X}$ containing X_1 and different from $\{X_1, X_2,\dots, X_n\}$ we obtain at least one singularity of $\mathcal{X}'$ with the same indices, where X_1 is replaced by one of the sets U', V'. So, if $\mathcal{X}$ has q singularities of index 1, then $\mathcal{X}'$ has at least q-1 and at most 2(q-1) singularities containing either U' or V', that we call again of index 1. Instead each singularity of $\mathcal{X}$ non containing X_1 determines a singularity of $\mathcal{X}'$ with the same indices.

1.9 *Proposition* Let S be a connected compact topological space and $\mathcal{X}=\{X_1,\dots,X_p\}$ an independent open covering with q singularities containing X_1. We can construct an independent open refinement $\tilde{\mathcal{X}}=\{\tilde{U}_{1,1},\dots, \tilde{U}_{1,h}, \tilde{X}_2,\dots, \tilde{X}_p\}$ of $\mathcal{X}$ which has no singularities containing some $\tilde{U}_{1,r}$.

Proof: Let $s_1=\{X_1, X_{i_{2,1}},\dots, X_{i_{m_1,1}}\}$, $s_2=\{X_1, X_{i_{2,2}},\dots, X_{i_{m_2,2}}\},\dots$, $s_q=\{X_1, X_{i_{2,q}},\dots, X_{i_{m_q,q}}\}$ be the singularities of $\mathcal{X}$ with index 1. Applying

Lemma 1.7, we eliminate s_1 and we obtain an independent open covering $X^{(1)} = \{U_1^{(1)}, U_2^{(1)}, X_2^{(1)}, \dots, X_p^{(1)}\}$, which has at most $2(q-1)$ singularities of index 1, i.e. containing one of the subsets $U_1^{(1)}$, $U_2^{(1)}$ and generated from $s_2, \dots, s_q$.

For the singularities generated from s_2 we have two possibilities:

(i) only one of the collections $\{U_1^{(1)}, X_{i_2,2}^{(1)}, \dots, X_{i_{m_2},2}^{(1)}\}$ and $\{U_2^{(1)}, X_{i_2,2}^{(1)}, \dots, X_{i_{m_2},2}^{(1)}\}$ is a singularity of $X^{(1)}$;

(ii) both of them are singularities of $X^{(1)}$.

Applying Lemma 1.7 once in case (i) and twice in case (ii), we obtain an independent open covering $X^{(2)}$ of form:

$\{U_1^{(2)}, U_2^{(2)}, U_3^{(2)}, X_2^{(2)}, \dots, X_p^{(2)}\}$ in case (i);

$\{U_1^{(2)}, U_2^{(2)}, U_3^{(2)}, U_4^{(2)}, X_2^{(2)}, \dots, X_p^{(2)}\}$ in case (ii).

The covering $X^{(2)}$ has at most $4(q-2)$ singularities with index 1, i.e. containing one of the sets $U_r^{(2)}$ and generated from $s_3, \dots, s_q$. The other singularities of $X^{(2)}$ have the same indices of those of X.

Afterwards we eliminate successively the singularities generated from s_3, from $s_4, \dots$, from s_q applying an analogous process. So we obtain the independent open covering $\tilde{X}$ we were looking for.

1.10 *Theorem* Let S be a connected compact topological space. Any finite open covering has a finite open refinement which is independent and non singular.

Proof: Given an open covering $A=\{A_1, \dots, A_p\}$ of S, we take an independent open refinement $X=\{X_1, \dots, X_p\}$ of A.

We denote by $S_1, S_2, \dots, S_{p-2}$ the sets of the singularities of X whose lowest index is $1,2,\dots,p-2$ respectively.

If $S_1 \neq \emptyset$, applying Proposition 1.9, we obtain a refinement $\tilde{X}^{(1)} = \{\tilde{U}_{1,1}^{(1)}, \dots, \tilde{U}_{1,h_1}^{(1)}, \tilde{X}_2^{(1)}, \dots, \tilde{X}_p^{(1)}\}$ of X whose singularities are generated from $S_2, \dots, S_{p-2}$. Instead, if $S_1 = \emptyset$, we take $\tilde{X}^{(1)} = X$.

Then, similarly, we construct a refinement

$\tilde{X}^{(2)} = \{\tilde{U}_{1,1}^{(2)}, \dots, \tilde{U}_{1,h_1}^{(2)}, \tilde{U}_{2,1}^{(2)}, \dots, \tilde{U}_{2,h_2}^{(2)}, \tilde{X}_3^{(2)}, \dots, \tilde{X}_p^{(2)}\}$

of $\tilde{X}^{(1)}$ whose singularities are generated from $S_3, \dots, S_{p-2}$.

In this way, after p-2 steps, we obtain an open refinement of X which is non singular and independent.

2. Isomorphism between the pretopological spaces $G_{N(X)}$ and $G'(Z)$.

Let S be a connected compact space and $X=\{X_1, \dots, X_p\}$ an independent open covering of S, such that $X_i \neq \emptyset$ for $i=1,2,\dots,p$. Then let $Y=\{Y_1, \dots, Y_p\}$ be an independent shrinking of X (see Proposition 1.5).

For each positive integer $n \leq p$ and any n-tuple $(i_1, \dots, i_n)$ of indices of X such that $i_1 < i_2 < \dots < i_n$ and $X_{i_1 \dots i_n} \neq \emptyset$, we put:

$$A_{(i_1 \dots i_n)} = X_{i_1 \dots i_n} - \bigcup\{\overline{Y_j} / j \notin \{i_1, \dots, i_n\}\};$$

$$B_{[i_1 \dots i_n]} = \bigcup\{A_{(j_1 \dots j_m)} / \{j_1, \dots, j_m\} \subseteq \{i_1, \dots, i_n\}\}.$$

2.1 *Lemma* $X_{i_1\dots i_n} = \cup\{A_{(j_1\dots j_m)} / \{j_1,\dots,j_m\}\supseteq\{i_1,\dots,i_n\}\}$.

2.2 *Lemma* Under the assumption $B_{[\emptyset]}=\emptyset$, we have $B_{[i_1\dots i_n]}\cap B_{[j_1\dots j_m]}=B_{[h_1\dots h_s]}$, where $\{h_1,\dots,h_s\} = \{i_1,\dots,i_n\}\cap\{j_1,\dots,j_m\}$.

2.3 *Definition* We denote by A_X the collection of all subsets of S of form $A_{(i_1\dots i_n)}$ and by B_X the collection of the $B_{[i_1\dots i_n]}$ with maximal sets of indices.

2.4 *Lemma* Any $A_{(i_1\dots i_n)}\in A_X$ is nonempty. Moreover A_X is an open covering of S and refines X.

2.5 *Lemma* B_X is an open covering of S.

2.6 *Lemma* Let $X_j\in X$ and $B_{[i_1\dots i_n]}\in B_X$. We have $X_j\cap B_{[i_1\dots i_n]}\neq\emptyset$ if and only if $j\in\{i_1,\dots,i_n\}$. Moreover, if $j\in\{i_1,\dots,i_n\}$, then $X_j\subseteq St(B_{[i_1\dots i_n]}, B_X)$ and $B_{[i_1\dots i_n]}\subseteq St(X_j, X)$.

2.7 *Definition* For each $i\in\{1,2,\dots,p\}$, let $Z_i = Y_i - \bigcup_{j\neq i}\overline{Y_j}$. We put $Z = B_X\vee\{Z_1,\dots,Z_p\}$.

2.8 *Lemma* $Z_i\neq\emptyset$ for each $i\in\{1,2,\dots,p\}$. Moreover $Z_i\cap Z_j=\emptyset$ whenever $i\neq j$.

Now let us consider the pf-space S_Z and the graph $G'(Z)$ that we obtain from Z (see [2], §6).

2.9 *Theorem* Given an open covering $X=\{X_1,\dots,X_p\}$ of S, let Z be the open covering of S associated to X with the foregoing process. Then the graph $G_{N(X)}$ of the edges of the nerve $N(X)$ of X is isomorphic to the graph $G'(Z)$.

Proof: Each vertex of $G'(Z)$ corresponds to a maximal collection of elements of Z with a nonempty intersection. Since in each of such collections we find exactly one element $Z_i\in Z$, the set of the vertices of $G'(Z)$ is bijective to the collection $\{Z_i\}(i=1,2,\dots,p)$, and we denote by w_i the vertex corresponding to Z_i. Clearly $\{w_1, w_2,\dots, w_p\}$ is bijective to the set $\{X_1, X_2,\dots, X_p\}$ of the vertices of $N(X)$. Moreover, given two distinct indices i,j, in $G'(Z)$ there is the edge w_iw_j iff there is some $B_{[i_1\dots i_n]}\in Z$ such that $\{i,j\}\subseteq\{i_1,\dots,i_n\}$, and hence iff $X_i\cap X_j\neq\emptyset$.

2.10 *Corollary* Under the same assumptions, if the covering X is non singular, then the nerve $N(X)$ of X is isomorphic to the complex $K_{G'(Z)}$ of the graph $G'(Z)$.

Proof: Since X is non singular, $N(X)$ is a complete complex (see [1], §3).

3. Isomorphism between the inverse systems $(S_X, |p_{XX'}|, Cov(S))$ and $(G_{N(X)}, |\Phi_{XX'}|, Cov(S))$.

Let $R=\{A_i\}(i\in J)$ and $R'=\{A'_h\}(h\in H)$ be finite open coverings such that $R\leq R'$, and let $\phi:H\to J$ be a function such that $A'_h\subseteq A_{\phi(h)}$ for any $h\in H$.

3.1 *Definition* We denote by $\overline{\phi}$ the function from $G_{N(R')}$ to $G_{N(R)}$ given by $\overline{\phi}(A'_h) = A_{\phi(h)}$ for any $h\in H$.

3.2 *Lemma* $\overline{\phi}:G_{N(R')}\to G_{N(R)}$ is a precontinuous map. Moreover, if $\phi':H\to J$ is another function such that $A'_h\subseteq A_{\phi'(h)}$ for any $h\in H$, then $\overline{\phi'}$ and $\overline{\phi}$ are homotopic.

Proof: Clearly $\overline{\phi}$ is precontinuous, and the function $H:G_{N(R')}\times I\to G_{N(R)}$ given by:

$$H(A'_h, t) = \begin{cases} A_{\phi(h)} & \text{if } t\in[0, 1/2] \\ A_{\phi'(h)} & \text{if } t\in[1/2, 1] \end{cases}$$

is a prehomotopy of $\overline{\Phi}$ to $\overline{\phi'}$.

3.3 *Definition* A function $\tilde{\Phi}:G'(R')\to G'(R)$ is called induced by $\phi:H\to J$, if, for any vertex $v'_{h_1\dots h_n}$ of $G'(R')$, we have $\tilde{\Phi}(v'_{h_1\dots h_n})=v_{i_1\dots i_m}$ with $\{i_1,\dots,i_m\}\supseteq \phi(\{h_1,\dots,h_n\})$.

3.4 *Lemma* Under the foregoing assumptions, we have:

(i) any function $\tilde{\Phi}:G'(R')\to G'(R)$ induced by ϕ is precontinuous;

(ii) any two functions $\tilde{\Phi}$ and $\tilde{\Phi}'$ from $G'(R')$ to $G'(R)$ induced by ϕ are homotopic;

(iii) if $\psi:H\to J$ is another function such that $A'_h\subseteq A_{\psi(h)}$ for any $h\in H$, and if $\tilde{\Psi}:G'(R')\to G'(R)$ is a function induced by ψ, then $\tilde{\Psi}$ and $\tilde{\Phi}$ are homotopic.

Proof: Since the pretopological spaces S_R and $G'(R)$ belong to the same homotopy type, we find two precontinuous maps $p:S_R\to G'(R)$ and $q:G'(R)\to S_R$ such that $qp\sim 1_{S_R}$ and $pq\sim 1_{G'(R)}$ in the following way (see [2], §6).

For any vertex $v_{i_1\dots i_n}$ of $G'(R)$, we put $q(v_{i_1\dots i_n})=x_{i_1\dots i_n}$ where $x_{i_1\dots i_n}$ belongs to $A_{i_1\dots i_n}-\bigcup\{A_j/j\in J-\{i_1,\dots,i_n\}\}$.

To define $p:S_R\to G'(R)$, we consider the graph $G^{\cup}(R)$ (2), and we put $p=\alpha\pi$ where $\pi:S_R\to G^{\cup}(R)$ is the canonical projection and $\alpha:G^{\cup}(R)\to G'(R)$ is a function such that $\alpha(v_{i_1\dots i_n})$ is a vertex $v_{i_1\dots i_m}$ of $G'(R)$ with $\{i_1,\dots,i_m\}\supseteq\{i_1,\dots,i_n\}$.

Similarly we obtain $p':S_{R'}\to G'(R')$ and $q':G'(R')\to S_{R'}$.

Now we construct a finite open covering $\tilde{R}=\{\tilde{A}_i\}(i\in J)$ of S such that $R\leq\tilde{R}\leq R'$, putting:

$$\tilde{A}_i = A_i - \{x_{h_1\dots h_n}=q'(v'_{h_1\dots h_n}) \,/\, i\notin\phi(\{h_1,\dots,h_n\})\},$$

where $v'_{h_1\dots h_n}$ denotes a vertex of $G'(R')$.

Clearly $x_{h_1\dots h_n}\in\tilde{A}_i$ iff $i\in\phi(\{h_1,\dots,h_n\})$; moreover the point $x_{i_1\dots i_m}\in\tilde{A}_i$ iff $i\in\{i_1,\dots,i_m\}$.

Afterwards we define $\tilde{p}:S_{\tilde{R}}\to G'(\tilde{R})$ and $\tilde{q}:G'(\tilde{R})\to S_{\tilde{R}}$ like p and q respectively, and we consider the precontinuous maps $p_{\tilde{R}R'}:S_{R'}\to S_{\tilde{R}}$ and $p_{R\tilde{R}}:S_{\tilde{R}}\to S_R$ given by the the identity in S.

Now we define a precontinuous map $\tilde{\Phi}:G'(R')\to G'(R)$ in the following way:

$$\begin{tikzcd} & & & G^{\cup}(\tilde{R}) \arrow[dr, "\tilde{\alpha}"] & & & & \\ G'(R') \arrow[r, "q'"] \arrow[rrrrrrr, bend right, "\tilde{\Phi}"'] & S_{R'} \arrow[r, "p_{\tilde{R}R'}"] & S_{\tilde{R}} \arrow[ur, "\tilde{\pi}"] \arrow[rr, "\tilde{p}"] & & G'(\tilde{R}) \arrow[r, "\tilde{q}"] & S_{\tilde{R}} \arrow[r, "p_{R\tilde{R}}"] & S_R \arrow[r, "q"] & G'(R) \end{tikzcd}$$

---

(2) The vertices of $G^{\cup}(R)$ are the classes of the equivalence relation σ in S, given by $x\sigma y$ iff $I_x=I_y$, where $I_x=\{i\in J/x\in A_i\}$ and J is the set of the indices of R. We will write $v_{i_1\dots i_n}$ to denote the equivalence class $[x]$ such that $I_x=\{i_1,\dots,i_n\}$. We recall that in $G^{\cup}(R)$ there is the edge $v_{i_1\dots i_n}v_{j_1\dots j_m}$ if and only if $\{i_1,\dots,i_n\}\cap\{j_1,\dots,j_m\}\neq\emptyset$.

We easily see that $\tilde{\phi}(v'_{h_1\dots h_n})=v_{i_1\dots i_m}$ with $\{i_1,\dots,i_m\}\supseteq\phi(\{h_1,\dots,h_n\})$, i.e. $\tilde{\phi}$ is induced by ϕ.

Then $\tilde{\phi}$ is unique up to homotopies, since $\tilde{\phi}\sim pp_{RR'}q'$, where $p_{RR'}:S_{R'}\to S_R$ is the identity in S.

Finally, also $\tilde{\psi}$ is homotopic to $pp_{RR'}q'$; and hence $\tilde{\psi}$ and $\tilde{\phi}$ are homotopic.

3.5 *Remark.* For any precontinuous map $\tilde{\phi}:G'(R')\to G'(R)$ induced by $\phi:H\to J$, we obtain the following homotopy commutative diagram:

$$\begin{tikzcd} S_{R'} \arrow[r, "p_{RR'}"] \arrow[d, "p'"'] & S_R \arrow[d, "p"] \\ G'(R') \arrow[r, "\tilde{\phi}"] & G'(R) \end{tikzcd}$$

3.6 *Definition* Let Cov'(S) denote the collection of the finite independent non singular coverings of S, whose elements are nonempty open sets.

3.7 *Proposition* Cov'(S) is cofinal in Cov(S).

Proof: Observe that any $A\in$Cov(S) has a refinement R which is a finite open covering of S; then recall Theorem 1.10.

3.8 *Definition* Let Cov"(S) denote the collection of all finite open coverings Z associated to some $X\in$Cov'(S) (see §2).

3.9 *Proposition* Cov"(S) is cofinal in Cov(S).

Proof: Given $R\in$Cov(S), take a finite open star-refinement R' of R and $X'\in$Cov'(S) such that $R'\leq X'$. It is easy to see that any covering Z' associated to X' refines R.

3.10 *Proposition* Let $X=\{X_i\}(i\in J)\in$Cov'(S) and let $Z\in$Cov"(S) be associated to X. If we take $X'=\{X'_h\}(h\in H)$ in Cov'(S) such that X' star-refines A_X, then any covering Z' associated to X' refines Z. Moreover, if Λ is the set of the indices of A_X and $\chi:H\to\Lambda$ is any function such that $St(X'_h,X')\subseteq A_{\chi(h)}$ for each $h\in H$, then, taking $\phi(h)\in\in\chi(h)$, we can define a function $\phi:H\to J$ such that:

(I) $X'_h\subseteq X_{\phi(h)}$ for any $h\in H$;

(II) for any $B'_{[h_1\dots h_n]}\in Z'$ there is $B_{[i_1\dots i_m]}\in Z$ such that $B'_{[h_1\dots h_n]}\subseteq\subseteq B_{[i_1\dots i_m]}$ and $\{i_1,\dots,i_m\}\supseteq\phi(\{h_1,\dots,h_n\})$;

(III) the function $\tilde{\phi}_{ZZ'}:G'(Z')\to G'(Z)$, that we obtain putting $\tilde{\phi}_{ZZ'}(w'_h)=w_{\phi(h)}$ for any $h\in H$, is induced by ϕ.

Proof: Ad (I). Observe that $X'_h\subseteq A_{\chi(h)}\subseteq X_i$ for any $i\in\chi(h)$.

Ad (II). $B'_{[h_1\dots h_n]}\subseteq St(X'_{h_r},X')\subseteq A_{\chi(h_r)}\subseteq\cap\{X_j\ /\ j\in\chi(h_r)\}$ for $r=1,2,\dots,n$.

Therefore $B'_{[h_1\dots h_n]}\subseteq\cap\{X_j\ /\ j\in\bigcup_{r=1}^{n}\chi(h_r)\}$. Hence $B_{[\bigcup_{i=1}^{n}\chi(h_r)]}$ is nonempty; so there is $B_{[i_1\dots i_m]}\in Z$ such that $\{i_1,\dots,i_m\}\supseteq\bigcup_{r=1}^{n}\chi(h_r)\supseteq\phi(\{h_1,\dots,h_n\})$ and $B_{[i_1\dots i_m]}\supseteq A_{\chi(h_r)}\supseteq B'_{[h_1\dots h_n]}$.

Ad (III). Let w'_h be a vertex of $G'(Z')$. w'_h corresponds to the nonempty intersection of Z'_h and of all $B_{[h_1\dots h_n]}\in Z'$ such that $h\in\{h_1,\dots,h_n\}$. By (II), for each of such $B'_{[h_1\ldots h_n]}$ there is $B_{[i_1\ldots i_m]}\in Z$ such that $\{i_1,\ldots,i_m\}\supseteq\phi(\{h_1,\ldots,h_n\})\ni$ $\ni\phi(h)$ and $B'_{[h_1\ldots h_n]}\subseteq B_{[i_1\ldots i_m]}$. Moreover each of such $B'_{[h_1\ldots h_n]}$ contains Z'_h. Hence the vertex $w_k=\tilde{\Phi}_{ZZ'}(w'_h)$ of $G'(Z)$ must correspond to a maximal nonempty intersection of a collection of elements of Z containing all the $B_{[i_1\ldots i_m]}$ we just mentioned. For example w_k may correspond to the collection containing $Z_{\phi(h)}$ and all $B_{[i_1\ldots i_m]}\in Z$ such that $\phi(h)\in\{i_1,\ldots,i_m\}$.

3.11 *Remark.* Similarly, let $X\in\mathrm{Cov}'(S)$ and let $Z\in\mathrm{Cov}''(S)$ be associated to X. If we take $Z'\in\mathrm{Cov}''(S)$ such that Z' star-refines A_X, then any $X'\in\mathrm{Cov}'(S)$, to which we can associate Z', is a refinement of X. Moreover we obtain the statements analogous to the ones from Proposition 3.10.

3.12 *Proposition* Under the foregoing assumptions, we obtain the following homotopy commutative diagram:

$$\begin{tikzcd}
S_{Z'} \arrow[r, "p_{ZZ'}"] \arrow[d, "p'"'] & S_Z \arrow[d, "p"] \\
G'(Z') \arrow[r, "\tilde{\Phi}_{ZZ'}"] \arrow[d, "f'"'] & G'(Z) \arrow[d, "f"] \\
G_N(X') \arrow[r, "\bar{\Phi}_{XX'}"] & G_N(X)
\end{tikzcd}$$

where p, p' are the precontinuous maps from Lemma 3.4, and f, f' are the isomorphisms from Theorem 2.9.

Proof: $pp_{ZZ'}\sim\tilde{\Phi}_{ZZ'}p'$ by Remark 3.5, and $\bar{\Phi}_{ZZ'}f' = f\tilde{\Phi}_{ZZ'}$.

3.13 *Theorem* The inverse systems $(S_X,\ [p_{XX'}],\ \mathrm{Cov}(S))$ and $(G_N(X),\ [\bar{\Phi}_{XX'}],\ \mathrm{Cov}(S))$, where $[p_{XX'}]$ and $[\bar{\Phi}_{XX'}]$ are the homotopy classes represented by $p_{XX'}$ and $\bar{\Phi}_{XX'}$ respectively, are isomorphic.

Proof: First we define a function $\Phi:\mathrm{Cov}'(S)\to\mathrm{Cov}''(S)$, taking for each $X\in\mathrm{Cov}'(S)$ an element $Z=\Phi(X)$ of $\mathrm{Cov}''(S)$ which is associated to X (see §2).

Then, for each $X\in\mathrm{Cov}'(S)$, we consider the precontinuous map $h_X:S_X\to G_{N(X)}$ given by:

$$\begin{tikzcd}
S_Z \arrow[r, "p"] \arrow[rr, bend right, "h_X"'] & G'(Z) \arrow[r, "f"] & G_N(X)
\end{tikzcd}$$

where p and f are the precontinuous maps before mentioned.

Given $X\le X'$ in $\mathrm{Cov}'(S)$, take $X''\in\mathrm{Cov}'(S)$ such that X'' star-refines both A_X and $A_{X'}$. Under this assumption, the following diagram is homotopy commutative:

$$\begin{tikzcd}
 & S_{Z''} \arrow[dl, "p_{ZZ''}"'] \arrow[dr, "p_{Z'Z''}"] \arrow[dd, "h_{X''}"] & \\
S_Z \arrow[dd, "h_X"'] & & S_{Z'} \arrow[dd, "h_{X'}"] \\
 & G_N(X'') \arrow[dl, "\bar{\Phi}_{XX''}"'] \arrow[dr, "\bar{\Phi}_{X'X''}"] & \\
G_N(X) & & G_N(X') \arrow[ll, "\bar{\Phi}_{XX'}"]
\end{tikzcd}$$

Hence (h_X,Φ) is a morphism from $(S_Z,\ [p_{ZZ'}],\ \mathrm{Cov}''(S))$ to $(G_{N(X)},\ [\bar{\Phi}_{XX'}],\ \mathrm{Cov}'(S))$. With a similar process we define a morphism (k_Z,Ψ) from $(G_{N(X)},\ [\bar{\Phi}_{XX'}],\ \mathrm{Cov}'(S))$ to $(S_Z,\ [p_{ZZ'}],\ \mathrm{Cov}''(S))$. Precisely we define $\Psi:\mathrm{Cov}''(S)\to\mathrm{Cov}'(S)$, taking for each

$Z\in Cov''(S)$ an element $X=\Psi(Z)$ of $Cov'(S)$ such that Z is associated to X. Then we consider the precontinuous map $k_Z: G_{N(X)} \to S_Z$ given by $k_Z = qf^{-1}$, where $f: G'(Z) \to G_{N(X)}$ and $q: G'(Z) \to S_Z$ are the before mentioned functions.
Afterwards, each of the morphisms (h_X, Φ) and (k_Z, Ψ) is the inverse of the other.
Finally recall Propositions 3.7 and 3.9.

4. Shape groups and Čech homology groups of a connected compact topological space S.

To calculate the shape groups $\Pi_n(S,a)$ based at a point $a\in S$, we have to fix, for each covering X, an open set $X\in X$ such that $a\in X$.
Therefore we have to consider some pointed open coverings of the pointed space (S,a), such that there exists exactly one element of each covering X containing a.
We denote such an element by X_1, and we choose the characteristical point x_1 of X_1 taking $x_1=a$. So a is a point of the element $A_{(1)}\in A_X$, and a belongs to the open set $Z_1\in Z$ and to each $B_{[1 i_2 \dots i_m]} \in B_X$. Then, "mutatis mutandis", we obtain that the inverse systems $((S_X,a), [p_{XX'}], Cov(S))$ and $((G_{N(X)},x_1), [\overline{\phi}_{XX'}], Cov(S))$ are isomorphic.
So, for each dimension n the inverse systems $(\Pi_n(S_X,a), p^*_{XX'}, Cov(S))$ and $(Q_n(G_{N(X)},x_1), \overline{\phi}^*_{XX'}, Cov(S))$ are isomorphic.
Afterwards, if $X, X' \in Cov(S)$ and $X \leq X'$, since X and X' are non singular and the complexes $N(X)$ and $N(X')$ are complete, the following diagram commutes:

$$\begin{array}{ccc}
\Pi_n(|N(X')|, x_1') & \xrightarrow{|\overline{\Phi}_{XX'}|^*} & \Pi_n(|N(X)|, x_1) \\
\Big\downarrow \bar{\mu}' & & \Big\downarrow \mu \\
Q_n(G_{N(X')}, x_1') & \xrightarrow{\overline{\Phi}^*_{XX'}} & Q_n(G_{N(X)}, x_1)
\end{array}$$

where μ and μ' are the isomorphisms given by the canonical projections from the polyhedron $|N(X)|$ to the graph $G_{N(X)}$ of the edges of $N(X)$ and from $|N(X')|$ to $G_{N(X')}$ respectively (see [1], §3).
Hence the inverse systems $(\Pi_n(S_X,a), p^*_{XX'}, Cov(S))$ and $(\Pi_n(|N(X)|,x_1), |\overline{\phi}_{XX'}|^*, Cov(S))$ are isomorphic. Therefore:

$$\varprojlim (\Pi_n(S_X,a), p^*_{XX'}, Cov(S)) \simeq \check{\Pi}_n(S,a) \simeq \varprojlim(\Pi_n(|N(X)|,x_1), |\overline{\phi}_{XX'}|^*, Cov(S)).$$

In the case of Čech homology groups, for any $X\in Cov(S)$ and each dimension n, we consider the homology group $H_n(N(X))$ of the simplicial complex $N(X)$ and the singular homology group $H_n(G_{N(X)})$ of the graph $G_{N(X)}$ (see [5]).
Given $X, X' \in Cov(S)$ such that $X \leq X'$, we obtain the following commutative diagram:

$$\begin{array}{ccc}
H_n(N(X')) & \xrightarrow{\overline{\Phi}^{XX'}_*} & H_n(N(X)) \\
\Big\downarrow \nu' & & \Big\downarrow \nu \\
H_n(G_{N(X')}) & \xrightarrow{\overline{\Phi}^{XX'}_*} & H_n(G_{N(X)})
\end{array}$$

where ν and ν' are the isomorphisms considered in [5], §5.

Hence:

$$\varprojlim (H_n(S_X), p_*^{XX'}, \mathrm{Cov}(S)) \simeq \check{H}_n(S) \simeq \varprojlim (H_n(N(X)), \bar{\phi}_*^{XX'}, \mathrm{Cov}(S)).$$

5. Examples.

5.1 Let S be the polyhedron $|K|$ of a finite simplicial complex K of dimension m. In this case we can calculate the groups $\Pi_n(S,a)$ and $H_n(S)$ more simply in the following way.

For any $i\in N$, we take the derived $K^{(i)}$ of K, and we denote by $V^{(i)}$ the vertex set of $K^{(i)}$ and by $\sigma_p^{(i)}$ a p-dimensional simplex whatever of $K^{(i)}$. Then we put:

$$r_i = \frac{1}{m} \inf\{d(x_h^{(i)}, x_k^{(i)})\}, \quad \text{where } x_h^{(i)}, x_k^{(i)} \in V^{(i)};$$

$$\mathcal{R}_i = \{V(\sigma_p^{(i)}, r_i)\}(\sigma_p^{(i)} \in K^{(i)};\ 0\le p\le m), \quad \text{where } V(\sigma_p^{(i)}, r_i) = \{y\in S \,/\, d(y, \sigma_p^{(i)})<r_i\};$$

$$\Gamma = \{\mathcal{R}_i\}(i\in N).$$

It is easy to see that each $\mathcal{R}_i$ is an open covering of S, and that the graph $G'(\mathcal{R}_i)$ is the graph of the edges of the complex $K^{(i)}$.

The set Γ is cofinal in Cov(S); so we have:

$$\check{\Pi}_n(S,a) = \varprojlim (\Pi_n(S_{\mathcal{R}_i}, a), p^*_{\mathcal{R}_i\mathcal{R}_j}, \Gamma);$$

$$\check{H}_n(S) = \varprojlim (H_n(S_{\mathcal{R}_i})), p_*^{\mathcal{R}_i\mathcal{R}_j}, \Gamma).$$

Since, for $i>0$, $\Pi_n(S_{\mathcal{R}_i},a) \simeq \Pi_n(|K|,a)$, $H_n(S_{\mathcal{R}_i}) \simeq H_n(K)$, and all functions $p^*_{\mathcal{R}_i\mathcal{R}_j}$ and $p_*^{\mathcal{R}_i\mathcal{R}_j}$ are isomorphisms, we obtain:

$$\check{\Pi}_n(S,a) = \Pi_n(S,a);$$

$$\check{H}_n(S) = H_n(K).$$

5.2 Let (S,d) be a compact metric space.

For any $\varepsilon>0$ we consider the symmetrical pf-space $S_\varepsilon=(S,\mathcal{P}_\varepsilon)$ where $\mathcal{P}_\varepsilon=\{\widehat{V(x,\varepsilon)}\}(x\in S)$ and $V(x,\varepsilon)=\{y\in S \,/\, d(x,y)<\varepsilon\}$. If $\varepsilon'<\varepsilon$, we consider the precontinuous map $p_{\varepsilon\varepsilon'}:S_{\varepsilon'}\to S_\varepsilon$ given by $p_{\varepsilon\varepsilon'}(x)=x$ for any $x\in S$.

Then we easily see that, for each dimension n, we have:

$$\check{\Pi}_n(S,a) = \varprojlim (\Pi_n(S_\varepsilon,a), p^*_{\varepsilon\varepsilon'}, E),$$

$$\check{H}_n(S) = \varprojlim (H_n(S_\varepsilon), p_*^{\varepsilon\varepsilon'}, E),$$

where E is the directed set that we obtain taking the set R^+ of all positive real numbers with the inverted order.

5.3 Let S be the Warsaw circle, i.e. the following subspace of R^2.

Given the points $a=(0,1)$, $b=(0,-2)$, $c=(\frac12,-1)$, $d=(\frac12,0)$, we take the segments ab, bc, cd and all points $(x,y)\in R^2$ such that $x\in\,]0,\frac12]$ and $y=\sin(\pi/2x)$.

Let $\Phi:[\frac12,1]\to ab\cup bc\cup cd$ be an homeomorphism such that $\Phi(1)=a$ and $\Phi(\frac12)=d$, and let $f:]0,1]\to S$ be the continuous surjection given by:

$$f(x) = \begin{cases} (x, \sin(\pi/2x)) & \text{if } 0<x\le \frac12; \\ \Phi(x) & \text{if } \frac12\le x\le 1. \end{cases}$$

Then for any $\varepsilon>0$ we consider the pretopological space S_ε from 5.2 and the precontinuous loop $\psi_\varepsilon : [0,1] \to S_\varepsilon$ based at a, given by:

$$\psi_\varepsilon(x) = \begin{cases} a & \text{if } 0 \leq x \leq \lambda \\ \Phi(x) & \text{if } \lambda \leq x \leq 1 \end{cases}$$

where $\lambda = 1/(4n+1)$ and n is the lowest positive integer such that $1/(4n+1) < \varepsilon$. The group $\Pi_1(S,a)$ is isomorphic to (Z,+), and we observe that its generator can be associated to the sequence of the prehomotopy classes represented by the loops ψ_ε of S_ε.

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