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BILINEAR MAPPINGS AND OPERATOR IDEALS*

A. Braunß and H. Junek

Summary

There are several possibilities to generate ideals of multilinear mappings by common operator ideals. We discuss here some of these procedures and study their mutual relations. For sake of simplicity we restrict us to ideals of bilinear mappings.

1. Operator Ideals and Ideals of Bilinear Mappings

In connection with the theory of holomorphic functions defined on Banach spaces on the one side, and the theory of tensor products on the other side, ideals of multi-linear forms have been studied more or less extensively during the last years (cf. /3/,/4/). Several results concerning concrete multi-ideals have been obtained in this way. However, no general theory of multi-ideals does exist up to now. Encouraged by the reach theory of operator ideals Pietsch outlined an axiomatic theory of multi-ideals and suggested the further developement in /1/. This paper can be considered as a contribution in this direction. Only for sake of simplicity we restrict ourself to the study of vector-valued bilinear forms. We shall use here the notion used in /1/ and /5/. Let us recall the basic definitions.

Let E and F be any Banach spaces. By $\mathcal{L}^1(E,F)$ we denote the Banach space of all linear bounded operators mapping E into F. We put $\mathcal{L}^1 := \bigcup \mathcal{L}^1(E,F)$, where the union is taken over all pairs E,F of Banach spaces. By $\mathcal{F}$ we denote the subclass of all finite rank operators. A subclass $\mathfrak{A} \subseteq \mathcal{L}^1$ with the components

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$\mathfrak{A}(E,F) := \mathfrak{A} \cap \mathcal{L}^1(E,F)$ is called an operator ideal, if the following conditions are satisfied:

(i) Each component $\mathfrak{A}(E,F)$ is a linear subspace of $\mathcal{L}^1(E,F)$ containing $\mathcal{F}(E,F)$.

(ii) If $R \in \mathcal{L}^1(E_0,E)$, $T \in \mathfrak{A}(E,F)$, and $S \in \mathcal{L}^1(F,F_0)$, then $STR \in \mathfrak{A}(E_0,F_0)$.

A non-negative function $T \longrightarrow \| T \,|\, \mathfrak{A} \|$ defined on an operator ideal $\mathfrak{A}$ is called a norm if the following comes true

(i) $\| - | \mathfrak{A} \|$ is a norm on each component $\mathfrak{A}(E,F)$ and $\| a \otimes y \,|\, \mathfrak{A} \| = \|a\| \cdot \|y\|$ for $a \in E'$ and $y \in F$.

(ii) If $R \in \mathcal{L}^1(E_0,E)$, $T \in \mathfrak{A}(E,F)$, and $S \in \mathcal{L}^1(F,F_0)$, then

$$\|STR \,|\, \mathfrak{A}\| \leq \|S\| \cdot \|T \,|\, \mathfrak{A}\| \cdot \|R\| .$$

A pair $(\mathfrak{A}, \| - | \mathfrak{A} \|)$ is called a normed ideal if the components $\mathfrak{A}(E,F)$ are complete with respect to the norm $\| - | \mathfrak{A} \|$. If the components of $\mathfrak{A}$ are complete with respect to the uniform operator norm, then $\mathfrak{A}$ is called a closed ideal. For more details we refer to /5/. As examples let us mention the following ideals.

$\mathfrak{G}, \mathfrak{K}, \mathfrak{W}, \mathfrak{X}$ denote the closed ideals of all approximable, compact, weakly compact and separable rank operators, respectively.

$\mathfrak{N}$ denotes the normed ideal of all nuclear operators. Recall that an operator $S \in \mathcal{L}(E,F)$ is said to be nuclear, if there is a representation

$$S = \sum_{i=1}^{\infty} a_i \otimes y_i \text{ for some } a_i \in E',\ y_i \in F \text{ with } \sum_{i=1}^{\infty} \|a_i\| \, \|y_i\| < \infty .$$

In this case we put $\| S \,|\, \mathfrak{N} \| = \inf \sum_{i=1}^{\infty} \| a_i\| \, \| y_i \|$, where the infimum is taken over all possible representations of S.

$\mathfrak{H}$ is the ideal of all operators which are factorizable through some Hilbert space. This ideal can be normed by the definition $\|T|\mathfrak{H}\| = \inf \|R\| \, \|S\|$, where the infimum is taken over all possible factorizations $T = R \cdot S$ through some Hilbert space. Finally, $\mathfrak{I}$ and $\mathfrak{P}$ denote the ideals of all integral and all absolutely summing operators, respectively. They are normed ideals as well.

Now let us turn to ideals of bilinear mappings. For any Banach spaces E,F,G we denote by $\mathcal{L}^2(E,F;G)$ the Banach space of all bounded G-valued bilinear mappings defined on $E \times F$ equipped with the uniform norm. This space is metric isomorphic to the Banach space $\mathcal{L}(E, \mathcal{L}(F,G))$ and we will identify these spaces in all the following. As in the case of operators, $\mathcal{L}^2$ denotes the class of all bounded bilinear mappings. A bilinear form $M \in \mathcal{L}^2(E,F;G)$ is called to be of finite rank ($M \in \mathcal{F}(E,F;G)$) if it is a linear combination of bilinear forms of the following type: Given $a \in E'$, $b \in F'$, and $z \in G$, then a bilinear form $M = a \otimes b \otimes z$ is defined by $M(x,y) = \langle x,a \rangle \langle y,b \rangle z$.

A subclass $\mathcal{A}$ of $\mathcal{L}^2$ with the components $\mathcal{A}(E,F;G) := \mathcal{A} \cap \mathcal{L}^2(E,F;G)$ is called a biideal if the following conditions are satisfied:

(i) Each component $\mathcal{A}(E,F;G)$ is a linear subspace of $\mathcal{L}^2(E,F;G)$ containing $\mathcal{F}(E,F;G)$.

(ii) If $M \in \mathcal{A}(E,F;G)$, $R \in \mathcal{L}(E_0,E)$, $S \in \mathcal{L}(F_0,F)$, and $T \in \mathcal{L}(G,G_0)$, then $TM(R -, S -) \in \mathcal{A}(E_0,F_0;G_0)$.

Given two operator ideals $\mathfrak{A}$ and $\mathfrak{B}$, there are several possibilities to generate biideals. In this paper we shall study two of them. No confusion will appear if we write $\mathcal{L}$ instead of $\mathcal{L}^1, \mathcal{L}^2$.

<u>1.1 Definition</u> (i) Let $\mathfrak{A}$ and $\mathfrak{B}$ be operator ideals. Let E,F,G be any Banach spaces. Then $\mathcal{L}(\mathfrak{A},\mathfrak{B})(E,F;G)$ is the set of all bilinear forms $M \in \mathcal{L}(E,F;G)$ for which there are operators $S \in \mathfrak{A}(E,F_1)$, $T \in \mathfrak{B}(F,F_1)$ and a bilinear mapping $N \in \mathcal{L}(E_1,F_1;G)$ such that M admits the factorization $M = N(S,T)$.

(ii) Suppose that $(\mathfrak{B}, \|-|\mathfrak{B}\|)$ is even a normed operator ideal. For all Banach spaces E,F,G define

$$[\mathfrak{A},\mathfrak{B}](E,F;G) = \mathfrak{A}(E,\mathfrak{B}(F,G)) .$$

It is easy to see that both constructions lead to biideals. Since $\mathcal{L}(\mathfrak{A},\mathfrak{B})$ is defined by a factorization, the structure of this ideal is very clear. In opposite to this the ideal $[\mathfrak{A},\mathfrak{B}]$ is rather difficult to handle. Next we are interested in finding conditions for the coincidence of the ideals $\mathcal{L}(\mathfrak{A},\mathfrak{B})$ and $[\mathfrak{A},\mathfrak{B}]$. For this purpose we give a slight generalization of Definition 1.1.

<u>1.2 Definition.</u> Let $\mathcal{A}$ be any biideal and let $\mathfrak{A}$ and $\mathfrak{B}$ be any operator ideals. For any Banach spaces E,F,G let $\mathcal{A}(\mathfrak{A},\mathfrak{B})(E,F;G)$ be the set of all bilinear mappings $M \in \mathcal{L}(E,F;G)$ which admit a representation

$$M = N(S,T)$$

for some operators $S \in \mathfrak{A}(E,E_1)$, $T \in \mathfrak{B}(F,F_1)$, and some bilinear mapping $N \in \mathcal{A}(E_1,F_1;G)$.

It is obvious that $\mathcal{A}(\mathfrak{A},\mathfrak{B})$ is also a biideal.

1.3 Proposition. If $\mathfrak{A}$, $\mathfrak{B}$ are any operator ideals and if $\mathfrak{B}$ is even a normed ideal, then

$$\mathcal{L}(\mathfrak{A},\mathfrak{B}) \subseteq [\mathfrak{A},\mathfrak{B}].$$

Proof. Suppose $M \in \mathcal{L}(\mathfrak{A},\mathfrak{B})(E,F;G)$ and let $M = N(S,T)$ be some representation of M, where $S \in \mathfrak{A}(E,E_1)$, $T \in \mathfrak{B}(F,F_1)$, and $N \in \mathcal{L}(E_1,F_1;G)$. Since $Nx \in \mathcal{L}(F_1,G)$ for all $x \in E_1$, we get $NxT \in \mathfrak{B}(F,G)$. Next we prove $N(-,T-) \in \mathcal{L}(E_1,\mathfrak{B}(F,G))$. In fact, the function $N(-,T-)\colon E_1 \longrightarrow \mathfrak{B}(F,G)$ is linear in the first coordinate and we have

$$\begin{aligned} \| N(x,T-) \mid \mathfrak{B}(F,G) \| &= \| N(x,-)\cdot T \mid \mathfrak{B}(F,G) \| \\ &\leq \| Nx \mid \mathcal{L}(F_1,G) \| \cdot \| T \mid \mathfrak{B}(F,F_1) \| \\ &\leq \| N \| \cdot \| x \| \cdot \| T \mid \mathfrak{B}(F,F_1) \| \end{aligned}$$

for all $x \in E_1$.
This shows $N(-,T-) \in \mathcal{L}(E_1,\mathfrak{B}(F,G))$. Since $S \in \mathfrak{A}(E,E_1)$, this proves $N(S,T) \in \mathfrak{A}(E,\mathfrak{B}(F,G))$.

2. Factorization Results

Let $\mathfrak{A}$ and $\mathfrak{B}$ be any normed operator ideals. In this section we will give necessary and sufficient conditions for the validity of the equation

$$\mathcal{L}(\mathfrak{A},\mathfrak{B}) = [\mathfrak{A},\mathfrak{B}]. \qquad (*)$$

2.1 Proposition. Let $(\mathfrak{A}, \| - \mid \mathfrak{A} \|)$ be any normed operator ideal. Then $[\mathfrak{F},\mathfrak{A}] = \mathcal{L}(\mathfrak{F},\mathfrak{A})$ holds true.

Proof. Let E,F, and G be arbitrary Banach spaces. For any $a \in E'$ and $S \in \mathfrak{A}(F,G)$ we have $a \times S$ (,). Obviously, the forms $a \otimes S$, $a \in E'$, $S \in \mathfrak{A}(F,G)$ generate the space $[\mathfrak{F},\mathfrak{A}](E,F;G)$. This proves the assertion.

2.2 Proposition. For each operator ideal $\mathfrak{A}$ holds $[\mathfrak{A},\mathcal{L}] = \mathcal{L}(\mathfrak{A},\mathcal{L})$.

Proof. Let
$B: \mathcal{L}(F,G)\times F \longrightarrow G$ defined by $B(T,y) = Ty$, $y \in F$, $T \in \mathcal{L}(F,G)$, be the canonical valuation map. Suppose
$M \in [\mathfrak{A},\mathcal{L}](E,F;G) = \mathfrak{A}(E,\mathcal{L}(F,G))$. Since
$M(x,y) = B(Mx,y) = B(M,1_F)(x,y)$ for all $x \in E$ and $y \in F$, we obtain
$M = B(M,1_F) \in \mathcal{L}(\mathfrak{A},\mathcal{L})$.

Next we shall give a necessary condition for the validity of equation (*) in terms of the transposed bilinear map, which is defined as follows. For every $M \in \mathcal{L}(E,F;G)$, $x \in E$, and $y \in F$ we put $M^T(y,x) = M(x,y)$. Then $M^T \in \mathcal{L}(F,E;G)$. Obviously, T is a metric isomorphism between the Banach spaces $\mathcal{L}(E,F;G)$ and $\mathcal{L}(F,E;G)$. For every biideal $\mathcal{A}$ we denote the complete image of $\mathcal{A}$ under T in $\mathcal{L}^2$ by $\mathcal{A}^T$. Clearly, $\mathcal{A}^T$ is a biideal, as well.

The next statements are evident.

<u>2.3 Proposition.</u> Let $\mathfrak{A}$ and $\mathfrak{B}$ be normed operator ideals. Then $\mathcal{L}(\mathfrak{A},\mathfrak{B})^T = \mathcal{L}(\mathfrak{B},\mathfrak{A})$.

<u>2.4 Corollary.</u> Let $\mathfrak{A}$ and $\mathfrak{B}$ be any normed operator ideals. Then $[\mathfrak{B},\mathfrak{A}] = \mathcal{L}(\mathfrak{B},\mathfrak{A})$ implies $[\mathfrak{B},\mathfrak{A}]^T \subseteq [\mathfrak{A},\mathfrak{B}]$.

In preparation of the next result we need the notion of the dual operator ideal (cf. /5/).

<u>2.5 Definition.</u> Let $\mathfrak{A}$ be a (normed) operator ideal. An operator $S \in \mathcal{L}(E,F)$ belongs to the dual operator ideal $\mathfrak{A}^{dual}$ if $S' \in \mathfrak{A}(F',E')$. In the normed case the dual ideal norm $\| - \mid \mathfrak{A}^{dual} \|$ is defined by $\| S \mid \mathfrak{A}^{dual} \| = \| S' \mid \mathfrak{A} \|$.

<u>2.6 Proposition.</u> Let $\mathfrak{A}$ and $\mathfrak{B}$ be normed operator ideals. Then $[\mathfrak{B},\mathfrak{A}]^T \subseteq [\mathfrak{A},\mathfrak{B}]$ implies $\mathfrak{B} \subseteq \mathfrak{A}^{dual}$.

Proof. Suppose $T \in \mathfrak{B}(E,F)$. Then we have $K_F T \in \mathfrak{B}(E,F'')$, where K_F denotes the canonical injection from F into its bidual F''. The operator $K_F T$ can considered as a bilinear form
$K_F T \in \mathfrak{B}(E,\mathcal{L}(F',\mathbb{C})) = \mathfrak{B}(E,\mathfrak{A}(F',\mathbb{C}))$. Its transposed map $(K_F T)^T$ corresponds to the dual operator $T' \in \mathcal{L}(F',E')$ by the equation

$$(K_F T)^T(b,x) = K_F T(x,b) = \langle b, K_F Tx \rangle = \langle Tx, b \rangle = \langle x, T'b \rangle$$

for $x \in E$, $b \in F'$. Using the assumption, we get

$$T' = (K_F T)^T \in [\mathfrak{A}, \mathfrak{B}](F', E; \mathbb{C}) = \mathfrak{A}(F', E'), \text{ i.e. } T \in \mathfrak{A}^{dual} .$$

Now we are going to give sufficient conditions for the equation (*). The proof of the next proposition is based on the following.

2.7 Lemma. Let $\mathfrak{A}$ and $\mathfrak{B}$ be any operator ideals. Suppose that $\mathfrak{A}$ is even normed. Then $[\mathfrak{B}, \mathfrak{A}]^T \subseteq [\mathfrak{A}, \mathfrak{L}]$ implies $[\mathfrak{B}, \mathfrak{A}] \subseteq \mathfrak{L}(\mathfrak{L}, \mathfrak{A})$.

Proof. Let $M \in [\mathfrak{B}, \mathfrak{A}]$ be given. Then we have $M^T \in [\mathfrak{A}, \mathfrak{L}] = \mathfrak{L}(\mathfrak{A}, \mathfrak{L})$ by 2.2, hence $M \in \mathfrak{L}(\mathfrak{L}, \mathfrak{A})$.

2.8 Proposition: Let $\mathfrak{A}$ and $\mathfrak{B}$ be operator ideals. Let $\mathfrak{A}$ be normed. If $[\mathfrak{B}, \mathfrak{A}]^T \subseteq [\mathfrak{A}, \mathfrak{L}]$ then $[\mathfrak{B} \circ \mathfrak{C}, \mathfrak{A}] \subseteq \mathfrak{L}(\mathfrak{C}, \mathfrak{A})$ for every operator ideal $\mathfrak{C}$. If, additionally, $\mathfrak{B} \circ \mathfrak{C} = \mathfrak{C}$ then $[\mathfrak{C}, \mathfrak{A}] = \mathfrak{L}(\mathfrak{C}, \mathfrak{A})$.

Proof. Using 2.7 we get

$$[\mathfrak{B} \circ \mathfrak{C}, \mathfrak{A}] = [\mathfrak{B}, \mathfrak{A}](\mathfrak{C}, \mathfrak{L}) \subseteq \mathfrak{L}(\mathfrak{L}, \mathfrak{A})(\mathfrak{C}, \mathfrak{L}) = \mathfrak{L}(\mathfrak{C}, \mathfrak{A}).$$

If $\mathfrak{B} \circ \mathfrak{C} = \mathfrak{C}$ then we have

$$[\mathfrak{C}, \mathfrak{A}] = [\mathfrak{B} \circ \mathfrak{C}, \mathfrak{A}] \subseteq \mathfrak{L}(\mathfrak{C}, \mathfrak{A}) \subseteq [\mathfrak{C}, \mathfrak{A}].$$

Therefore, $[\mathfrak{C}, \mathfrak{A}] = \mathfrak{L}(\mathfrak{C}, \mathfrak{A})$.

In the following we discuss sufficient conditions for the inclusion

$$[\mathfrak{B}, \mathfrak{A}]^T \subseteq [\mathfrak{A}, \mathfrak{L}] .$$

2.9 Proposition. Let $\mathfrak{A}$ be a normed operator ideal and let $\mathfrak{N}$ be the ideal of nuclear operators. Then $[\mathfrak{N}, \mathfrak{A}]^T \subseteq [\mathfrak{A}, \mathfrak{L}]$.

Proof. Suppose that $M \in [\mathfrak{N}, \mathfrak{A}](E, F; G)$. Then there are $a_i \in E'$ and $T_i \in \mathfrak{A}(F, G)$ such that

$$M = \sum_{i=1}^{\infty} a_i \otimes T_i \quad \text{and} \quad \sum_{i=1}^{\infty} \|a_i\| \|T_i | \mathfrak{A}\| < \infty .$$

By 2.1 and 2.3 we have

$$(\sum_{i=1}^{n} a_i \otimes T_i)^T \in [\mathfrak{A},\mathfrak{L}](F,E;G)$$

and

$$\sum_{i=1}^{n} \| (a_i \otimes T_i)^T | \mathfrak{A} \| \leq \sum_{i=1}^{n} \| a_i \| \cdot \| T_i | \mathfrak{A} \| < \infty .$$

This implies

$$\sum_{i=1}^{\infty} (a_i \otimes T_i)^T \in [\mathfrak{A},\mathfrak{L}] .$$

<u>2.10 Corollary.</u> Let $\mathfrak{A}$ and $\mathcal{L}$ be operator ideals, let $\mathfrak{A}$ be normed. Then

$$[\mathfrak{N}\cdot\mathcal{L},\mathfrak{A}] \subseteq \mathfrak{L}(\mathcal{L},\mathfrak{A}) ,$$

and

$$\mathfrak{N}\cdot\mathcal{L} = \mathcal{L}$$

then

$$[\mathcal{L},\mathfrak{A}] = \mathfrak{L}(\mathcal{L},\mathfrak{A}) .$$

<u>2.11 Proposition.</u> For every closed operator ideal $\mathfrak{A}$ we have

$$[\mathcal{G},\mathfrak{A}]^T \subseteq [\mathfrak{A},\mathfrak{L}] .$$

Proof. Suppose that $M \in [\mathcal{G},\mathfrak{A}](E,F;G) = \mathcal{G}(E,\mathfrak{A}(F,G))$. By definition of $\mathcal{G}$ there are finite rank operators $M_n \in \mathcal{F}(E,\mathfrak{A}(F,G))$ with $\lim \| M - M_n \| = 0$. Using 2.1 we have $M_n \in \mathfrak{L}(\mathcal{F},\mathfrak{A})$.

Hence $M_n^T \in \mathfrak{L}(\mathfrak{A},\mathcal{F}) \subseteq [\mathfrak{A},\mathfrak{L}]$. By the closedness of $\mathfrak{A}$ it follows from $\|\cdot\| - \lim M_n^T = M^T$ that $M^T \in [\mathfrak{A},\mathfrak{L}]$.

<u>2.12. Corollary.</u> Let $\mathfrak{A}$ be any closed operator ideal and let $\mathcal{L}$ be an arbitrary operator ideal. Then

$$[\mathcal{G}\cdot\mathcal{L},\mathfrak{A}] \subseteq \mathfrak{L}(\mathcal{L},\mathfrak{A}).$$

If $\mathcal{G}\cdot\mathcal{L} = \mathcal{L}$ then $[\mathcal{L},\mathfrak{A}] = \mathfrak{L}(\mathcal{L},\mathfrak{A})$.
Especially, $[\mathcal{G},\mathfrak{A}] = \mathfrak{L}(\mathcal{G},\mathfrak{A})$ and $[\mathcal{G},\mathcal{G}] = \mathfrak{L}(\mathcal{G},\mathcal{G})$.

<u>2.13 Definition.</u> A (normed) operator ideal $\mathfrak{A}$ is called to be injective if $S \in \mathcal{L}(E,F)$, $T \in \mathfrak{A}(E,G)$, and $\| Sx \| \leq \| Tx \|$ for all $x \in E$ imply $S \in \mathfrak{A}(E,F)$ and, in the normed case, $\| S | \mathfrak{A} \| \leq \| T | \mathfrak{A} \|$.

<u>2.14 Proposition.</u> Let $\mathfrak{A}$ be any injective closed operator ideal.

Then

$$[\mathfrak{K},\mathfrak{A}]^{T} \subseteq [\mathfrak{A},\mathcal{L}] .$$

Proof. Suppose that $M \in [\mathfrak{K},\mathfrak{A}](E,F;G)$. Let U_E denote the unit ball of E. Since $M(U_E)$ is precompact in $\mathfrak{A}(F,G)$, there is a sequence (T_i) of operators $T_i \in \mathfrak{A}(F,G)$ such that

$$\| T_i \| \longrightarrow 0 \quad \text{and} \quad M(U_E) \subseteq \overline{\operatorname{acx}} \{T_i : i \in \mathbb{N}\} .$$

The Banach space $c_0(G)$ is defined by
$c_0(G) = \{ (z_i): z_i \in G, \| (z_i) \| = \sup_{i \in \mathbb{N}} \| z_i \| < \infty \}$.

Next, define an operator

$$T: F \longrightarrow c_0(G) \quad \text{by} \quad Ty = (T_i y)_{i \in \mathbb{N}} \quad \text{for} \quad y \in F.$$

Then we get

$$\| Ty - \sum_{i=1}^{n} j_i T_i y \| = \sup_{i > n} \| T_i y \| \leq \sup_{i > n} \| T_i \| \| y \|$$

for each $y \in F$, where j_i denotes the canonical injection from G into the i^{th} coordinate of $c_0(G)$. Since $\lim_{n \to \infty} \sup_{i > n} \| T_i \| = 0$, it follows

$$T = \| \cdot \| - \lim_{n \to \infty} \sum_{i=1}^{n} j_i T_i \in \mathfrak{A}(F,G).$$

If we put $F_1 = \overline{T(F)}$, we obtain $T \in \mathfrak{A}(F,F_1)$ by the injectivity of $\mathfrak{A}$.

Now, we can define a bilinear map $N: T(F) \times E \longrightarrow G$ by $N(Ty,x) = M(x,y)$ for $x \in E$ and $y \in F$. Let $\varepsilon > 0$ and $x \in U_E$ by given Then there are numbers λ_i and an operator $R \in \mathfrak{A}(F,G)$ such that $\sum_{i=1}^{\infty} |\lambda_i| \leq 1$, $\| R \| \leq 1$, and $Mx = \sum_{i=1}^{\infty} \lambda_i T_i + \varepsilon R$. Now, it follows from $M(x,y) = \sum_{i=1}^{\infty} \lambda_i T_i y + \varepsilon Ry$ that

$\| M(x,y) \| \leq \sum_{i=1}^{\infty} |\lambda_i| \sup_i \| T_i y \| + \varepsilon \| y \|$.

Hence

$\| N(Ty,x) \| = \| M(x,y) \| \leq \| Ty \| + \varepsilon \| y \|$ for all $y \in F$, and $\varepsilon \longrightarrow 0$

proves the continuity of N.
Therefore, N can be extended to some bilinear map belonging to $\mathcal{L}(F_1,E;G)$. This extension is denoted by N, as well.
Now, the assertion follows from $M^T = N(T,1_E) \in [\mathfrak{A},\mathcal{L}](F,E;G)$.

2.15 Corollary. Let $\mathfrak{A}$ be any injective closed operator ideal and let $\mathcal{L}$ be an arbitrary operator ideal. Then $[\mathfrak{K}\circ\mathcal{L}, \mathfrak{A}] \subseteq \mathcal{L}(\mathcal{L},\mathfrak{A})$. If, additionally, $\mathfrak{K}\circ\mathcal{L} = \mathcal{L}$ then $[\mathcal{L},\mathfrak{A}] = \mathcal{L}(\mathcal{L},\mathfrak{A})$.

As a special case, we get $[\mathfrak{K},\mathfrak{A}] = \mathcal{L}(\mathfrak{K},\mathfrak{A})$ and $[\mathfrak{K},\mathfrak{K}] = \mathcal{L}(\mathfrak{K},\mathfrak{K})$.

3. Examples and Counterexamples

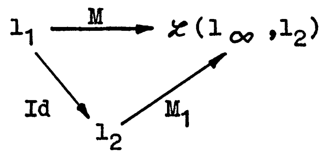
To provide us with examples of pairs $(\mathfrak{B},\mathfrak{A})$ of normed operator ideals which do not satisfy the equation (*) we will use the following two propositions.
The contrapositions of 2.4 and 2.6 yield.

3.1 Proposition. Let $\mathfrak{A}$ and $\mathfrak{B}$ be any normed ideals. If $\mathfrak{B} \nsubseteq \mathfrak{A}^{dual}$ then $[\mathfrak{B},\mathfrak{A}] \neq \mathcal{L}(\mathfrak{B},\mathfrak{A})$.

3.2 Proposition. $[\mathfrak{P},\mathfrak{I}]^T \nsubseteq [\mathfrak{N},\mathcal{L}]$. Especially, $[\mathfrak{N},\mathfrak{N}]^T \nsubseteq [\mathfrak{N},\mathcal{L}]$, $[\mathfrak{P},\mathfrak{N}]^T \nsubseteq [\mathfrak{N},\mathcal{L}]$, and $[\mathfrak{P},\mathfrak{I}]^T \nsubseteq [\mathfrak{I},\mathcal{L}]$.

Proof. Choose $(E,F,G) = (l_1,l_\infty,l_2)$ and define
$M: l_1 \times l_\infty \longrightarrow l_2$ by $M(\xi,\lambda) = (\xi_i\lambda_i)_{i\in\mathbb{N}}$, $\xi \in l_1$, $\lambda \in l_\infty$.

Then $M: l_1 \longrightarrow \mathfrak{L}(l_\infty, l_2) = \mathfrak{Y}(l_\infty, l_2)$. The diagram



where $M_1(\xi) = D_\xi$, $\xi \in l_2$, and $D_\xi(\mu) = (\xi_i \mu_i)_{i\in N}$, yields a factorization of M by $Id \in \mathfrak{L}(l_1, l_2) = \mathfrak{P}(l_1, l_2)$ by Grothendieck's Theorem (cf. /5, 22.4.4/).

This shows $M \in [\mathfrak{P}, \mathfrak{Y}](l_1, l_\infty; l_2)$.
The transposed bilinear map $M^T: l_\infty \longrightarrow \mathfrak{L}(l_1, l_2)$ is given by $M^T(\lambda) = D_\lambda$ for $\lambda \in l_\infty$. Since $\| D_\lambda : l_1 \longrightarrow l_2 \| = \| \lambda \|_\infty$, M^T is even an isometric embedding, and this map cannot be weakly compact, since l_∞ is not reflexive. This proves $M^T \notin [\mathfrak{W}, \mathfrak{L}]$.

Now, 2.4 together with 3.1, 3.2, and some well known relations between operator ideals yield several pairs of operator ideals, which do not satisfy the equation (*).
Simultaneously, this table shows that the positive results given in section 2 are optimal.

$\mathfrak{B} \backslash \mathfrak{A}$	$\mathfrak{G}$	$\mathfrak{K}$	$\mathfrak{X}$	$\mathfrak{W}$	$\mathfrak{P}$	$\mathfrak{Y}$
$\mathfrak{F}$	+	+	+	+	+	+
$\mathfrak{G}$	+	+	+	+	-	-
$\mathfrak{K}$	-	+	+	+	-	-
$\mathfrak{X}$	-	-	-	-	-	-
$\mathfrak{W}$	-	-	-	-	-	-
$\mathfrak{P}$	-	-	-	-	-	-
$\mathfrak{Y}$	-	-	-	-	-	-

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