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COISOTROPIC BUNDLES AND INDUCED REPRESENTATIONS

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0. Introduction

This paper deals with some problems from symplectic geometry connected with a symplectic analogue of the induction procedure. A procedure of "symplectic induction" was introduced by Weinstein [We2] in connection with a gauge-invariant description of phase spaces of classical particles in Yang-Mills fields and was further developed by Guillemin and Sternberg [GS2]. According to geometric quantization (see e.g. [B], [GS1], [Ki], [Ko], [We1]), a symplectic model of a unitary representation of a Lie group G is a Hamiltonian G-space. Given a Lie subgroup H of G, Weinstein's procedure associates to each Hamiltonian H-space a Hamiltonian G-space which is a symplectic model of the representation of G induced by the representation of H corresponding to the Hamiltonian H-space. The "induced" Hamiltonian G-space has an additional structure of a coisotropic bundle over G/H. Basic facts concerning coisotropic bundles and Weinstein's procedure are briefly summarized in sections 1 and 2. These sections contain also some results on classification of Lagrangian bundles,which are special cases of coisotropic bundles.

Principal results of this paper concerning symplectic induction in the case of semisimple Lie groups are contained in the next two sections. The main technical device is the use of complex symplectic geometry, which allows us to replace (real) coisotropic bundles by (complex) Lagrangian bundles. In section 3 we study some geometrical properties of holomorphic Lagrangian bundles over complex flag manifolds. Results of this section are applied in section 4 to the study of some class of coisotropic bundles over real flag manifolds of a real semisimple Lie group G which correspond to the representations of G induced by finite dimensional representations of parabolic subgroups of G. In particular, we establish a relationship between the bundles of this class and coadjoint orbits of G thus extending some results of [GS2].

This paper is a considerably extended version of my talk during the Conference. This talk deals with holomorphic Lagrangian bundles only, which constitute, approximately, the material of section 3. Since then a detailed exposition of the results of this section has been prepared (see [L]). The entirely new section 4 is a preliminary version of a publication which is now in preparation.

1. Coisotropic bundles

In this section M is a fixed real or complex manifold. All fiber bundles to be considered have M as a base space (in the complex case they are assumed to be holomorphic). For the basic notions of symplectic geometry used here the reader is referred to [AM], [GS1], [We1] and [Wo] (all these references deal with real symplectic geometry; transition to the complex case is obvious).

(1.1) A coisotropic bundle is a quadruple

$$\lambda = (E, \pi, M, \omega),$$

where $\pi : E \longrightarrow M$ is a fiber bundle and ω is a symplectic form on E such that each fiber E_m is a coisotropic submanifold of (E, ω). A morphism of coisotropic bundles is a fiber preserving symplectomorphism; an M-morphism is one which induces the identity on M. E carries a natural isotropic foliation whose restriction to each E_m coincides with the kernel foliation of $\omega|_{E_m}$. We shall assume that this foliation is a fibration $\rho : E \longrightarrow N$. Thus E has two compatible structures of a fiber bundle, that is, we have a commutative diagram

$$\text{(1.1.1)} \qquad \begin{array}{ccc} E & & \\ {\scriptstyle\rho}\downarrow & \searrow{\scriptstyle\pi} & \\ N & \xrightarrow[\sigma]{} & M. \end{array}$$

Moreover, the fibers of σ are symplectic manifolds.

In the special case where the fibers of π are Lagrangian submanifolds we obtain the notion of a Lagrangian bundle. In this case ρ coincides with π.

(1.2) Polarizations ([B], [GS1], [We1], [Wo]). A polarization of a real symplectic manifold (X, ω) is a complex involutive Lagrangian subbundle F of the complexified tangent bundle $TX \otimes \mathbb{C}$.

If $\bar{F} = F$ (where bar denotes the complex conjugation in $TX \otimes \mathbb{C}$), F is called a real polarization. Thus real polarizations are in a one-one correspondence with Lagrangian foliations of X. In particular, any structure of a Lagrangian bundle on X determines a real polarization of X.

At the other extreme are Kähler polarizations which are characterized by $F \cap \bar{F} = \{0\}$. They are in one-one correspondence with complex structures on X with respect to which ω has type (1, 1).

A polarization F is strongly admissible if $F \cap TX$ and $(F + \bar{F}) \cap TX$ are involutive subbundles of TX and the corresponding foliations are fibrations. Such polarizations are in one-one correspondence with structures of coisotropic bundle on X such that, for each $m \in M$, $\sigma^{-1}(m)$ (see (1.1.1)) carries a Kähler polarization which depends smoothly on m.

In some cases we can find a complexification $(X^{\mathbb{C}}, \omega^{\mathbb{C}})$ of (X, ω) such that a polarization F of (X, ω) induces a structure of holomorphic Lagrangian bundle on $(X^{\mathbb{C}}, \omega^{\mathbb{C}})$. On the other hand, if (X, ω) is a real form of a complex symplectic manifold $(X^{\mathbb{C}}, \omega^{\mathbb{C}})$, each structure of holomorphic Lagrangian bundle on $(X^{\mathbb{C}}, \omega^{\mathbb{C}})$ determines a polarization of (X, ω).

(1.3) The isotropic fibers of a coisotropic bundle λ carry an additional structure. Namely, the M-morphism of bundles

(1.3.1) $$T^*M \times_M E \longrightarrow TE : (p,e) \longmapsto \xi^p(e) = ((T_e\pi)^*(p))^{\#},$$

where sharp denotes the isomorphism $T^*E \longrightarrow TE$ induced by ω, is an infinitesimal action of T^*M (viewed as a bundle of Abelian Lie algebras) on E, that is, for each $m \in M$ and all $p, p' \in T^*_mM$, the vector fields ξ^p and $\xi^{p'}$, defined on E_m, commute. Moreover, (1.3.1) is a vector bundle isomorphism onto the subbundle $\ker T\varsigma$ (i.e. the action is free). Thus the isotropic fibers of λ are parallelizable affine manifolds.

We say that a coisotropic bundle λ is an affine coisotropic bundle (AC bundle) if the infinitesimal action (1.3.1) is induced by a (necessarily unique) action

$$T^*M \times_M E \longrightarrow E$$

of T^*M (now viewed as a bundle of Abelian Lie groups) on E. (This is so iff all isotropic fibers of λ are simply connected and all vector fields ξ^p are complete.) Thus the isotropic bundle $\varsigma : E \longrightarrow N$ of an AC bundle is an affine bundle; the corresponding vector bundle is σ^*T^*M.

The remainder of this section is devoted to the classification of holomorphic affine Lagrangian bundles (AL bundles) over a fixed complex manifold M (in the real case the classification is well known; cf. [Wo], Prop. 4.4.2 and the Remarks that follow it).

(1.4) The simplest AL bundle is the cotangent bundle

$$\tau^*_M = (T^*M, \pi_M, M, \omega_M)$$

with its canonical symplectic form. The underlying affine bundle is T^*M itself. It turns out that any AL bundle is locally isomorphic to τ^*_M. More precisely, we have the following.

(1.5) Proposition. Let λ be an AL bundle over M. For each $m \in M$, there is an open neighborhood $U \ni m$ in M together with a U-isomorphism $\varphi_U : \lambda|_U \to \tau^*_U$.

See [L], (1.9) for a simple proof.

(1.6) The above proposition allows us to classify holomorphic AL bundles over M. In fact, it follows from the theory of fiber bundles (see e.g. [G], Prop. 5.1.1) that the M-isomorphism classes of fiber bundles over M which are locally isomorphic to a given model bundle are in one-one correspondence with the elements of the first cohomology space of M with values in the sheaf of germs of M-automorphisms of the model bundle. In the case of τ^*_M, this sheaf is isomorphic to $\mathcal{Z}^1$, the sheaf of germs of closed holomorphic 1-forms on M; the isomorphism is obtained by assigning to each $\alpha \in \mathcal{Z}^1(U)$, U open in M, the U-automorphism $p \mapsto p + \alpha_m$, $m = \pi_M(p)$. Thus we obtain the following.

(1.7) Proposition. There is a natural bijection between the set of M-isomorphism classes of holomorphic AL bundles over M and $H^1(M, \mathcal{Z}^1)$.

2. Hamiltonian coisotropic bundles and symplectic induction

In this section G and H denote real or complex Lie groups and $\mathfrak{g}$ and $\mathfrak{h}$ their Lie algebras.

(2.1) A Hamiltonian G-space (X, ω, J) consists of a symplectic space (X, ω), an action of G on (X, ω) by symplectomorphisms and a G-equivariant map $J : X \to \mathfrak{g}^*$ such that

$$i(\xi_X)\omega = dJ^*(\xi) \quad \forall \xi \in \mathfrak{g},$$

where $\xi \mapsto \xi_X$ is the infinitesimal action of $\mathfrak{g}$ on X and J^* is a map from $\mathfrak{g}$ to analytic functions on X defined by

$$(J^*(\xi))(x) = \langle J(x), \xi \rangle \quad \forall \xi \in \mathfrak{g} \ \forall x \in X.$$

J is called a <u>momentum mapping</u> for the action of G.

A morphism $\varphi : (X, \omega, J) \longrightarrow (X, \omega', J')$ of Hamiltonian G-spaces is a G-equivariant symplectomorphism satisfying

$$J' \circ \varphi = J.$$

Each coadjoint orbit of G in $\mathfrak{g}^*$ has a natural structure of a homogeneous Hamiltonian G-space. If (X, ω, J) is a homogeneous Hamiltonian G-space, $J : X \longrightarrow J(X)$ is a morphism and a covering.

Hamiltonian G-spaces are symplectic analogues of unitary representations of G; the homogeneous ones are counterparts of irreducible representations. See [B], [Ki], [Ko] for details.

<u>(2.2)</u> An <u>(affine)</u> <u>Hamiltonian coisotropic G-bundle</u> ((A)HC G-bundle) over a G-space M is a pair (λ, J), where $\lambda = (E, \pi, M, \omega)$ is an (affine) coisotropic bundle on which G acts by automorphisms in such a way that the total space E is a Hamiltonian G-space, and J is the momentum mapping for this action. A morphism of (A)HC G-bundles is a map which is both a morphism of the underlying (affine) coisotropic bundles and a morphism of the underlying Hamiltonian G-spaces. We write $[\![(\lambda, J)]\!]$ for the M-isomorphism class of (λ, J).

There is a natural relationship between HC G-bundles and G-invariant polarizations of Hamiltonian G-spaces which is a G-equivariant counterpart of the relationship between coisotropic bundles and polarizations described at the end of (1.2).

<u>(2.3)</u> <u>Marsden-Weinstein reduction.</u> This is a general method of producing new symplectic spaces from a given Hamiltonian space. We shall use it below to construct AHC G-bundles over a G-homogeneous base. Let (X, ω, J) be a Hamiltonian H-space. Fix a regular value $f \in \mathfrak{h}^*$ of J and let H_f denote the stabilizer of f. Then $J^{-1}(f)$ is a H_f-invariant submanifold of X so the orbit space

$$X_f = J^{-1}(f)/H_f$$

is well defined. Assume that X_f has a structure of manifold such that the natural projection $\nu_f : J^{-1}(f) \longrightarrow X_f$ is a submersion (this assumption is satisfied, for instance, if the action of H_f on $J^{-1}(f)$ is free and proper). Then a theorem of Marsden and Weinstein ([AM], 4.3.1) asserts that there is a symplectic form ω_f on X_f which is uniquely determined by

$$\nu_f^* \omega_f = i_f^* \omega ,$$

where $i_f : J^{-1}(f) \longrightarrow X$ denotes the inclusion. The symplectic space (X_f, ω_f) is called the (Marsden-Weinstein) reduction of X at f.

There is an alternative description of X_f which is sometimes more useful. Let O_f be the H-orbit through f and let $O_f^- = \{- f \mid f \in O_f\}$. Then X_f is symplectically isomorphic to the reduction of the product Hamiltonian H-space $X \times O_f^-$ at zero. When viewed this way, X_f is usually denoted by X_{O_f} and called the (Marsden-Weinstein) reduction of X with respect to O_f. More generally, we may define the reduction X_Y of X with respect to an arbitrary Hamiltonian H-space Y replacing O_f by Y in the above construction.

(2.4) Reduction of a cotangent bundle to a Lie group. T^*G has two natural structures of left and right affine Hamiltonian Lagrangian (AHL) G-bundle, the corresponding actions of G being cotangent to the actions of G on itself by left and right translations, respectively. If H is a Lie subgroup of G, we may view T^*G as a right Hamiltonian H-space. Change the right action of H into a left one letting each $h \in H$ act on G as the right translation by h^{-1}. For each Hamiltonian H-space Y, the assumptions of the Marsden-Weinstein theorem are fulfilled so we may form the reduced space $(T^*G)_Y$. Since the actions of G and H on T^*G commute and the momentum mapping for the action of G (resp. H) is invariant under H (resp. G), the structure of a (left) AHL G-bundle on T^*G induces a natural structure of AHC G-bundle on $(T^*G)_Y$. We shall denote this bundle by $\lambda(Y)$. In the special case where Y is a coadjoint orbit O_f through f we shall write λ_f rather than $\lambda(O_f)$.

The structure of λ_f can be described more closely. In fact, using the left trivialization of T^*G we obtain an isomorphism

$$(T^*G)_f \cong G \times_{H_f} r^{-1}(f),$$

where $r : \mathfrak{g}^* \longrightarrow \mathfrak{h}^*$ is the restriction map. It follows that the space of isotropic fibers is isomorphic to

$$G \times_H O_f \cong G \times_H H/H_f \cong G/H_f$$

and the space of coisotropic fibers is isomorphic to G/H. Moreover, the commutative diagram (1.1.1) becomes

$$\begin{array}{ccc} G \times_{H_f} r^{-1}(f) & & \\ \downarrow & \searrow & \\ G/H_f & \longrightarrow & G/H \end{array} ,$$

where all arrows represent the natural G-equivariant projections.

It is easy to see that (λ_f, J_f), where J_f is the momentum mapping for the action of G on λ_f, is an AHL G-bundle iff f is H-invariant. Actually, we can prove a stronger result.

<u>(2.5)</u> <u>Proposition.</u> Keep the above notation and let also $(\mathfrak{h}^*)^H$ denote the subspace of H-invariant elements in $\mathfrak{h}^*$.

(a) For each $f \in (\mathfrak{h}^*)^H$, (λ_f, J_f) is an AHL G-bundle.

(b) If (λ, J) is an AHL G-bundle over G/H, then $r(J(E_o))$ (where o stands for the coset H) consists of a single element f, which therefore must belong to $(\mathfrak{h}^*)^H$, and the map

$$\varphi_J : E \longrightarrow G/H \times \mathfrak{g}^* : e \longmapsto (\pi(e), J(e))$$

induces a (G/H)-isomorphism between (λ, J) and (λ_f, J_f) (the latter being identified with a subbundle of $G/H \times \mathfrak{g}^*$).

(c) The map $f \longmapsto [(\lambda_f, J_f)]$ is a bijection of $(\mathfrak{h}^*)^H$ onto the set of (G/H)-isomorphism classes of AHL G-bundles over G/H.

See [L], (2.6) for a proof.

<u>(2.6)</u> <u>Symplectic induction.</u> It is easy to see that the correspondence $Y \longmapsto \lambda(Y)$ establishes a covariant functor from the category of Hamiltonian H-spaces to the category of Hamiltonian G-spaces. In the real case, this functor is a symplectic counterpart of the induction functor. The analogy between these functors is based on the theory of geometric quantization which in some cases allows us to "quantize" the actions of H on Y and G on $\lambda(Y)$ in such a way that the resulting unitary representation of G is equivalent to the representation induced by the representation of H corresponding to Y. One such case will be considered in section 4.

Holomorphic AHC bundles seem to be natural candidates for symplectic models of holomorphically induced representations. More about this will be said in section 4.

3. Holomorphic Lagrangian bundles over complex flag manifolds

For the proofs of the following results the reader is referred to [L]. Throughout this section G denotes a connected complex semisimple Lie group and P a parabolic subgroup of G; the Lie algebra of P is denoted by $\mathfrak{p}$. (We shall sometimes view G and P as linear algebraic groups.) By a Lagrangian bundle we shall always mean a holomorphic Lagrangian bundle over the complex flag manifold G/P (o will stand for the coset P). The interest in this case is motivated by the following theorem whose part (a) is due to Ozeki and Wakimoto ([OWa], Th.2.2) while part (b) follows from standard properties of invariant polarizations (see e.g.[B], Chap.IV).

(3.1) Theorem. Let (X,π,M,ω,J) be a HL G-bundle with homogeneous total space X (here we make no assumptions about M). Then:

(a) M is a flag manifold, M = G/P for some P;

(b) $r(J(X_0)) = \{f\}$ with $f \in (\mathfrak{p}^*)^P$ and the map

$$\varphi_J : x \mapsto (\pi(x), J(x))$$

induces a morphism from (X,π,M,ω,J) to (λ_f, J_f). Moreover, $\varphi_J(X)$ is a Zariski open G-orbit on the total space E_f of λ_f. (E_f has a unique structure of algebraic variety compatible with its manifold structure.)

It is natural to ask whether this theorem can be reversed, i.e., whether every AHL G-bundle over G/P has a Zariski open G-orbit on its total space. The affirmative answer to this question follows easily from a theorem of Richardson [Ri] which asserts that each parabolic subgroup of a linear reductive algebraic group over an algebraically closed field has a Zariski open orbit acting by the adjoint representation on the nilradical of its Lie algebra.

(3.2) Proposition. For each $f \in (\mathfrak{p}^*)^P$, G has a Zariski open orbit X_f on the total space E_f of the bundle (λ_f, J_f).

(3.3) Corollary. $J_f(E_f) = \overline{J_f(X_f)}$ (Zariski closure).

The next result, which does not seem to be related to the representation theory, illustrates the difference between real and complex symplectic geometry - it has no analogue in the real case.

(3.4) Theorem. Any affine Lagrangian bundle over G/P has a unique structure of an affine Hamiltonian Lagrangian G-bundle so that the natural map $(\mathfrak{p}^*)^P \longrightarrow H^1(G/P, \mathcal{X}^1)$ (induced by forgetting about the G-action) is a bijection (actually a linear isomorphism).

We sketch the proof. The elements of $(\mathfrak{p}^*)^P$ can be thought of as biinvariant 1-forms on P. Since these are closed, we can define a map $(\mathfrak{p}^*)^P \longrightarrow H^1(P,\mathbb{C})$ which is easily seen to be an isomorphism. Next, the spectral sequence of the fibration $G \longrightarrow G/P$ gives rise to an isomorphism $H^1(P,\mathbb{C}) \longrightarrow H^2(G/P,\mathbb{C})$. Finally, there is a map $H^1(G/P, \mathcal{X}^1) \longrightarrow H^2(G/P,\mathbb{C})$ resulting from the long exact sequence of cohomology groups corresponding to the short exact sequence of sheaves

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O} \longrightarrow \mathcal{X}^1 \longrightarrow 0,$$

where $\mathcal{O}$ denotes the sheaf of germs of holomorphic functions on G/P. Since $H^1(G/P,\mathcal{O}) = H^2(G/P,\mathcal{O}) = 0$, this map is an isomorphism. Now it is quite easy to show that the diagram

$$\begin{array}{ccc} (\mathfrak{p}^*)^P & \longrightarrow & H^1(P,\mathbb{C}) \\ \downarrow & & \downarrow \\ H^1(G/P, \mathcal{X}^1) & \longrightarrow & H^2(G/P,\mathbb{C}) \end{array}$$

consisting of the above described maps commutes, which clearly implies that $(\mathfrak{p}^*)^P \longrightarrow H^1(G/P, \mathcal{X}^1)$ is an isomorphism. Thus any affine Lagrangian bundle over G/P has a structure of an affine Hamiltonian Lagrangian G-bundle which is unique up to an M-automorphism. The group of (G/P)-automorphisms of any AL bundle over G/P being trivial (since it is isomorphic to $H^0(G/P, \mathcal{X}^1) = 0$), this structure is in fact unique.

Finally we shall characterize those AHL G-bundles whose total spaces are G-homogeneous.

(3.5) Theorem. For an AHL G-bundle (λ_f, J_f), $f \in (\mathfrak{p}^*)^P$, over G/P, the following conditions are equivalent:

(i) the total space E_f of λ_f is G-homogeneous;

(ii) E_f is a Stein manifold;

(iii) the orbit $J_f(X_f)$, where X_f is the unique Zariski open G-orbit on E_f, consists of semisimple elements (here we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of the killing form).

4. Coisotropic bundles over real flag manifolds

Here G has the same meaning as in the preceding section. Moreover, we assume that G is defined over $\mathbb{R}$ (as an algebraic group) and let $G_{\mathbb{R}}$ denote the corresponding group of real points. $G_{\mathbb{R}}$ is viewed as a real algebraic group or as a real Lie group. G_0 denotes the connected component of identity of the real Lie group $G_{\mathbb{R}}$. For simplicity reasons the results of this section are formulated usually in terms of $G_{\mathbb{R}}$. $Q_{\mathbb{R}}$ denotes a parabolic subgroup of $G_{\mathbb{R}}$, that is, the group of real points of a parabolic subgroup Q of G which is defined over $\mathbb{R}$. The homogeneous space $G_{\mathbb{R}}/Q_{\mathbb{R}}$ is called a real flag manifold. We write $\mathfrak{q}_{\mathbb{R}}$ for the Lie algebra of $Q_{\mathbb{R}}$ and $\mathfrak{n}_{\mathbb{R}}$ for the nilradical of $\mathfrak{q}_{\mathbb{R}}$. We denote by $M_{\mathbb{R}}$ (resp. $\mathfrak{m}_{\mathbb{R}}$) the compact component of a Langlands decomposition of $Q_{\mathbb{R}}$ (resp. $\mathfrak{q}_{\mathbb{R}}$).

(4.1) Suppose that $Q_{\mathbb{R}}$ has a compact coadjoint orbit of positive dimension on $\mathfrak{q}_{\mathbb{R}}$. It can be shown that this orbit must be contained in $(\mathfrak{q}_{\mathbb{R}}/\mathfrak{n}_{\mathbb{R}})^*$ (which we identify with a subspace of $\mathfrak{q}_{\mathbb{R}}^*$). Moreover, if choose a Langlands decomposition of $\mathfrak{q}_{\mathbb{R}}$, the compact orbit is also an $M_{\mathbb{R}}$-orbit and if we identify it with an adjoint orbit in the Levi component of $\mathfrak{q}_{\mathbb{R}}$, then this adjoint orbit generates a compact ideal. Thus $Q_{\mathbb{R}}$ has a compact coadjoint orbit of positive dimension iff some (and hence any) Levi component of $\mathfrak{q}_{\mathbb{R}}$ has a compact ideal.

(4.2) We shall be considering AHC $G_{\mathbb{R}}$-bundles λ_f over $G_{\mathbb{R}}/Q_{\mathbb{R}}$ (see (2.4)) for which the space of isotropic fibers is compact. These split into two classes: real AHL $G_{\mathbb{R}}$-bundles (which exist for any $Q_{\mathbb{R}}$) and bundles for which O_f is a compact orbit of positive dimension (which exist only for certain $Q_{\mathbb{R}}$ as we saw above). For the reasons of simplicity we assume that the orbit O_f is connected (in the usual topology). The total space of λ_f will be denoted, as usual, by E_f.

(4.3) Complexification of λ_f. Let $f^{\mathbb{C}}$ denote the complexification of f. Then the AHC G-bundle $\lambda_{f^{\mathbb{C}}}$ over G/Q is a complexification of λ_f. That is, if we view the total space $E_{f^{\mathbb{C}}}$ as an algebraic variety, then it is defined over $\mathbb{R}$ in such a way that E_f coincides with the set $E_{f^{\mathbb{C}}}(\mathbb{R})$ of real points of E_f .

(4.4) It is well known that the compact orbit O_f has an invariant polarization which is necessarily Kähler. This polarization is determined by a parabolic subalgebra of the reductive algebra $\mathfrak{q}/\mathfrak{n}$ ($\mathfrak{q}$ being the Lie algebra of Q and $\mathfrak{n}$ the nilradical of $\mathfrak{q}$). This parabolic subalgebra can be written uniquely as $\mathfrak{p}/\mathfrak{n}$, where $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ contained in $\mathfrak{q}$. It is easily seen that $f^{\mathbb{C}}|\mathfrak{p}$ is left invariant by the parabolic subgroup P of G corresponding to $\mathfrak{p}$. Thus $f^{\mathbb{C}}|\mathfrak{p}$ defines a holomorphic AHL G-bundle $\lambda_{f^{\mathbb{C}}|\mathfrak{p}}$ over G/P (cf. (2.4) and (2.5)).

(4.5) Proposition. The Hamiltonian G-spaces $E_{f^{\mathbb{C}}}$ and $E_{f^{\mathbb{C}}|\mathfrak{p}}$ are isomorphic.

Proof. Let J_Q and J_P denote the momentum mappings for the actions of Q and P on T^*G, respectively. Then $J_P = r_{\mathfrak{p}\mathfrak{q}} \circ J_Q$, where $r_{\mathfrak{p}\mathfrak{q}} : \mathfrak{q}^* \to \mathfrak{p}^*$ is the restriction map. Thus

$$J_Q^{-1}(f^{\mathbb{C}}) \subset J_P^{-1}(f^{\mathbb{C}}|\mathfrak{p}).$$

and we obtain a commutative diagram

$$\begin{array}{ccc} J_Q^{-1}(f^{\mathbb{C}}) & \longrightarrow & J_P^{-1}(f^{\mathbb{C}}|\mathfrak{p}) \\ \downarrow & & \downarrow \\ E_{f^{\mathbb{C}}} & \longrightarrow & E_{f^{\mathbb{C}}|\mathfrak{p}} \end{array}$$

whose top horizontal arrow is the inclusion. It is easy to see that the bottom horizontal arrow is the desired isomorphism. □

(4.6) Suppose that the orbit O_f is quantizable, i.e., it gives rise to a unitary representation U of $Q_{\mathbb{R}}$ (U is necessarily finite dimensional and irreducible). Let $\mathrm{Ind}(G_{\mathbb{R}}, Q_{\mathbb{R}}, U)$ be the representation of $G_{\mathbb{R}}$ induced by U. According to (2.6), λ_f is a symplectic model of $\mathrm{Ind}(G_{\mathbb{R}}, Q_{\mathbb{R}}, U)$. Geometric quantization associates to λ_f the representation $\mathrm{Ind}_{hol}(G_{\mathbb{R}}, (Q_{\mathbb{R}})_f, \chi_f, \mathfrak{p})$ holomorphically induced by a unitary character χ_f of $(Q_{\mathbb{R}})_f$ corresponding to f (which exists because O_f is quantizable) and the parabolic subalgebra $\mathfrak{p}$ (see (4.4); we assume that $\mathfrak{p}$ has been chosen in such a way that the corresponding polarization of O_f is positive, which is always possible). $\mathrm{Ind}_{hol}(G_{\mathbb{R}}, (Q_{\mathbb{R}})_f, \chi_f, \mathfrak{p})$ can be obtained in two steps (cf. [B]), the first being the holomorphic induction from $(Q_{\mathbb{R}})_f$ to $Q_{\mathbb{R}}$ which gives U (here we use the Borel--Weil-Bott theorem) and the second being the ordinary induction from $Q_{\mathbb{R}}$ to $G_{\mathbb{R}}$. It follows that Ind $(G_{\mathbb{R}}, Q_{\mathbb{R}}, U)$ and $\mathrm{Ind}_{hol}(G_{\mathbb{R}}, (Q_{\mathbb{R}})_f, \chi_f, \mathfrak{p})$ are equivalent.

These facts suggest that $\lambda_{f^{\mathbb{C}}}|_{\mathfrak{p}}$ is a symplectic model of $\mathrm{Ind}_{hol}(G_{\mathbb{R}}, (Q_{\mathbb{R}})_f, \chi_f, \mathfrak{p})$ and the isomorphism (4.5) is a symplectic counterpart of the equivalence

$$\mathrm{Ind}(G_{\mathbb{R}}, Q_{\mathbb{R}}, U) \sim \mathrm{Ind}_{hol}(G_{\mathbb{R}}, (Q_{\mathbb{R}})_f, \chi_f, \mathfrak{p}).$$

(4.7) Theorem. $G_{\mathbb{R}}$ has a Zariski open orbit X_f on the total space E_f of λ_f.

Proof. It follows from (4.5) and (3.2) that G has a Zariski open orbit $X_{f^{\mathbb{C}}}$ on $E_{f^{\mathbb{C}}}$. Since E_f is Zariski dense in $E_{f^{\mathbb{C}}}$, $X_f = X_{f^{\mathbb{C}}} \cap E_f$ is nonempty. It is easy to see that X_f is a Zariski open $G_{\mathbb{R}}$-orbit. □

Remark. X_f splits into a finite number of G_o-orbits.

(4.8) It follows from (4.7) that $J_f(E_f)$ is a Zariski closure of a coadjoint orbit (cf. (3.3)). If E_f is $G_{\mathbb{R}}$-homogeneous, it is isomorphic to a Zariski closed coadjoint orbit (cf. (3.5)). If we identify $\mathfrak{g}_{\mathbb{R}}^*$ with $\mathfrak{g}_{\mathbb{R}}$ by means of the Killing form, Zariski closed coadjoint orbits become identified with Zariski closed adjoint orbits. Such orbits are semisimple (i.e. consist of semisimple elements); however not every semisimple adjoint orbit is Zariski closed. Indeed, we can prove the following fact (which generalizes some results of Rothschild [Ro]).

(4.9) Theorem. Let X be an adjoint $G_{\mathbb{R}}$-orbit. Then the following conditions are equivalent:

(i) X is Zariski closed;
(ii) X is isomorphic (as a Hamiltonian $G_{\mathbb{R}}$-space) to some E_f;
(iii) X is semisimple and has an invariant polarization with a compact space of isotropic fibers.

Remark. If X satisfies the above conditions, then it is connected in the usual topology hence it is also a G_o-orbit.

(4.10) If $Q_{\mathbb{R}}$ is a minimal parabolic subgroup, the induced representations corresponding to the bundles λ_f belong to the unitary principal series. Symplectic analogues of some of those representations were studied by Guillemin and Sternberg who obtained a special case of (4.7). More precisely, they have proved

([GS2], Th. 3.1) that, for some choice of f, E_f is isomorphic to a coadjoint orbit. (4.7) seems also to be related to the results of Wakimoto [Wa] who used non semisimple orbits to realize the principal series representations.

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