Dirk Werner

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EXTREME POINTS IN SPACES OF OPERATORS AND VECTOR-VALUED MEASURES

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1. Introduction.

If one wants to prove that an operator T acting between Banach spaces X and Y is an extreme operator, that is an extreme point of the unit ball of the space of bounded linear operators $L(X,Y)$, it suffices to check that T' (the adjoint of T) maps extreme functionals on Y onto extreme functionals on X. An operator with this property is called a nice operator [8]. It is even enough that T' maps a weak*-dense subset of $\operatorname{ex} B_{Y'}$ into $\operatorname{ex} B_{X'}$, but this is not a substantial generalization. (B_Z denotes the unit ball of a Banach space Z, $\operatorname{ex} C$ the set of extreme points of a convex set C.)

Consider some examples:

1. The identity operator $id: X \to X$ is always a nice, hence extreme operator. Using this and the canonical isometry between $L(X,X'')$ and $L(X',X')$ one can easily see that the natural injection $i_X: X \to X''$ is always extreme.
2. To see that there are non-nice extreme operators consider the injection operator from l^1 into l^p $(1 < p < \infty)$. Moreover, no operator from some space $L^1(\mu)$ into some space $L^p(\nu)$ (p as above) is nice, but $B_{L(L^1,L^p)}$ is compact with respect to the weak*-operator topology so that there are many extreme operators.
3. A fairly easy description of nice operators is possible in the setting of spaces of continuous functions. The main tool here is to represent operators into CL (L a compact Hausdorff space) as vector-valued functions. If X is a Banach space, we associate with an operator from X into CL a function from L into X' in the following way: $T \mapsto T^*$, $T^*(l) := \delta_l \circ T$. This mapping induces an isometric isomorphism between $L(X,CL)$ and the space of weak*-continuous functions $C(L,X'(\text{weak*}))$ and between the space of compact operators $K(X,CL)$ and the space $C(L,X')$ of norm-continuous functions [7, p.490].

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Now, $T \in L(X,CL)$ is nice if and only if the representing function assumes only extremal values: $T^*(L) \subseteq \text{ex } B_{X'}$. More can be said if $X = CK$. $T \in L(CK,CL)$ is nice if and only if T has the form $Tf = \lambda \cdot f \circ \psi$ with $\psi: L \to K$ continuous, $\lambda \in CL$ with modulus one. Equivalently, T is nice if and only if T is essentially multiplicative ($Tf \cdot Tg = T\underline{1} \cdot T(fg)$).

As we pointed out before, in general it is not necessary for an extreme operator to be nice. In the case of operators between CK-spaces, however, the situation is much more involved. Several authors have treated the question if an extreme operator T from CK into CL is necessarily nice, cf. [3], [4], [8], [9], [11]. Although a variety of properties of K, L or T is known that ensure that T is nice (e.g. K metrizable, real scalars; or L extremally disconnected, no matter if the scalars are real or complex), the general answer is no since Sharir has constructed counterexamples both for the real and complex case [12], [13].
All these results may be translated into results concerning measure valued functions, thanks to the isometry $T \mapsto T^*$. In this paper we shall consider operators from $C(K,E)$ to $C(L)$, where E is a Banach space. By Singer's theorem, the dual of $C(K,E)$ may be thought of as the space $M(K,E')$ of E'-valued regular Borel measures of finite variation, equipped with the total variation norm, cf. [15], [16] or [10]. Thus, we shall investigate functions on L the values of which are vector measures. The most far-reaching results can be obtained for compact operators, represented by elements of $C(L,M(K,E'))$. As it turns out, the proofs work for the space $C(L,M(K,Z))$, too, when Z is not a dual Banach space.
Throughout, X, Y, Z, E denote Banach spaces, K, L denote compact Hausdorff spaces. The results apply to real as well as complex Banach spaces.

2. Characterization of extreme points.

Proposition 1: Let T be an extreme point of the unit ball of $K(C(K,E),CL)$. Then, for $l \in L$, $T^*(l)$ is a point measure with norm one: $T^*(l) = p \otimes \delta_k$, $p \in E'$ with norm one.

The case E = scalars has been settled in [8]. Proposition 1 is a consequence of the following result.

Proposition 1*: Let f be an extreme point of the unit ball of $C(L,M(K,Z))$. Then, for $l \in L$, $f(l)$ is a point measure with norm one.

Proof: First of all $\|f(l)\| = 1$ since $f(l) \pm (1-\|f(l)\|)m$ lie in the unit ball of $M(K,Z)$ whenever $\|m\| \le 1$. Next, consider the function $F: L \to M(K)$, $F(l) := |f(l)|$. Since $\| |m_1| - |m_2| \| \le \| m_1 - m_2 \|$ for vector measures m_i of bounded variation, F is a norm-continuous function which has probability measures as values. We claim:

$$F \in \operatorname{ex} \{ g \mid g \in C(L, MK), \text{ all } g(l) \text{ are probability measures} \}.$$

Once this claim is established, we finish the proof as follows. F defines an operator $S: CK \to CL$ by $(S\varphi)(l) := \int \varphi \, d(F(l))$, of course $S^* = F$. If F is extreme, S, too, is extreme and positive. By a result due to Phelps [9] S is nice, that means $F(l)$ is a point measure. It follows that $f(l)$ is a point measure.

To prove the claim let $u: L \to M(K)$ be a continuous function such that $F(l) \pm u(l) \ge 0$ and $(F(l) \pm u(l))(K) = 1$ for all $l \in L$. In this case $u(l)$ is absolutely continuous with respect to $F(l)$ so that there is a Borel function h_l with $u(l) = h_l \cdot F(l)$. We may (and shall) assume that h_l is real-valued and $-1 \le h_l(k) \le 1$ for all $k \in K$. Let $m(l) := h_l \cdot f(l) \in M(K,Z)$. Then (a) $l \mapsto m(l)$ is continuous, and (b) $\|f(l) \pm m(l)\| \le 1$ for all $l \in L$. We conclude that $m = 0$ and consequently $u = 0$.

It remains to prove (a) and (b).

(a) $\|m(l) - m(l')\| = \|h_l \cdot f(l) - h_{l'} \cdot f(l')\|$

$\le \|h_l(f(l) - f(l'))\| + \|(h_l - h_{l'})f(l')\|$

The first term $= \int |h_l| \, d|f(l) - f(l')|$ [6, p.173]

$\le \|f(l) - f(l')\|$.

The second term $= \int |h_l - h_{l'}| \, dF(l')$ [6, p.173]

$= \int_A \cdots + \int_B \cdots$

with $A := \{k \mid k \in K, h_l(k) \ge h_{l'}(k)\}$, $B := K \setminus A$.

Now $\int_A \cdots = \int_A h_l \, dF(l') - \int_A h_{l'} \, dF(l')$

$= \int_A h_l \, dF(l) - \int_A h_{l'} \, dF(l') + \int_A h_l \, d(F(l') - F(l$

$\le |u(l) - u(l')|(A) + |F(l) - F(l')|(A)$.

B may be treated analogously so that

$$\|m(l) - m(l')\| \le \|f(l) - f(l')\| + \|u(l) - u(l')\| + \|F(l) - F(l')\|$$

Therefore $m(.)$ is continuous.

(b) $\|f(l) \pm m(l)\| = \|(1 \pm h_l) \cdot f(l)\|$

$= \int |1 \pm h_l| \, dF(l)$ [6, p.173]

$= \int dF(l) \pm \int h_l \, dF(l)$

$= (F(l) \pm u(l))(K) = 1$.

In order to tackle the problem if an extreme operator from $C(K,E)$ into CL is nice we need to know the extreme functionals on $C(K,E)$. They were first described by Singer [14], but the proofs in the literature are quite complicated, cf. also [17]. Here we shall present a very simple argument.

<u>Theorem 2:</u> A vector measure $m \in M(K,Z)$ is an extreme point of the unit ball if and only if $m = z \otimes \delta_k$ for some $z \in \operatorname{ex} B_Z$, $k \in K$.

Proof: Assume m is extreme. If it were not a point measure, the total variation measure $|m|$ wouldn't be either. Therefore $0 < |m|(A) =: a < 1$ for some Borel set A. Then we can represent m as a non-trivial convex combination of norm-one measures: $m = a\,(a^{-1} \cdot \underline{1}_A m) + (1-a)\,((1-a)^{-1} \cdot \underline{1}_{A^c} m)$. We infer that m is of the form $z \otimes \delta_k$, since m is extreme we have $z \in \operatorname{ex} B_Z$.
On the other hand, if m is as stated, we shall show that it is extreme. Indeed, m is extreme in the subspace $Z_0 := \{x \otimes \delta_k \mid x \in Z\}$. But Z_0 is a very well complemented subspace, namely, it is the range of a projection P satisfying $\|n\| = \|Pn\| + \|n - Pn\|$ for all $n \in M(K,Z)$, a so-called L-projection. Here, $Pn := n(\{k\}) \otimes \delta_k$ defines the required projection. It follows easily from the defining norm-condition that $\operatorname{ex} B_{Z_0} \subseteq \operatorname{ex} B_{M(K,Z)}$, and m is extreme.

Specializing to $Z = E'$ we get Singer's theorem on extreme functionals. We may state it in the form: $Q: C(K,E) \longrightarrow E$, $Qf := f(k)$ ($k \in K$ fixed, but arbitrary) is a nice operator. Looking at the above argument we see that the proof used the facts that Q' is an isometric embedding and that the polar of $\operatorname{Ker} Q$ (that is Z_0) is the range of an L-projection. A space with this property is called an M-ideal [1],[2]. So we have actually shown:

<u>Proposition 3:</u> Suppose $Q: X \to Y$ is a quotient map such that $\operatorname{Ker} Q$ is an M-ideal. Then Q is a nice operator. Moreover, $p \in \operatorname{ex} B_{Y'}$ iff $Q'p \in \operatorname{ex} B_{X'}$.

Note that Proposition 3 includes [5, Theorem 1(b)].
It is known that an M-ideal J of a Banach space X satisfies the following intersection property IP(n) for each $n \in \mathbb{N}$:
Whenever $U_1, \dots, U_n$ are open balls in X with non-void intersection U such that $J \cap U_i \neq \emptyset$ for all i, then $J \cap U \neq \emptyset$.
Conversely, if a closed subspace J of X satisfies IP(3), then J

is an M-ideal [1, Th. 5.9],[2, Th. 2.17]. Alfsen and Effros [1, p.125] exhibit an example of a subspace of some 3-dimensional real Banach space which fails to be an M-ideal but satisfies IP(2). It is, however, an open problem, whether such an example exists in a complex Banach space [20]. Here it is interesting to note that Proposition 3 holds if Ker Q satisfies the slightly weaker condition of having IP(2). This is shown in [18] by means of a direct application of the intersection property.

Proposition 3 can be used to prove the following extension theorem for extreme operators.

Proposition 4: Suppose $Q: X \to Y$ is a quotient map and that Ker Q is an M-ideal of X. If $S: Y \to CL$ is an extreme operator, then $T := S \circ Q: X \to CL$ is an extreme operator, and $T^*(1)$ is extreme iff $S^*(1)$ is extreme. (Analogously for compact operators.)

Proof: We have only to show that T is extreme, the other statements are proved in Proposition 3. Let $\|T \pm U\| \leq 1$. Then we have $\|Q'(S^*(l)) \pm U^*(l)\| \leq 1$ for all $l \in L$. By assumption on Q Q' is an isometric isomorphism from Y' onto $W := (\mathrm{Ker}\ Q)^0 \subseteq X'$, and there is a decomposition $X' = W \oplus_1 W^\perp$ with some closed subspace $W^\perp \subseteq X'$. Using this we may write $U^*(l) = w(l) + w^\perp(l) \in W \oplus W^\perp$, and therefore $1 \geq \|Q'(S^*(l)) \pm w(l)\| + \|w^\perp(l)\|$. Since S is extreme, a theorem due to Sharir [11] yields $1 = \|S^*(l)\| = \|Q'(S^*(l))\|$ on a dense subset H of L. Hence $w^\perp(l) = 0$ on H, equivalently $U^*(H) \subseteq W$. But U^* is weak*-continuous, and W is weak*-closed so that $U^*(L) \subseteq W$. Thus, for $l \in L$ there exists $v(l) \in Y'$ such that $U^*(l) = Q'(v(l))$. Now v is seen to be weak*-continuous, hence $v = 0$ and $U = 0$ because $\|Q'(S^*(l)) \pm Q'(v(l))\| = \|S^*(l) \pm v(l)\| \leq 1$. The (easier) proof for compact operators is established in the same way.

Corollary 5: Let $S: E \to CL$ be an extreme operator (resp. extreme compact operator). Then for every compact Hausdorff space K there exists an extreme operator (resp. extreme compact operator) $T: C(K,E) \to CL$ with $T^*(1)$ extreme iff $S^*(1)$ extreme.

Proof: Choose any $k \in K$ and consider $Q: C(K,E) \to E$, $Qf := f(k)$. Proposition 4 gives the result, the M-ideal property has been observed in the proof of Theorem 2. Cf. also [2, prop. 10.1].

In [3] it is shown that for every compact Hausdorff space L there exists a Banach space E and an extreme operator $S: E \to CL$ such that $S^*(l)$ is extreme if and only if l is an isolated point of L. If L is the unit interval, E may be chosen to be 4-dimensional. The following theorem shows that the space $C(K,E)$ perfectly reflects the properties of CK and E as far as the characterization of compact extreme operators into CL is concerned; for merely bounded operators we shall need additional assumptions.

Theorem 6: The following statements are equivalent:

(a) Every extreme point of the unit ball of $K(E,CL)$ is nice.

(b) Every extreme point of the unit ball of $K(C(K,E),CL)$ is nice, where K is an arbitrary compact Hausdorff space.

Proof: That (b) implies (a) is the contents of Corollary 5. Assume (a) and let $T: C(K,E) \to CL$ be an extreme point of the unit ball of compact operators. By Proposition 1 we have $T^*(l) = p(l) \otimes \delta_{\psi(l)}$. Since T^* is norm-continuous, ψ is locally constant: If $l \in L$, then $\|T^*(l) - T^*(l')\| < 1$ for l' in some neighbourhood U of l. We conclude $\psi(l') = \psi(l)$ for all $l' \in U$. L is compact, so there are finitely many pairwise disjoint clopen sets $L_1,\dots,L_n$ covering L and $k_1,\dots,k_n \in K$ with $\psi(l) = k_i$ for $l \in L_i$. Now $p(.)$ is seen to be norm-continuous, hence it represents a compact operator from E to CL. By assumption (a) we have only to show that it is extreme. Let $u: L \to E'$ be a norm-continuous function such that $\|p(l) \pm u(l)\| \le 1$ for all $l \in L$. Define $w(l) := \sum 1_{L_i}(l) \cdot u(l) \otimes \delta_{k_i}$. Then w is norm-continuous, $\|T^*(l) \pm w(l)\| \le 1$ for all l, and hence $w = 0$. We infer $u = 0$.

A consequence of the above proof is:

Corollary 7: If T is an extreme point of the unit ball of $K(CK,CL)$, then T is a finite rank operator.

Proof: We have $T^*(l) = p(l) \cdot \delta_{\psi(l)}$, $p = T1 \in CL$. As pointed out above, there are pairwise disjoint clopen sets $L_1,\dots,L_n$ covering L and $k_1,\dots,k_n \in K$ with $\psi(l) = k_i$ for $l \in L_i$. It follows

$$(Tf)(l) = T^*(l)(f) = p(l)\, f(k_i) \quad \text{for } l \in L_i,\ f \in CK.$$

Hence $$Tf = \sum f(k_i)\, 1_{L_i} \cdot T1$$

and $$\operatorname{range}(T) \subseteq \operatorname{span}\{1_{L_i} \cdot T1 \mid i = 1,\dots,n\} .$$

Theorem 8: Suppose that every extreme operator from E to CL is nice. Let $T: C(K,E) \longrightarrow CL$ be an extreme operator such that $T^*(l)$ is a non-zero point measure for all $l \in L$. Then T is nice.

Proof: T^* is of the form $T^*(l) = p(l) \otimes \delta_{\psi(l)}$, $p(l) \in E' \setminus \{0\}$. The function $l \mapsto p(l)$ is seen to be weak*-continuous, hence represents a bounded linear operator $S: E \longrightarrow CL$. In order to show that T is nice it is sufficient to show that S is nice by Theorem 2. For this it is enough to prove that S is extreme according to our present assumptions. In fact, let u be a weak*-continuous function from L into E' such that for all $l \in L$ $\|p(l) \pm u(l)\| \leq 1$. Then $\|T^*(l) \pm u(l) \otimes \delta_{\psi(l)}\| \leq 1$ for all l. But $l \mapsto u(l) \otimes \delta_{\psi(l)}$ is weak*-continuous since $\psi: L \longrightarrow K$ and the map $(B_{E'}, \text{weak*}) \times K \longrightarrow (B_{C(K,E)'}, \text{weak*})$, $(q,k) \mapsto q \otimes \delta_k$ are continuous. Therefore $u = 0$, and S is extreme.
(To prove that ψ is continuous fix $l_0 \in L$. We shall show that $\delta_{\psi(.)}$ is continuous at l_0. Choose $x \in E$ so that $p(l_0)(x) = 1$. Then for $g \in CL$ $T^*(l)(g \otimes x) = p(l)(x)\, g(\psi(l))$, and the weak*-continuity of T^* and p together with $p(l_0)(x) \neq 0$ imply the weak*-continuity of $\delta_{\psi(.)}$. Since l_0 is arbitrary we are done.)

It can be shown that the conditions in Theorem 8 concerning T are fulfilled whenever K is dispersed and E is finite dimensional, cf. [18].
To finish with, let us rephrase Theorem 6 in terms of vector-valued functions.

Theorem 6*: The following statements are equivalent:

(a) Every extreme function in $C(L,Z)$ assumes only extreme values.
(b) Every extreme function in $C(L,M(K,Z))$ assumes only extreme values, where K is an arbitrary compact Hausdorff space.

In [19] it is proved that an abstract L-space Z satisfies the above conditions.

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MATHEMATISCHES INSTITUT, FREIE UNIVERSITÄT BERLIN, ARNIMALLEE 2-6. D-1000 BERLIN 33