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AN APPROACH TO GENERALIZING BANACH SPACES: NORMED ALMOST LINEAR SPACES

G. Godini

INTRODUCTION

This paper is a sequel to [2] in which we have introduced the normed almost linear spaces, a generalization of normed linear spaces. *All spaces involved in this paper are over the real field* R. Roughly speaking, a *normed almost linear space* (nals) is a set X together with two mappings s:X x X → X and m:R x X → X which satisfy some of the axioms of a linear space - called an *almost linear space* (als) - and on the set X there exists a functional ||.||:X → R - called a *norm* - which satisfies all the axioms of an usual norm on a linear space (ls), as well as some additional ones, which in the case of a normed linear space (nls) are consequences of the axioms of the norm. Due to the fact that we have weakened the axioms of a ls, but we have strengthened the axioms of the norm, some results involving only algebric structure, which are not true in an als, hold in a nals (see Section 1). Since the norm of a nals X does not generate a metric on X, in [2] we considered the strong normed almost linear spaces, which also generalize the normed linear spaces. Roughly speaking, a *strong normed almost linear space* (snals) is a nals X together with a semi-metric on X which is related in a certain way to the norm of X.

To support the idea that the nals is a good concept, we introduced in [2] the concept of a dual space of a nals X, where the functionals on X are no longer linear but "almost linear", which is also a nals. When X is a nls, then the dual space defined by us is the usual dual space X^*.

The nals and snals were not introduced for the sake of generalization. We have proved in [2] that they constitute the natural framework for the theory of best simultaneous approximation, by showing that this theory is a particular case of the theory of best approximation in a nals (snals).

The present paper has a more general interest, since here we want to extend for a nals (snals) some general results from the theory of normed linear spaces ([1]). Now, in the theory of normed linear spaces an important tool is the Hahn-Banach theorem. A similar theorem is no longer true in a nals. Consequently,

This paper is in final form and no version of it will be submitted for publication elsewhere.

we do not know whether the dual space of a nals X may be reduced to the only functional f=0. Though the algebric dual of an als X may be {0}, in all our examples when X is a nals, the dual space of X is not {0}. The main objective of this paper is to give sufficient conditions on the nals X in order that its dual space have non-zero almost linear functionals.

We draw attention that in the definition of the norm of a nals (the same for the semi-metric of a snals), in [2] we have considered all the axioms given in this paper, as well as an additional one. Since this latter axiom is surely of no use for solving our main problem (whether the dual space of a nals is, or is not {0}), here we omit it. On the other hand, the dual space defined by us, as well as all the examples of (strong) normed almost linear spaces in Section 4 satisfy all the axioms required in [2].

This paper is organized as follows. Section 1 contains basic results, the most of them being used throughout this paper. Section 2 deals with bases in almost linear spaces. Not all of them have a basis, and when they do then there exist a norm and a metric such that they are snals. Section 3 is devoted to the question whether the dual space of a nals contains non-zero almost linear functionals. If X has a basis then this is surely true, and we also give some sufficient conditions for an affirmative answer to the above question. We also examine the extension property of almost linear functionals defined on an almost linear subspace of the nals X. Finally, Section 4 contains examples related to the subject matter of this paper.

We did not change the terminology (and notation) from the theory of normed linear spaces ([1]), except for the linear functional which we extended it in two ways to an als.

The most part of the results of this paper makes sense only when the nals (als) X is not a ls. From our results which make sense in a nls (ls) E, we recover either trivial or known results in E. That is why *throughout this paper, if otherwise not stated, the als* X *is not a ls*.

1. BASIC PROPERTIES OF A NORMED ALMOST LINEAR SPACE

In 1.1 - 1.5 below, we recall some of the definitions and remarks of [2].

1.1. DEFINITION. An *almost linear space* (als) is a set X together with two mappings $s:X \times X \to X$ and $m:R \times X \to X$ satisfying the conditions L_1 - L_8 given below. For $x,y \in X$ and $\lambda \in R$ we denote $s(x,y)$ by $x+y$ and $m(\lambda,x)$ by λx, when these will not lead to misunderstandings. Let $x,y,z \in X$ and $\lambda,\mu \in R$. L_1). $(x+y)+z=x+(y+z)$; L_2). $x+y=y+x$; L_3). There exists an element $0 \in X$ such that $x+0=x$ for each $x \in X$; L_4). $1x=x$; L_5). $\lambda(x+y)=\lambda x+\lambda y$; L_6). $0x=0$; L_7). $\lambda(\mu x)=(\lambda\mu)x$; L_8). $(\lambda+\mu)x=\lambda x+\mu x$ for $\lambda\geq 0$, $\mu\geq 0$.

We denote $-1x$ by $-x$, when this will not lead to misunderstanding, and in the sequel x-y means x+(-y).

1.2. DEFINITION. A nonempty set Y of an als X is called an *almost linear subspace* of X, if for each $y_1, y_2 \in Y$ and $\lambda \in R$, $s(y_1,y_2) \in Y$ and $m(\lambda,y_1) \in Y$. An almost linear subspace Y of X is called a *linear subspace* of X if $s: Y \times Y \to Y$ and $m: R \times Y \to Y$ satisfiy all the axioms of a ls.

For an als X we introduce the following two sets.

$$V_X=\{x \in X: x-x=0\} \tag{1.1}$$

$$W_X=\{x \in X: x=-x\} \tag{1.2}$$

By L_1 - L_8 it follows that V_X is a linear subspace of X, and it is the largest one. The set W_X is an almost linear subspace of X and we have $W_X=\{x-x: x \in X\}$. Notice that $V_X \cap W_X=\{0\}$. Clearly, the als X is a ls, iff $V_X=X$, iff $W_X=\{0\}$.

1.3. DEFINITION. A *norm* on the als X is a functional $||.||: X \to R$ satisfying the conditions N_1 - N_3 below. Let $x,y,z \in X$ and $\lambda \in R$. N_1). $||x-z|| \le ||x-y|| + ||y-z||$; N_2). $||\lambda x||=|\lambda|\ ||x||$; N_3). $||x||=0$ iff $x=0$.

Using N_1 we get

$$||x+y|| \le ||x||+||y|| \qquad (x,y \in X) \tag{1.3}$$

$$||x-y|| \ge |\ ||x||-||y||\ | \qquad (x,y \in X) \tag{1.4}$$

By the above axioms it follows that $||x|| \ge 0$ for each $x \in X$.

1.4. DEFINITION. An als X together with $||.||: X \to R$ satisfying N_1 - N_3 i called a *normed almost linear space* (nals).

Clearly, any nls is a nals. Since the norm of a nals does not generate a metric on X (for $x \in X \setminus V_X$ we have $||x-x|| \ne 0$), we shall sometimes work in a particular class of normed almost linear spaces defined below.

1.5. DEFINITION. A *strong normed almost linear space* (snals) is a nals X together with a semi-metric ρ on X which satisfies M_1 and M_2 below.

M_1 $\quad |\ ||x||-||y||\ | \le \rho(x,y) \le ||x-y|| \qquad (x,y \in X)$

M_2 $\quad \rho(x+z,y+z) \le \rho(x,y) \qquad (x,y,z \in X)$

As we have observed in [2], if X is a nls then the only semi-metric on X satisfying M_1 and M_2 is that generated by the norm (which is a metric on X).

Now we shall give some basic facts which hold in a nals (snals).

1.6. LEMMA. *Let* X *be a nals and let* $x,y,z \in X$. *If*

$$x+y=x+z \tag{1.5}$$

then $||y||=||z||$ *In particular if* $x=x+y$ *then* $y=0$. *If* X *is a snals, then* (1.5) *implies that* $\rho(y,z)=0$.

Proof. By (1.5) we get $x+y+z=x+2z=x+2y$, and so, $x+2^n y=x+2^n z$ for each $n\in N$. Hence

$$y+2^{-n}x=z+2^{-n}x \qquad (n\in N) \tag{1.6}$$

Using (1.4), (1.6) and (1.3), we obtain that $||y||-2^{-n}||x||\le||y+2^{-n}x||=||z+2^{-n}x||\le\le||z||+2^{-n}||x||$, for each $n\in N$. Therefore $||y||\le||z||$, and similarly $||z||\le||y||$ whence $||y||=||z||$. If X is a snals, then by (1.6), M_2 and M_1 we obtain $\rho(y,z)\le\le\rho(y,2^{-n}x+y)+\rho(2^{-n}x+y,z)=\rho(y,2^{-n}x+y)+\rho(2^{-n}x+z,z)\le\rho(0,2^{-n}x)+\rho(2^{-n}x,0)\le2^{-n}||x||++2^{-n}||x||=2^{-n+1}||x||$ for each $n\in N$, whence $\rho(y,z)=0$.

Remarks. a). In an als X the relation $x=x+y$ does not always imply $y=0$ (see 4.1 b), 4.3 b). b). In a snals X where ρ is not a metric on X the relation (1.5) does not always imply $y=z$ (see 4.6 b)).

1.7. LEMMA. *Let* X *be a nals and let* $x\in X$, $w\in W_X$. *Then* $\max\{||x||,||w||\}\le\le||x+w||$.

Proof. We have $2||w||=||w-w||\le||w-x||+||x-w||=2||x+w||$, and $2||x||==||x-(-x)||\le||x-w||+||w+x||=2||x+w||$, whence the conclusion follows.

1.8. LEMMA. *Let* X *be a nals and let* $x,y\in X$. *If* $x+y\in V_X$, *then both* $x,y\in\in V_X$.

Proof. If $x+y\in V_X$ then $(x-x)+(y-y)=0$. Since $x-x\in W_X$, by Lemma 1.7 it follows that $||x-x||=||y-y||=0$, and so $x-x=y-y=0$, i.e., $x,y\in V_X$.

Remark. In an als X the relation $x+y\in V_X$, does not always imply $x,y\in V_X$ (see 4.2 b)).

1.9. LEMMA. *Let* X *be a nals, and let* $x,y\in X$, $x\notin V_X$, $\alpha\in R$, $|\alpha|\ge1$ *such that* $x=\alpha x+y$. *If* $\alpha\ge1$, *then* $\alpha=1$ *and* $y=0$; *if* $\alpha\le-1$, *then* $\alpha=-1$ *and* $y\in V_X$.

Proof. Suppose $\alpha\ge1$. Then $x=x+(1-\alpha)x+y$, whence by Lemma 1.6, we obtain $(1-\alpha)x+y=0$. By Lemma 1.8 it follows that $(1-\alpha)x\in V_X$, and since $x\notin V_X$, we must have $\alpha=1$, and so $y=0$.

Suppose $\alpha\le-1$. Then $x=\alpha(\alpha x+y)+y$, and so $x=\alpha^2x+(\alpha y+y)$. Since $\alpha^2\ge1$, by the above case we obtain $\alpha^2=1$ and $\alpha y+y=0$. Therefore $\alpha=-1$ and $y-y=0$, i.e., $y\in V_X$.

Remarks. a) Lemma 1.9 is no longer true in an als (see 4.1 b), 4.2 b)). b) In a nals X the relations $x=\alpha x+y$, $x,y\in X$, $x\notin V_X$ and $0<|\alpha|<1$ are not contradictory (see 4.4 b)).

1.10. LEMMA. *Let* X *be a nals. If* $w_1+v_1=w_2+v_2$, $w_i\in W_X$, $v_i\in V_X$, $i=1,2$, *then* $w_1=w_2$ *and* $v_1=v_2$.

Proof. Suppose $w_1+v_1=w_2+v_2$. Then $w_1=w_2+v$, where $v=v_2-v_1$. Hence $w_1=w_2-v$, and so $w_2=w_2-2v$. By Lemma 1.6 it follows that $v=0$ and so $w_1=w_2$ and $v_1=v_2$.

Remark. Lemma 1.10 is no longer true in an als (see 4.3 b)).

1.11. LEMMA. *Let* X *be a snals where* ρ *is a metric, and let* $x\in X$. *If* $x+w+v\in W_X+V_X$ *for some* $w\in W_X$ *and* $v\in V_X$ *then* $x\in W_X+V_X$.

Proof. Let $w_1 \in W_X$ and $v_1 \in V_X$ such that

$$x+w+v=w_1+v_1 \tag{1.7}$$

Let $w_2=x-x \in W_X$. Using (1.7) we obtain

$$w_1+v_1-x=w_2+w+v \tag{1.8}$$

Multiplying (1.8) by -1 and adding the obtained relation to (1.7), we get $(w+w_1)+$ $+(2x+v-v_1)=(w+w_1)+(w_2+v_1-v)$. Since ρ is a metric on X, by Lemma 1.6 we obtain that $2x=w_2+2(v_1-v)$, and so $x \in W_X+V_X$.

1.12. LEMMA. *Let* X *be a snals where* ρ *is a metric,* Y *an almost linear subspace of* X *and* $x_o \in X$. *Suppose that*

$$\{\lambda x_o+y:\ \lambda>0,\ y \in Y\} \cap Y=\emptyset \tag{1.9}$$

Then the relations $\lambda_1 x_o+y_1=\lambda_2 x_o+y_2$, $\lambda_i \geq 0$, $y_i \in Y$, $i=1,2$ *imply that* $\lambda_1=\lambda_2$ *and* $y_1=y_2$.

Proof. Suppose $\lambda_1 x_o+y_1=\lambda_2 x_o+y_2$, $\lambda_i \geq 0$, $y_i \in Y$, $i=1,2$. If $\lambda_1=0$ then by (1.9) it follows that $\lambda_2=0$ and so $y_1=y_2$. Without loss of generality we can suppose now $\lambda_1 \geq \lambda_2>0$. Then $\lambda_2 x_o+(\lambda_1-\lambda_2)x_o+y_1=\lambda_2 x_o+y_2$, whence by Lemma 1.6 we get $(\lambda_1-\lambda_2)x_o+y_1=$ $=y_2$. By (1.9) it follows that $\lambda_1=\lambda_2$, whence $y_1=y_2$.

1.13. LEMMA. *Let* X *be a nals and let* $x,x_n \in X$, $n \in N$ *be such that* $\lim||x_n+x||=0$. *Then* $x \in V_X$.

Proof. We have $||x-x|| \leq ||x-(-x_n)||+||-x_n-x||=2||x_n+x||$ for each $n \in N$. Therefore $||x-x||=0$ and so $x \in V_X$.

Immediate consequences of the above lemma are the following two results.

1.14. LEMMA. *Let* X *be a nals* $x \in X \setminus V_X$, $x_n \in X$, $\alpha_n \in R$, $n \in N$. *If* $\lim||x_n+\alpha_n x||=0$ *then* $\lim \alpha_n=0$.

1.15. LEMMA. *Let* X *be a nals and let* $x,x_n \in X$, $\lambda_n \in R$, $n \in N$, $\lim \lambda_n=\infty$. *If the sequence* $\{||\lambda_n x+x_n||\}_{n=1}^{\infty}$ *is bounded, then* $x \in V_X$.

2. BASES IN ALMOST LINEAR SPACES

2.1. DEFINITION. A subset B of the als X is called a *basis* of X if for each $x \in X \setminus \{0\}$ there exist *unique* sets $\{b_1,\ldots,b_n\} \subset B$, $\{\lambda_1,\ldots,\lambda_n\} \subset R \setminus \{0\}$ (n depending on x) such that $x=\sum_{i=1}^{n}\lambda_i b_i$, where $\lambda_i>0$ for $b_i \notin V_X$.

Clearly, if B is a basis of X then $0 \notin B$.

In contrast to the case of a ls, there exists almost linear spaces (even snals) which have no basis. In Section 4 one can find examples of spaces which have or which have not bases.

2.2. LEMMA. *If the als* X *has a basis* B, *then the sets* $\{-b:\ b \in B\}$ *and*

$\{\alpha_b b\colon b\in B,\ \alpha_b\neq 0,\ \alpha_b>0$ *for* $b\notin V_X\}$ *are also bases of* X.

Proof. The proof is straightforward.

2.3. LEMMA. *Let* X *be an als with a basis and let* $x_1,x_2\in X$. *If* $x_1+x_2\in V_X$ *then* $x_i\in V_X$, $i=1,2$.

Proof. Suppose $x_1+x_2\in V_X$ and let $x_3=-x_1$ and $x_4=-x_2$. Since X has a basis B, there exist $b_1,\dots,b_n\in B$, $b_i\neq b_j$ for $i\neq j$, such that $x_i=\sum_{j=1}^n \alpha_{ij}b_j$, where $\alpha_{ij}\geq 0$ if $b_j\notin V_X$, $1\leq i\leq 4$. By hypothesis we get that $\sum_{i=1}^4 x_i=0$ and so $\sum_{j=1}^n(\sum_{i=1}^4\alpha_{ij})b_j=0$. Suppose $b_1\notin V_X$. Then $b_1=(1+\sum_{i=1}^4\alpha_{i1})b_1+\sum_{j=2}^n(\sum_{i=1}^4\alpha_{ij})b_j$. Since $b_1\in B$, it follows that $1+\sum_{i=1}^4\alpha_{i1}=1$. But $\alpha_{i1}\geq 0$, $1\leq i\leq 4$, and so $\alpha_{i1}=0$, $1\leq i\leq 4$. Consequently for each $b_j\notin V_X$, $1\leq j\leq n$, we get $\alpha_{ij}=0$, $1\leq i\leq 4$, which shows that $x_i\in V_X$, $1\leq i\leq 4$.

2.4. LEMMA. *Let* X *be an als with a basis* B. *Then* $B\cap V_X$ *is a basis of* V_X.

Proof. Use Lemma 3.

2.5. LEMMA. *Let* X *be an als. The set* $B\subset X$ *is a basis of* X *iff* $B\cap V_X$ *is a basis of* V_X, *and for each* $x\in X\setminus V_X$ *there exist unique* $b_1,\dots,b_n\in B\setminus V_X$, $v\in V_X$ *and* $\lambda_1,\dots,\lambda_n>0$ *such that* $x=\sum_{i=1}^n\lambda_i b_i+v$.

Proof. Use Lemmas 2.4, 2.3 and Definition 2.1.

2.6. LEMMA. *Let* B *be a basis of the als* X. *Then for each* $b\in B\setminus V_X$ *there exist unique* $\psi(b)\in B\setminus V_X$, $v(b)\in V_X$ *and* $\lambda(b)>0$ *such that* $-b=\lambda(b)\psi(b)+v(b)$.

Proof. Let $b\in B\setminus V_X$. Then $-b\notin V_X$ and by Lemma 2.5 we get

$$-b=\sum_{i=1}^k\lambda_i b_i+v \tag{2.1}$$

where $b_1,\dots,b_k\in B\setminus V_X$, $k\geq 1$, $b_i\neq b_j$ for $i\neq j$, $v\in V_X$ and $\lambda_i>0$, $1\leq i\leq k$, are uniquely determined. Clearly the lemma is proved if we show that $k=1$. Let $e_1,\dots,e_m\in B\setminus V_X$, $e_i\neq e_j$ for $i\neq j$, $v_i\in V_X$, $\mu_{ij}\geq 0$, $1\leq i\leq k$, $1\leq j\leq m$, such that

$$-b_i=\sum_{j=1}^m\mu_{ij}e_j+v_i \qquad (1\leq i\leq k) \tag{2.2}$$

Multiplying (2.1) by -1 and using (2.2) we get

$$b=\sum_{j=1}^m(\sum_{i=1}^k\lambda_i\mu_{ij})e_j+\sum_{i=1}^k\lambda_i v_i+v \tag{2.3}$$

Since $b\in B\setminus V_X$, there exists an index $j_o\in\{1,\dots,m\}$ - say $j_o=1$ - such that $b=e_1$ and we must have $\sum_{i=1}^k\lambda_i\mu_{ij}=0$, $2\leq j\leq m$. Since $\lambda_i>0$ and $\mu_{ij}\geq 0$ it follows that $\mu_{ij}=0$ for each $1\leq i\leq k$ and each $2\leq j\leq m$. Consequently, we get by (2.2)

$$-b_i=\mu_{i1}e_1+v_i \qquad (1\leq i\leq k) \tag{2.4}$$

and $\mu_{i1}>0$ since $-b_i\notin V_X$, $1\leq i\leq k$. Suppose $k>1$. By (2.4) for $i=1,2$ we get that $e_1=(-b_1-v_1)/\mu_{11}=(-b_2-v_2)/\mu_{21}$ and so $b_1=(\mu_{11}/\mu_{21})b_2+((v_2/\mu_{21})-(v_1/\mu_{11}))$, contradicting Lemma 2.5.

Let $\psi:B\setminus V_X \to B\setminus V_X$ be defined as in Lemma 2.6. Then ψ is well-defined and we have:

2.7. LEMMA. *The mapping* $\psi:B\setminus V_X \to B\setminus V_X$ *defined as above is injective and* $\psi(\psi(b))=b$ *for each* $b\in B\setminus V_X$. *In particular* ψ *is surjective.*

Proof. Let $b_1,b_2\in B\setminus V_X$ such that $\psi(b_1)=\psi(b_2)=b\in B\setminus V_X$. Then $-b_i=\lambda_i b+v_i$, $\lambda_i>0$, $v_i\in V_X$, $i=1,2$, and similarly with the proof given at the end of Lemma 2.6, this contradicts Lemma 2.5.

Let now $b\in B\setminus V_X$. Then $-b=\lambda\psi(b)+v$, where $\lambda>0$, $v\in V_X$ and $\psi(b)\in B\setminus V_X$ are given by Lemma 2.6. Then $-\psi(b)=(b/\lambda)+(v/\lambda)$, and so, again by Lemma 2.6 we get $\psi(\psi(b))=b$.

The main result of this section is the following.

2.8. THEOREM. *Let* B *be a basis of the als* X. *Then there exists a basis* B' *of* X *with the property that for each* $b'\in B'\setminus V_X$ *we have* $-b'\in B'\setminus V_X$. *Moreover* $\mathrm{card}\,(B\setminus V_X)=\mathrm{card}\,(B'\setminus V_X)$.

Proof. Let $B'=\{b-\psi(b):b\in B\setminus V_X\}\cup(B\cap V_X)$. Then for $b\in B\setminus V_X$ we get by Lemma 2.3 that $b'=b-\psi(b)\in B'\setminus V_X$. Hence by Lemma 2.7 we obtain that $-b'=\psi(b)-\psi(\psi(b))\in B'\setminus V_X$. To show that B' is a basis, we use Lemma 2.5. Clearly, $B'\cap V_X=B\cap V_X$ is a basis of V_X (by Lemma 2.4). Let now $x\in X\setminus V_X$. Then there exist unique $b_1,\dots,b_n\in B\setminus V_X$, $n\geq 1$, $b_i\neq b_j$ for $i\neq j$, $v\in V_X$ and $\lambda_1,\dots,\lambda_n>0$ such that $x=\sum_{i=1}^n\lambda_i b_i+v$. By Lemmas 2.6 and 2.7, for each $b\in B\setminus V_X$ we have $-\psi(b)=\mu(b)b+v(b)$, where $\psi(b)>0$ and $v(b)\in V_X$ are uniquely determined. Then $b-\psi(b)=(\mu(b)+1)b+v(b)$, whence

$$b=\frac{b-\psi(b)}{\mu(b)+1}-\frac{v(b)}{\mu(b)+1}\qquad (b\in B\setminus V_X)\tag{2.5}$$

Let $b_i'=b_i-\psi(b_i)\in B'\setminus V_X$, $1\leq i\leq n$, and let us put $\mu(b_i)=\mu_i$ and $v(b_i)=v_i$. We have by (2.5) that

$$x=\sum_{i=1}^n\frac{\lambda_i}{\mu_i+1}b_i'+\bar{v}$$

where $\bar{v}\in V_X$. We show now that this representation is unique. Suppose $x=\sum_{i=1}^n\lambda_i b_i'+\bar{v}_1=\sum_{i=1}^n\nu_i b_i'+\bar{v}_2$, where $b_i'\in B'\setminus V_X$, $b_i'\neq b_j'$ for $i\neq j$, λ_i, $\nu_i\geq 0$, $1\leq i\leq n$, $\bar{v}_1,\bar{v}_2\in V_X$. Then there exist $b_i\in B\setminus V_X$, $1\leq i\leq n$, such that $b_i'=b_i-\psi(b_i)$. Here $b_i\neq b_j$ for $i\neq j$ since $b_i'\neq b_j'$, $i\neq j$. Using (2.5) where $\mu(b_i)=\mu_i$ and $v(b_i)=v_i$, we get $x=\sum_{i=1}^n\lambda_i((\mu_i+1)b_i+v_i)+\bar{v}_1=\sum_{i=1}^n\nu_i((\mu_i+1)b_i+v_i)+\bar{v}_2$. By Lemma 2.5 it follows that $\lambda_i(\mu_i+1)=\nu_i(\mu_i+1)$, $1\leq i\leq n$ and $\sum_{i=1}^n\lambda_i v_i+\bar{v}_1=\sum_{i=1}^n\nu_i v_i+\bar{v}_2$. Since $\mu_i>0$, it follows from the former equality that $\lambda_i=\nu_i$, and so $\bar{v}_1=\bar{v}_2$. Hence the mapping $\chi:B\setminus V_X\to B'\setminus V_X$ defined by $\chi(b)=b-\psi(b)$, $b\in B\setminus V_X$ is a one-to-one mapping, and so $\mathrm{card}\,(B\setminus V_X)=\mathrm{card}\,(B'\setminus V_X)$, which completes the proof.

2.9. COROLLARY. *If the als* X *has a basis then* W_X *has a basis.*

Proof. Let B be a basis of X. By the above theorem we can suppose that for each $b\in B\setminus V_X$ we have $-b\in B\setminus V_X$. Let $B_1=\{b-b:b\in B\setminus V_X\}\subset W_X$. We show that B_1

is a basis of W_X . Let $w \in W_X \setminus \{0\}$. By Lemma 2.5, $w=\sum_{i=1}^{n}\lambda_i b_i+v$, where $b_i \in B\setminus V_X$, $b_i \neq b_j$ for $i\neq j$, $\lambda_i>0$, $1\leq i\leq n$, $v\in V_X$. Then $-w=\sum_{i=1}^{n}\lambda_i(-b_i)-v$ and so $w=(1/2)(w-w)= =\sum_{i=1}^{n}(\lambda_i/2)(b_i-b_i)$. To show the uniqueness of this representation, suppose $w=\sum_{i=1}^{k}\lambda_i(b_i-b_i)=\sum_{i=1}^{k}\mu_i(b_i-b_i)$, $b_i\in B\setminus V_X$, $b_i-b_i\neq b_j-b_j$ for $i\neq j$, and $\lambda_i,\mu_i>0$, $1\leq i\leq k$. Then $b_i\neq\pm b_j$ for $i\neq j$, and since for each $b\in B\setminus V_X$, $-b\in B\setminus V_X$ we must have $\lambda_i=\mu_i$, $1\leq i\leq k$.

Remarks. a) The converse to Corollary 2.9 is not true (see 4.6 c), 4.8 c)). b) An almost linear subspace Y of an als X with a basis, has not in general a basis (see 4.8 c)).

Another consequence of Theorem 2.8 is

2.10. COROLLARY. *If* X *is an als with a basis, then there exist a norm* $||.||$ *and a metric* ρ *on* X *for which* X *is a snals.*

Proof. Choose a basis B with the property from Theorem 2.8. For an element $x\in X\setminus\{0\}$, use the unique representation given by Definition 2.1, $x=\sum_{i=1}^{n}\lambda_i b_i$ and define $||x||=\sum_{i=1}^{n}|\lambda_i|$. Observing that if $x=\sum_{i=1}^{n}\lambda_i b_i=\sum_{i=1}^{k}\lambda_i b_i+\sum_{i=k+1}^{n}\lambda_i b_i$, $b_i\in B\setminus V_X$ for $1\leq i\leq k$, $b_i\in B\cap V_X$ for $k+1\leq i\leq n$ and $\lambda_i>0$ for $1\leq i\leq k$, then the unique representation for $-x$ is $-x=\sum_{i=1}^{k}\lambda_i(-b_i)+\sum_{i=k+1}^{n}(-\lambda_i)b_i$, it is easy to show that $||.||$ satisfies N_1-N_3. Let now $x,y\in X$. Then $x=\sum_{i=1}^{n}\lambda_i b_i$, $y=\sum_{i=1}^{n}\mu_i b_i$, $\lambda_i,\mu_i\geq 0$ for $b_i\in B\setminus V_X$, $b_i\neq b_j$ for $i\neq j$, and define $\rho(x,y)=\sum_{i=1}^{n}|\lambda_i-\mu_i|$. Then ρ is a metric on X satisfying M_1 and M_2. Therefore X is a snals.

Though the norm and the metric defined as above are not easy to be handled, we can use their existence to conclude that all the results of Section 1 involving algebraic structure are also true in an als with a basis. We shall make references only to two of them, which we collect in a lemma.

2.11. LEMMA. *Let* X *be an als with a basis.*

i) *The relations* $x+y=x+z$, $x,y,z\in X$ *imply that* $y=z$.

ii) *The relations* $w_1+v_1=w_2+v_2$, $w_i\in W_X$, $v_i\in V_X$, $i=1,2$ *imply that* $w_1=w_2$ *and* $v_1=v_2$.

2.12. COROLLARY. *Let* X *be an als. If* W_X *has a basis then* W_X+V_X *has a basis.*

Proof. Let B_1 be a basis of W_X and B_2 a basis of the linear space V_X. By Lemma 2.11 ii), $B_1\cup B_2$ is a basis of W_X+V_X.

3. ALMOST LINEAR FUNCTIONALS AND THE DUAL SPACE

Up to 3.7 (except for 3.4) we recall definitions and results from [2].

3.1. DEFINITION. Let X be an als. A functional $f:X\to R$ is called an *almost linear functional* if the conditions (3.1)-(3.3) are satisfied.

$$f(x+y)=f(x)+f(y) \qquad (x,y\in X) \tag{3.1}$$

$$f(\lambda x)=\lambda f(x) \qquad (\lambda\geq 0,\ x\in X) \tag{3.2}$$

(3.3) $f(w)\geq 0$ $(w\in W_X)$

The functional $f:X\to R$ is called a *linear functional* on X if it satisfies (3.1), and (3.2) for each $\lambda\in R$. Then (3.3) is also satisfied.

Let $X^{\#}$ be the set of all almost linear functionals defined on the als X. For $f_1,f_2\in X^{\#}$, let $s(f_1,f_2)$ be the functional on X defined by $s(f_1,f_2)(x)=$ $=f_1(x)+f_2(x)$, $x\in X$, and for $f\in X^{\#}$ and $\lambda\in R$ let $m(\lambda,f)$ be the functional on X defined by $m(\lambda,f)(x)=f(\lambda x)$, $x\in X$. Then $s(f_1,f_2)\in X^{\#}$, $m(\lambda,f)\in X^{\#}$, and $s:X^{\#}\times X^{\#}\to$ $\to X^{\#}$, $m:R\times X^{\#}\to X^{\#}$ satisfy L_1-L_8, where $0\in X^{\#}$ is the functional which is 0 at each $x\in X$. Therefore $X^{\#}$ is an als. Notice that for each $f\in X^{\#}$ we have that $f|V_X$ is linear. We denote $s(f_1,f_2)$ by f_1+f_2 and $m(\lambda,f)$ by $\lambda\circ f$.

3.2. LEMMA. *Let* X *be an als and let* $f\in X^{\#}$. *We have* $f\in V_{X^{\#}}$ *iff* f *is linear on* X, *iff* $-1\circ f=-f$, *iff* $f|W_X=0$.

3.3. DEFINITION. Let X be an als. An almost linear subspace Γ of $X^{\#}$ is said to be *total* over X if the relations $x_1,x_2\in X$, $f(x_1)=f(x_2)$ for each $f\in\Gamma$ imply that $x_1=x_2$.

The als $X^{\#}$ may be not total over X (see Section 4).

3.4. LEMMA. *Let* X *be an als. If* $X=W_X$ *then* $X^{\#}=W_{X^{\#}}$. *If* $X=V_X$ *then* $X^{*}=V_{X^{\#}}$. *If in addition* $X^{\#}$ *is total over* X *then the converse to the above statements is also true.*

Proof. Suppose $X=W_X$ and let $f\in X^{\#}$. Then for each $x\in X$ we have $(-1\circ f)(x)=$ $=f(-x)=f(x)$ and so $-1\circ f=f$, i.e., $f\in W_{X^{\#}}$. Suppose $X=V_X$. Then $W_X=\{0\}$ and for each $f\in X^{\#}$ we have $f|W_X=0$. By Lemma 3.2 it follows that $f\in V_{X^{\#}}$.

Assume now that $X^{\#}$ is total over X and let $x\in X$. If $X^{\#}=W_{X^{\#}}$ then for each $f\in X^{\#}$ we have that $-1\circ f=f$ and so $(-1\circ f)(x)=f(-x)=f(x)$, whence by our assumption it follows that $x=-x$, i.e., $x\in W_X$. If $X^{\#}=V_{X^{\#}}$ then by Lemma 3.2, we get $f(x-x)=0=f(0)$ for each $f\in X^{\#}$ and so $x-x=0$, i.e., $x\in V_X$.

Let now X be a nals and for $f\in X^{\#}$ define

(3.4) $||f||=\sup\{|f(x)|:\ x\in X,\ ||x||\leq 1\}$

Let $X^{*}=\{f\in X^{\#}:||f||<\infty\}$.

3.5. THEOREM. X^{*} *together with* $||.||$ *defined by* (3.4) *is a nals.*

3.6. DEFINITION. The space X^{*} together with $||.||$ defined by (3.4) is called the *dual* space of the nals X.

Remark. We recall that for any nals X, the dual space X^{*} is a snals for the metric ρ defined by

$$\rho(f_1,f_2)=\sup\{|f_1(x)-f_2(x)|:\ x\in X,\ ||x||\leq 1\}\qquad (f_1,f_2\in X^{*})$$

3.7. LEMMA. *For any nals* X, $V_{X^{*}}$ *is a Banach space.*

Proof. Since V_{X^*} is a nls for the norm defined by (3.4), and by Lemma 3.2 each $f\in V_{X^*}$ is linear on X, the proof that V_{X^*} is complete, is similar with the proof that the dual space of a nls is complete.

Remark. The snals X^* where ρ is the metric defined in the above remark, is complete in the metric ρ.

In contrast to the case of a ls, when X is an als it is possible that $X^{\#}=\{0\}$ (see 4.1 d), 4.2 d)). On the other hand in all our examples when X is a nals, even $X^*\neq\{0\}$ and an open question is whether X^* may be $\{0\}$. The main part of this section is devoted to this question but unfortunately we were not able to prove or disprove it. Now, when the nals X has a basis, then $X^*\neq\{0\}$. (Hence by Corollary 2.10, for any als X with a basis $X^{\#}\neq\{0\}$). To show this we need the following lemma.

3.8. LEMMA. *Let* X *be a nals with a basis* B. *Then for each* $b_o\in B\setminus V_X$ *there exists* $f\in X^{\#}$ *such that* $f(b_o)=1$ *and* $f(b)=0$ *for each* $b\in B\setminus\{b_o\}$. *If* $b_o\in W_X$ *then* $f\in X^*$.

Proof. Let $x\in X\setminus\{0\}$. Then $x=\sum_{i=1}^{n}\lambda_i b_i$, where $b_i\neq b_j$ for $i\neq j$ and $\lambda_i>0$ for $b_i\in B\setminus V_X$. Define $f(x)=0$ if $b_o\notin\{b_1,\dots,b_n\}$ and $f(x)=\lambda_{i_o}$ if $b_{i_o}=b_o$ for some $i_o\in\{1,\dots,n\}$. Define also $f(0)=0$. Then f satisfies (3.1)-(3.3) (notice that (3.3) holds since $f\geq0$), and so $f\in X^{\#}$. Suppose now that $b_o\in W_X$. By Lemma 2.2 we can suppose $||b_o||=1$. Let $x\in X$ such that $f(x)>0$. Then $x=\lambda_o b_o+\sum_{i=1}^{k}\lambda_i b_i$ where $\lambda_o>0$, $b_i\neq b_j$ for $i\neq j$. By Lemma 1.7 we have $f(x)=\lambda_o=||\lambda_o b_o||\leq||x||$ and so $f\in X^*$, $||f||=1$.

3.9. THEOREM. *Let* X *be a nals such that* W_X *has a basis. Then* $X^*\neq\{0\}$.

Proof. Since W_X has a basis, by Lemma 3.8 there exists $f\in(W_X)^*\setminus\{0\}$. Let $x\in X$ and define $f_1(x)=f(x-x)$. Then $f_1\in X^{\#}$, $f_1\neq0$ and for each $x\in X$ we have that $0\leq f_1(x)\leq||f||\,||x-x||\leq2||f||\,||x||$, i.e., $f\in X^*\setminus\{0\}$.

3.10. COROLLARY. *If the nals* X *has a basis, then* $X^*\neq\{0\}$.

Proof. Use Corollary 2.9 and Theorem 3.9.

3.11. PROPOSITION. *Let* X *be a nals with a basis* B *such that* $\operatorname{card}(B\setminus V_X)<\infty$. *Then* $X^*=\{f\in X^{\#}:f\,|\,V_X\in(V_X)^*\}$.

Proof. Clearly we must prove only the inclusion $\supset$. Let $f\in X^{\#}$, $f|V_X\in(V_X)^*$. If $f\notin X^*$, then there exist $x_n\in X$, $||x_n||\leq1$, $n\in N$ such that $|f(x_n)|\to\infty$. Let $B\setminus V_X=\{b_1,\dots,b_k\}$. By Lemma 2.5, we have that $x_n=\sum_{i=1}^{k}\lambda_{ni}b_i+v_n$, $\lambda_{ni}\geq0$, $v_n\in V_X$, $n\in N$. By Lemma 1.15 the sequences $\{\lambda_{ni}\}_{n=1}^{\infty}$, $1\leq i\leq k$, are all bounded, and since $|f(x_n)|=|\sum_{i=1}^{k}\lambda_{ni}f(b_i)+f(v_n)|\to\infty$, it follows that $|f(v_n)|\to\infty$. Since $f|V_X\in(V_X)^*$ we must have $||v_n||\to\infty$. On the other hand $||v_n||\leq||x_n||+||\sum_{i=1}^{k}\lambda_{ni}b_i||$, for each $n\in N$, a contradiction since the right hand inequality is bounded. Therefore $f\in X^*$.

3.12. COROLLARY. *If the nals* X *has a basis* B *such that* card $B<\infty$ *then* $X^{\#}=X^*$.

As we have mentioned in the introduction, in a nals X a theorem of Hahn-Banach type is no longer true. In a nals X there could exist an almost linear subspace $Y\subset X$ and $f\in Y^*$ such that: a) f can not be extended to a functional $f_1\in X^{\#}$

(see 4.5 d)); b) f has a unique extension $f_1 \in X^{\#}$ but $f_1 \notin X^*$ (see 4.5 e)); c) f has a unique extension $f_1 \in X^*$ but $||f_1||>||f||$ (see 4.5 f)). In view of a), any conditions on X, $Y \subset X$ and $f \in Y^* \setminus \{0\}$ which guarantee the existence of an extension as those from b), c), or norm-preserving extension, are of interest. In the sequel we shall deal with this problem taking into account our main problem whether $X^* \neq \{0\}$.

The almost linear subspace $W_X \subset X$ has the property that for each $f \in (W_X)^*$ there exists a norm-preserving extension to X while for V_X this is an open question.

3.13. PROPOSITION. *Let* X *be a nals and let* $f \in (W_X)^*$. *Then there exists* $f_1 \in X^*$ *such that* $f_1 | W_X = f$, $||f_1|| = ||f||$ *and* $f_1 | V_X = 0$.

Proof. Clearly, the functional defined by $f_1(x) = f(x-x)/2$, $x \in X$ has all the required properties.

An immediate consequence of this result is:

3.14. COROLLARY. *Let* X *be a nals. If* $(W_X)^* \neq \{0\}$ *then* $X^* \neq \{0\}$.

In view of this result, to solve the problem whether for a nals X we have $X^* \neq \{0\}$, it is enough to solve it for a nals X such that $X = W_X$ (and X has no basis).

If the converse to Corrolary 3.14 were true in the class of nals X such that $X \neq V_X$ then for each nals X, $X^* \neq \{0\}$ as one can see from the next result. For this result our assumption from the introduction that X is a nals which is not a ls, is essential.

3.15. PROPOSITION. *The following assertions are equivalent*:

i) *There exists a nals* X *such that* $X^* = \{0\}$.

ii) *There exists a nals* X *such that* $X^* \neq \{0\}$ *and* $X^* = V_{X^*}$ (i.e., X^* *is a Banach space*).

Proof. i) $\Rightarrow$ ii). Suppose X is a nals such that $X^* = \{0\}$. Let $Y = \{(x,\alpha): x \in X, \alpha \in R\}$ and let $s: Y \times Y \to Y$ and $m: R \times Y \to Y$ be defined by $s((x_1,\alpha_1),(x_2,\alpha_2)) = (x_1+x_2, \alpha_1+\alpha_2)$ and $m(\lambda,(x,\alpha)) = (\lambda x, \lambda\alpha)$. Let $0 \in Y$ be the element $(0,0)$. Then Y is an als and we have $V_Y = \{(v,\alpha): v \in V_X, \alpha \in R\}$ and $W_Y = \{(w,0): w \in W_X\}$. Since $X \neq V_X$ then $Y \neq V_Y$. Define a norm on Y by $||(x,\alpha)||_1 = ||x|| + |\alpha|$. Then Y together with $||.||_1$ is a nals. Clearly the functional f_o defined on Y by $f_o((x,\alpha)) = \alpha$, $(x,\alpha) \in Y$, belongs to V_{Y^*} and $||f_o||_1 = 1$. We show that $Y^* = V_{Y^*}$. Let $f \in Y^* \setminus V_{Y^*}$. By Lemma 3.2 there exists $(w_o,0) \in W_Y$, $w_o \in W_X$ such that $f((w_o,0))>0$. Define the functional f_1 on X by $f_1(x) = f((x,0))$, $x \in X$. Then $f_1 \in X^*$ and by i), $f_1 = 0$, a contradiction since $f_1(w_o) = f((w_o,0))>0$. Therefore $V_{Y^*} = Y^*$.

ii) $\Rightarrow$ i). Let X be a nals such that $X^* = V_{X^*} \neq \{0\}$. Since X is not a ls, $W_X \neq \{0\}$ and we have $(W_X)^* = \{0\}$.

In the theory of Banach spaces it is well-known that there exist Banach spaces which have no preduals. Proposition 3.15 suggest - in case a nals X with $X^* = \{0\}$ exists - the following question. *Is it true that for each Banach space* E *there exists a nals* X *such that* $X^* \equiv E$? We can also ask the following question which makes sense for any solution to the main problem whether $X^* \neq \{0\}$. *Is it true* *that for each Banach space* E *there exists a nals* X *such that* $V_{X^*} \equiv E$?

We study now the extension property of functionals defined on the linear subspace V_X. Here we notice that in a nals it can happen that $V_X=\{0\}$ and $V_{X^*}\neq\{0\}$ (see 4.8 e)). When X is an als, it is possible that $V_X\neq\{0\}$ and $V_{X^*}=\{0\}$ (see 4.3 e)), but in all our examples when X is a nals, if $V_X\neq\{0\}$ then $V_{X^*}\neq\{0\}$. The same phenomenon appears in all our results on extensions of functionals defined on V_X, when we always get linear functionals on X.

3.16. PROPOSITION. *Let* X *be a nals with a basis* B.

i) *For each* $f\in(V_X)^{\#}$ *there exists* $f_1\in V_{X^{\#}}$, $f_1|V_X=f$.

ii) *If* card $(B\setminus V_X)<\infty$ *then for each* $f\in(V_X)^*$ *there exists* $f_1\in V_{X^*}$ *such that* $f_1|V_X=f$.

Proof. By Theorem 2.8 we can suppose that B has the property that for each $b\in B\setminus V_X$ we have $-b\in B\setminus V_X$.

i) Let $f\in(V_X)^{\#}\setminus\{0\}$ and let $x\in X\setminus V_X$. By Lemma 2.5, there exist unique $b_1,\dots,b_n\in B\setminus V_X$, $\lambda_i>0$, $1\le i\le n$ and $v\in V_X$ such that

$$x=\sum_{i=1}^{n}\lambda_i b_i+v \tag{3.5}$$

Define $f_1(x)=f(v)$ and for $v\in V_X$ define $f_1(v)=f(v)$. Then clearly $f_1\in X^{\#}$ and f_1 is an extension of f. To show that $f_1\in V_{X^{\#}}$, by Lemma 3.2 we must show that $f_1(-x)=-f_1(x)$ for each $x\in X\setminus V_X$. If x has the representation given in (3.5) then $-x=\sum_{i=1}^{n}\lambda_i(-b_i)-v$ and so $f_1(-x)=f(-v)=-f_1(x)$.

ii) Suppose card $(B\setminus V_X)<\infty$ and let $f\in(V_X)^*\setminus\{0\}$. Then by i) above there exists $f_1\in V_{X^{\#}}$, $f_1|V_X=f$, whence the result follows by Proposition 3.11.

3.17. COROLLARY. *Let* X *be a nals with a basis* B *such that* card $(B\setminus V_X)<\infty$. *Then* X^* *is total over* X.

Proof. Suppose $B\setminus V_X=\{b_1,\dots,b_n\}$ and let $x_1,x_2\in X$ such that $f(x_1)=f(x_2)$ for each $f\in X^*$. By Lemma 2.5 we have that $x_i=\sum_{j=1}^{n}\lambda_{ij}b_j+v_i$, $\lambda_{ij}\ge0$, $1\le j\le n$, $v_i\in V_X$, i=1,2. By Lemma 3.8, for each $b_j\in B\setminus V_X$ there exists $f_j\in X^{\#}$ such that $f_j(b_j)=1$ and $f_j(b)=0$ for $b\in B\setminus\{b_j\}$. By Proposition 3.11, $f_j\in X^*$, whence by our assumption it follows $\lambda_{1j}=\lambda_{2j}$ for $1\le j\le n$. Consequently, for each $f\in X^*$ we get $f(v_1)=f(v_2)$. Since V_X is a nls, by Proposition 3.16 ii) it follows that $v_1=v_2$. Therefore $x_1=x_2$.

3.18. PROPOSITION. *Let* X *be a nals such that* $X=W_X+V_X$. *Then for each* $f\in(V_X)^*$ *there exists a norm-preserving extension* $f_1\in V_{X^*}$.

Proof. Let $f\in(V_X)^*\setminus\{0\}$. By Lemma 1.10, for each $x\in X$ there exist unique $w\in W_X$ and $v\in V_X$ such that $x=w+v$. Define $f_1(x)=f(v)$. Clearly $f_1\in X^{\#}$ and by Lemma 3.2, $f_1\in V_{X^{\#}}$. By Lemma 1.7 we get $|f_1(x)|=|f(v)|\le||f||\,||v||\le||f||\,||x||$ and so $||f_1||=||f||$.

3.19. PROPOSITION. *Let* X *be a snals such that* ρ *is a metric and let* $x_o\in X\setminus(W_X+V_X)$. *Suppose*

$$X=\{\lambda x_o+\mu(-x_o)+w+v \ : \ \lambda,\mu\geq 0, \ w\in W_X \ , \ v\in V_X\}$$

i) *For each* $f\in(V_X)^*$ *there exists* $f_1\in V_X*$, $f_1|V_X=f$.

ii) $V_X*\neq\{0\}$.

iii) *For each* $f\in(W_X+V_X)^*$ *there exists* $f_1\in X^*$, $f_1|(W_X+V_X)=f$.

<u>Proof</u>. We show first that

$$X=X_1\cup X_2\cup(W_X+V_X) \tag{3.6}$$

where $X_1=\{\lambda x_o+w+v \ : \ \lambda>0, \ w\in W_X \ , \ v\in V_X\}$, $X_2=\{-\lambda x_o+w+v \ : \ \lambda>0, \ w\in W_X \ , \ v\in V_X\}$, and that we have $X_1\cap X_2=\emptyset$, $X_i\cap(W_X+V_X)=\emptyset$, $i=1,2$. Since the inclusion $\supset$ in (3.6) is obvious, let $x\in X$, say $x=\lambda x_o+\mu(-x_o)+w+v$, $\lambda,\mu\geq 0$, $w\in W_X$, $v\in V_X$. If $\lambda=\mu$, then since $\lambda(x_o-x_o)\in W_X$, it follows that $x\in W_X+V_X$. If $\lambda>\mu$, then $x=(\lambda-\mu)x_o+\mu(x_o-x_o)+w+v\in X_1$. Similarly, if $\lambda<\mu$ then $x\in X_2$. This proves (3.6). Since $\pm x_o\notin W_X+V_X$, by Lemma 1.11 it follows that $X_i\cap(W_X+V_X)=\emptyset$, $i=1,2$. Let now $x\in X_1\cap X_2$. Then there exist $\lambda_i>0$, $w_i\in W_X$, $v_i\in V_X$, $i=1,2$ such that $x=\lambda_1x_o+w_1+v_1=-\lambda_2x_o+w_2+v_2$. Hence, $(\lambda_1+\lambda_2)x_o+w_1+v_1=\lambda_2(x_o-x_o)+w_2+v_2\in W_X+V_X$, whence by Lemma 1.11 it follows $(\lambda_1+\lambda_2)x_o\in W_X+V_X$, a contradiction since $\lambda_1+\lambda_2>0$ and $x_o\notin W_X+V_X$. Therefore $X_1\cap X_2=\emptyset$. Using Lemma 1.12 (for $Y=W_X+V_X$) and Lemma 1.10 we get that any $x\in X$ can be uniquely represented in the form

$$x=\lambda x_o+w+v \qquad (\lambda\in R, \ w\in W_X \ , \ v\in V_X) \tag{3.7}$$

i) Let $f\in(V_X)^*\setminus\{0\}$. If $x\in X$ has the representation given by (3.7), define $f_1(x)=f(v)$. Clearly $f_1\in V_X\#$. If $f_1\notin V_X*$ then there exist $x_n\in X$, $||x_n||\leq 1$, $n\in N$ such that $|f_1(x_n)|\to\infty$. Suppose $x_n=\lambda_nx_o+w_n+v_n$, $\lambda_n\in R$, $w_n\in W_X$, $v_n\in V_X$, $n\in N$. Suppose that for an infinity of n we have $\lambda_n\geq 0$, and without loss of generality we can suppose $\lambda_n\geq 0$ for all $n\in N$. By Lemma 1.7 it follows that $||\lambda_nx_o+v_n||\leq$ $\leq||x_n||\leq 1$ for each $n\in N$, and so by Lemma 1.15 the sequence $\{\lambda_n\}_{n=1}^{\infty}$ is bounded. Then $||v_n||\leq 1+\lambda_n||x_n||$, $n\in N$, whence the sequence $\{||v_n||\}_{n=1}^{\infty}$ is bounded. We get the same conclusion if $\lambda_n\leq 0$, $n\in N$, since then we work with $-x_o$ instead of x_o. Now, since $|f_1(x_n)|=|f(v_n)|\to\infty$ and $f\in(V_X)^*$, we obtain that $||v_n||\to\infty$, a contradiction. Therefore $f_1\in V_X*$.

ii) If $V_X\neq\{0\}$ then by i) above we get $V_X*\neq\{0\}$. Suppose now $V_X=\{0\}$ and let $x\in X$. Then by (3.7) there exist unique $\lambda\in R$, $w\in W_X$, such that $x=\lambda x_o+w$. Define $f(x)=\lambda||x_o||$. Clearly we have $f\in V_X\#$. By Lemma 1.7 we get $f(x)=||\lambda x_o||\leq$ $\leq||\lambda x_o+w||=||x||$ and so $f\in V_X*\setminus\{0\}$.

iii) Let $f\in(W_X+V_X)^*\setminus\{0\}$. If $V_X=\{0\}$ then the result follows by Proposition 3.13. Suppose now $V_X\neq\{0\}$. By i) above, there exists $f_2\in X^*$ such that $f_2|V_X=1_{V_X}$ and $f_2|W_X=0$. By Proposition 3.13, there exists $f_3\in X^*$ such that $f_3|W_X=f|W_X$ and $f_3|V_X=0$. Let $f_1=f_2+f_3$. Then $f_1\in X^*$ and we have $f_1|(W_X+V_X)=f$.

3.20. PROPOSITION. *Let* $X=W_X$ *be a smile such that* ρ *is a metric,* Y *an* *almost linear subspace of* X *and* $x_o \in X \setminus Y$. *Suppose that* $X=\{\lambda x_o+y : \lambda\geq 0, y\in Y\}$ *and let* $f\in Y^* \setminus \{0\}$. *If there exist no* $y_1, y_2 \in Y$ *such that* $y_2=x_o+y_1$, *then there exists a norm-preserving extension of* f *to* X.

Proof. By hypothesis and Lemma 1.12 it follows that each $x\in X$ has a unique representation of the form $x=\lambda x_o+y$, $\lambda\geq 0$, $y\in Y$. Define $f_1(x)=f(y)$. Then $f_1\in X^{\#}$ and by Lemma 1.7 we have $0\leq f_1(x)=f(y)\leq ||f|| \, ||y|| \leq ||f|| \, ||x||$, i.e., $||f_1||=||f||$.

3.21. PROPOSITION. *Let* $X=W_X$ *be a nals,* Y *an almost linear subspace of* X *and* $x_o\in X\setminus Y$. *Suppose* $X=\{\lambda x_o+y : \lambda\geq 0, y\in Y\}$ *and let* $f\in Y^*\setminus\{0\}$. *If there exist* $y_1, y_2\in Y$ *such that* $y_2=x_o+y_1$ *and* $f(y_2)\geq f(y_1)$ *then there exists* $f_1\in X^*$, $f_1|Y=f$.

Proof. Suppose $y_2=x_o+y_1$, $y_1,y_2\in Y$ and $f(y_2)\geq f(y_1)$. Let $\beta=f(y_2)-f(y_1)\geq 0$, and for $x\in X$, $x=\lambda x_o+y$, $\lambda\geq 0$, $y\in Y$ define $f_1(x)=\lambda\beta+f(y)$. In order that f_1 be well-defined we must show that if $\lambda x_o+y=\mu x_o+z$, $\lambda,\mu\geq 0$, $y,z\in Y$ then

$$\lambda\beta+f(y)=\mu\beta+f(z) \tag{3.8}$$

Since (3.8) is clear if $\lambda=\mu=0$, suppose now $\lambda>0$. Then $\lambda x_o+y+\mu y_1=\mu x_o+\mu y_1+z=\mu y_2+z$ and so $x_o+y_3=y_4$ where $y_3=(y+\mu y_1)/\lambda\in Y$ and $y_4=(\mu y_2+z)/\lambda\in Y$. Then $x_o+y_1+y_3=y_1+y_4$ and since $x_o+y_1=y_2$ it follows that $y_2+y_3=y_1+y_4$. Hence $f(y_2)+f(y_3)=f(y_1)+f(y_4)$ i.e., $\beta=f(y_4)-f(y_3)$. Using the above expressions of y_3 and y_4 we obtain (3.8). Consequently f_1 is well-defined and we have that $f_1\in X^{\#}$.

Suppose $f_1\notin X^*$. Then there exist $x_n\in X$, $||x_n||\leq 1$, $n\in N$, such that $f_1(x_n)\to\infty$. Suppose $x_n=\lambda_n x_o+y_n$, $\lambda_n\geq 0$, $y_n\in Y$, $n\in N$. By Lemma 1.15, the sequence $\{\lambda_n\}_{n=1}^{\infty}$ is bounded and so, since $||y_n||\leq ||x_n||+\lambda_n||x_o||$ for each $n\in N$, the sequence $\{||y_n||\}_{n=1}^{\infty}$ is bounded. On the other hand $f_1(x_n)=\lambda_n\beta+f(y_n)\to\infty$ and so $f(y_n)\to\infty$, a contradiction since $\{||y_n||\}_{n=1}^{\infty}$ is bounded and $f\in Y^*$.

Remark. We can not improve the conclusion of Proposition 3.21 to obtain a norm-preserving extension (see 4.5 f)).

4. EXAMPLES

In this section we give examples of almost linear spaces, normed almost linear spaces and strong normed almost linear spaces, mainly for exhibiting counterexamples related to the content of this paper. Some examples are from [2], others are new and we send the interested reader for more examples, information and proofs to consult [2]. We draw attention that we do not know an example of a nals which is not a snals.

In all the examples below s and m are the mappings defined in Section 1. In the sequel we shall sometimes denote $s(x,y)$ by $x \dotplus y$ and $m(\lambda,x)$ by $\lambda\circ x$. The norm of a nals will be denoted by $|||\cdot|||$.

4.1. EXAMPLE. a) Let $X=\{x\in R : x\geq 0\}$. Define $s(x,y)=\max\{x,y\}$ and $m(\lambda,x)=x$ for $\lambda\neq 0$, $m(0,x)=0$. The element $0\in X$ is $0\in R$. Then X is an als. We have $V_X=\{0\}$ and

$W_X=X$. Clearly, there exists no norm on X.

b) Let $x,y\in X$, $0<y<x$. Then $x=x \dot{+} y$ and $x=\alpha o x \dot{+} y$ for $\alpha\neq 0$. Notice that the conclusion of Lemma 1.8 holds in X.

c) X has no basis.

d) We have $X^{\#}=\{0\}$.

4.2. EXAMPLE. a) Let L be a ls and let X=L where $s(x,y)=x+y$, $m(\lambda,x)=|\lambda|x$ and $0\in X$ is the element $0\in L$. Then X is an als and we have $V_X=\{0\}$ and $W_X=X$. There exists no norm on X.

b) Let $x\in L\setminus\{0\}$ and let $y=-x$ (this operation is understanded in L). Then $x,y\in X$ and we have $x \dot{+} y=0\in V_X$ and both $x,y\notin V_X$. We also have $x=2ox \dot{+} y$ and so the conclusion of Lemma 1.9 does not hold. Notice that in this example the relation (1.5) implies y=z.

c) X has no basis.

d) We have $X^{\#}=\{0\}$.

4.3. EXAMPLE. a) Let L be a ls dim L≥2, and let $\phi\in L^{\#}$, $\phi\neq 0$. Let $X=\{x\in L : \phi(x)\geq 0\}$ and let $X_+=\{x\in X : \phi(x)>0\}$, $X_o=\{x\in X : \phi(x)=0\}$. Define $s(x,y)=x+y$ if both $x,y\in X_+$ or both $x,y\in X_o$, $s(x,y)=s(y,x)=x$ if $x\in X_+$ and $y\in X_o$, and $m(\lambda,x)=|\lambda|x$ if $x\in X_+$, $m(\lambda,x)=\lambda x$ if $x\in X_o$. Let $0\in X$ be the element $0\in L$. Then X is an als and we have $V_X=X_o$, $W_X=X_+\cup\{0\}$. There exists no norm on X.

b) Let $w\in W_X\setminus\{0\}$. Then $w=w \dot{+} v$ for each $v\in V_X$.

c) X has no basis.

d) Let $f=\phi|X$. We have $X^{\#}=\{\lambda of : \lambda\in R\}=\{\lambda f : \lambda\geq 0\}$ and $X^{\#}$ is not total over X.

e) We have $V_X\neq\{0\}$ and $V_{X^{\#}}=\{0\}$.

4.4. EXAMPLE. a) Let R^2 be endowed with the Euclidean norm $\|\cdot\|$ and let $e_1=(1,0)$, $e_2=(0,1)$. Let $A_i=\{\lambda e_i : \lambda\geq 0\}$, i=1,2 and let $X=A_1\cup A_2$. Define $s(x,y)=x+y$ if both $x,y\in A_i$, i=1,2, $s(x,y)=s(y,x)=(\|x\|+\|y\|)e_2$ if $x\in A_i\setminus\{0\}$, $y\in A_j\setminus\{0\}$, $i\neq j$ and $m(\lambda,x)=|\lambda|x$. Let $0\in X$ be the element $0\in R^2$. Then X is an als and we have $V_X=\{0\}$, $W_X=X$. Let $|||x|||=\|x\|$. Then X together with $|||\cdot|||$ is a nals. It is a snals for the semi-metric $\rho(x,y)=|\,|||x|||-|||y|||\,|$.

b) Let $x=(0,2)\in X$, $y=(1,0)\in X$ and let $\alpha=1/2$. We have $x=(1/2)ox \dot{+} y$ and $y\neq x/2$.

c) X has no basis.

d) Let $f(x)=|||x|||$, $x\in X$. We have $X^{*}=\{\lambda of : \lambda\in R\}=\{\lambda f : \lambda\geq 0\}$ and X^{*} is not total over X.

4.5. EXAMPLE. a) Let L be a ls and $\phi\in L^{\#}$, $\phi\neq 0$. Let $X=\{x\in L : \phi(x)>0\}\cup\{0\}$. Define $s(x,y)=x+y$ and $m(\lambda,x)=|\lambda|x$. The element $0\in X$ is the element $0\in L$. Then X is an als and we have $V_X=\{0\}$ and $W_X=X$. Define $|||x|||=\phi(x)$. Then X is a nals. For the semi-metric defined by $\rho(x,y)=|\phi(x)-\phi(y)|$ it is a snals.

b) X has no basis if dim L≥2.

c) Let $f=\phi|X$. We have $X^{*}=X^{\#}=\{\lambda of : \lambda\in R\}=\{\lambda f : \lambda\geq 0\}$. Clearly X^{*} is not total over X if dim $L\geq 2$.

d) There exists a snals X_1, an almost linear subspace $Y\subset X_1$ and $f\in Y^*$, $f\neq 0$ such that f can not be extended to an almost linear functional $f_1\in X_1^{\#}$. Indeed, let $L=R^2$ and $\phi=(0,1)\in L^{\#}$ and define X as in a) above. Let $X_1=\{(\alpha,\beta)\in X : \alpha\geq 0,\beta\geq 0\}$ and $Y=\{(\alpha,\beta)\in X_1 : \beta\geq\alpha\}$. Then X_1 is an almost linear subspace of X and so it is a snals, and Y is an almost linear subspace of X_1. Let f be the functional defined on Y by $f((\alpha,\beta))=\beta-\alpha$, $(\alpha,\beta)\in Y$. Clearly $f\in Y^{\#}$ and we have $0\leq f((\alpha,\beta))=\beta-\alpha\leq\beta= =|||(\alpha,\beta)|||$. Therefore $f\in Y^*$. Suppose there exists $f_1\in X_1^{\#}$ such that $f_1|Y=f$. Let $y_1=(1,2)\in Y$, $y_2=(3,3)\in Y$ and $x_o=(2,1)\in X_1\setminus Y$. We have $y_2=x_o+y_1$ and so $f_1(y_2)= =f_1(x_o)+f_1(y_1)$. It follows that $f_1(x_o)=-1$, which is not possible since $x_o\in W_{X_1}=X_1$. Notice that for the snals $X_2=\{\lambda x_o+y : \lambda\geq 0,\ y\in Y\}$ and $f\in Y^*$ defined as above, we have $y_2=x_o+y_1$ and $f(y_2)<f(y_1)$ (see Proposition 3.21).

e) There exist a snals X_1, an almost linear subspace $Y\subset X_1$ and $f\in Y^*$ such that there exists a unique $f_1\in X_1^{\#}$ with $f_1|Y=f$ and $f_1\notin X_1^*$. Indeed, let X be as in d) above and let $X_1=\{(\alpha,\beta)\in X : \alpha\leq\beta\}$, $Y=\{(\alpha,\beta)\in X_1 : 0\leq\alpha\leq\beta\}$. Then X_1 is a snals and Y is an almost linear subspace of X_1. Let $f\in Y^*$ be defined by $f((\alpha,\beta))=\beta-\alpha$, $(\alpha,\beta)\in Y$. Then the functional $f_1((\alpha,\beta)=\beta-\alpha$, $(\alpha,\beta)\in X_1$ belongs to $X_1^{\#}$ and $f_1|Y=f$. Let $f_2\in X_1^{\#}$ such that $f_2|Y=f$, and let $x_1=(\alpha_1,\beta_1)\in X_1\setminus Y$. Then $\alpha_1<0$ and so $(-\alpha_1,-\alpha_1)\in Y$, and we also have that $(0,\beta_1-\alpha_1)\in Y$. Therefore $f_2((-\alpha_1,-\alpha_1))=0$ and $f_2((0,\beta_1-\alpha_1))=\beta_1-\alpha_1$. Since we have $(\alpha_1,\beta_1)+(-\alpha_1,-\alpha_1)=(0,\beta_1-\alpha_1)$ it follows that $f_2((\alpha_1,\beta_1))=\beta_1-\alpha_1=f_1((\alpha_1,\beta_1))$, i.e., $f_2=f_1$. Therefore f has a unique extension $f_1\in X_1^{\#}$. Let $x_n=(-n,1)\in X_1$, $n\in N$. We have $|||x_n|||=1$ and $f_1(x_n)=n+1$, i.e., $f_1\notin X_1^*$.

f) There exist a snals X_1, an almost linear subspace $Y\subset X_1$ anf $f\in Y^*$ such that there exists a unique $f_1\in X_1^*$, $f_1|Y=f$ and $|||f_1|||>|||f|||$. Indeed, let X be as in d) above and let $X_1=\{(\alpha,\beta)\in X : |\alpha|\leq\beta\}$, $Y=\{(\alpha,\beta)\in X_1 : \alpha\geq 0\}$. Then X_1 is a snals and Y is an almost linear subspace of X_1. Let $f\in Y^*$ be defined by $f((\alpha,\beta))= =\beta-\alpha$, $(\alpha,\beta)\in Y$. As in e) above $f_1\in X_1^{\#}$ defined by $f_1((\alpha,\beta))=\beta-\alpha$, $(\alpha,\beta)\in X_1$ is the unique extension of f to X_1. We have $|||f_1|||=2>|||f|||=1$. Observe that we have $X_1=\{\lambda x_o+y : \lambda\geq 0,\ y\in Y\}$ where $x_o=(-1,1)\in X_1$.

4.6. EXAMPLE. a) Let $(E,||\cdot||)$ be a nls and let X be the collection of all nonempty, bounded and convex subsets A of E. Define $s(A_1,A_2)=A_1+A_2=\{a_1+a_2 : a_i\in A_i\}$, $i=1,2$ and $m(\lambda,A)=\lambda A=\{\lambda a : a\in A\}$. Let $0\in X$ be the set $\{0\}$. Then X is an als, and we have $V_X=\{\{x\} : x\in E\}\equiv E$ and W_X is the set of those $A\in X$, A symmetric with respect to $0\in E$. For $A\in X$, let $|||A|||=\sup_{a\in A}||a||$. Then X together with $|||\cdot|||$ is a nals. It is a snals for the Hausdorff semi-metric defined by

$$(4.1)\qquad \rho(A_1,A_2)=\max\{\sup_{a_1\in A_1}\inf_{a_2\in A_2}||a_1-a_2||,\ \sup_{a_2\in A_2}\inf_{a_1\in A_1}||a_1-a_2||\}$$

b) Let a be an arbitrary non-zero element of E. Let $A_1=A_3=\{\alpha a : -1<\alpha<1\}$ and $A_2=\{\alpha a : -1\leq\alpha\leq 1\}$. Then $A_i\in X$, $i=1,2,3$ and we have $A_1+A_2=A_1+A_3$, $A_2\neq A_3$.

c) The snals X has no basis. Indeed, this is a consequence of b) above and Lemma 2.11 a). For E=R and X defined as in a) above, W_X has the basis {(-1,1) [-1,1]}.

d) We do not have a complete description of X^* and V_{X^*} but we know that they are both $\neq\{0\}$. Moreover for each $\phi\in(V_X)^*(=E^*)$, $\phi\neq 0$ there exist $f_1\in X^*\setminus V_{X^*}$ and $f_2\in V_{X^*}$, $|||f_1|||=|||f_2|||=|||\phi|||$ such that $f_1|V_X=f_2|V_X=\phi$. Indeed, define $f_1(A)=\sup_{a\in A}\phi(a)$, $A\in X$, and $f_2(A)=(f_1(A)-f_1(-A))/2$, $A\in X$. Then f_1,f_2 satisfy the required conditions. We do not know whether X^* is, or is not total over X.

4.7. EXAMPLE. a) Let $(E,||\cdot||)$ be a nls and let X be the collection of all nonempty, bounded, closed, convex subsets A of E. Define $s(A_1,A_2)=\overline{A_1+A_2}$, and define $m,0\in X$ as in Example 4.6 a). Then X is an als, and V_X,W_X have a similar description as in 4.6 a). Endowed with the same norm as in 4.6 a), the als X is a nals. Together with ρ defined by (4.1) it is a snals. Notice that now ρ is a metric on X.

b) Let E=R and define X as above. We have that $X=W_X+V_X$. Since a basis for W_X is the set $B_1=\{[-1,1]\}$, by Corollary 2.12, X has a basis. It seems to us that for dim E≥2 the corresponding X has no basis.

c) We can repeate word for word what was said in 4.6 d) but now we know that X^* is total over X (see [2]).

4.8. EXAMPLE. a) Let $(E,||\cdot||)$ be a nls and let $\phi\in E^*$ $||\phi||=1$, ϕ attains its norm. Then $H=\{x\in E : \phi(x)=0\}$ is proximinal in E, i.e., for each $x\in E$ the set $P_H(x)=\{h_o\in H : ||x-h_o||=\inf_{h\in H}||x-h||\}$ is nonempty. It is known (see e.g., [4]) that there exists a linear selection $p_H(x)\in P_H(x)$, $x\in E$. Let $X=\{x\in E : \phi(x)\geq 0\}$. Define $s(x,y)=x+y$, $m(\lambda,x)=\lambda x$ for $\lambda\geq 0$ and $m(-1,x)=x-2p_H(x)$. The element $0\in X$ is $0\in E$. Then X is an als and we have $V_X=H$, $W_X=\{x\in E : \phi(x)\geq 0, p_H(x)=0\}$. For $x\in X$ let $|||x|||=\phi(x)+||p_H(x)||$. Then X is a nals and for the semi-metric on X defined by $\rho(x,y)=|\phi(x)-\phi(y)|+|||p_H(x)||-||p_H(y)|||$ it is a snals. If H is a semi L-summand in E (i.e., for each $x\in E$ we have that $P_H(x)$ is a singleton and $||x||=||x-p_H(x)||+||p_H(x)||$ (see [3])) then $|||x|||=||x||$ for each $x\in X$ and for the metric on X defined by $\rho(x,y)=||x-y||$ (where x-y is understanded in E), X is a snals.

b) Let $x_o\in W_X\setminus\{0\}$. Then $W_X=\{\lambda x_o : \lambda\geq 0\}$ and so W_X has the basis $\{x_o\}$. Since $X=W_X+V_X$ by Corollary 2.12, X has a basis.

c) Suppose dim E≥2, X defined as in a) above, and let $Y=\{x\in E: \phi(x)>0\}\cup\{0\}$. Then Y is an almost linear subspace of X and Y has no basis. Notice that $W_Y=W_X$ has a basis.

d) Let $x_o\in W_X\setminus\{0\}$. Then $X^*=\{\phi_1|X : \phi_1\in E^*, \phi_1(x_o)\geq 0\}$ and $V_{X^*}=\{\phi_1|X : \phi_1\in E^*, \phi_1(x_o)=0\}$. Here X^* is total over X.

e) Let Y be defined as in c) above. We have $V_Y=\{0\}$ and for each $f\in V_{X^*}$ $f|Y\in V_{Y^*}$, i.e., $V_{Y^*}\neq\{0\}$.

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