Acta Universitatis Carolinae. Mathematica et Physica

Rainer Linde

Generalized Banach-Mazur distance

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 30 (1989), No. 2, 99--104

Persistent URL: http://dml.cz/dmlcz/701801

Generalized Banach-Mazur Distance

RAINER LINDE

Jena*)

Received 15 March 1989

For two isomorphic Banach spaces X and Y the Banach-Mazur distance is given by

$$d(X, Y) = \inf\{\|T\|\ \|T^{-1}\|: T: X \to Y \text{ isomorphism}\}.$$

It is well-known that the Banach-Mazur distance of two n-dimensional spaces is always less than or equal to n. By a result of Gluskin [4] there is a constant $c \geqq 1$ such that for every natural number n there exist Banach spaces X_n and Y_n with $\dim X_n = \dim Y_n = n$ and

$$d(X_n, Y_n) \geqq cn$$

We want to measure the distance of n-dimensional Banach spaces not by the norm but by any other operator ideal quasi-norms.

1. Notations

As usually we denote by $L(X, Y)$ the set of all linear and bounded operators from the Banach space X in the Banach space Y. For the definition of a quasi-normed operator ideal we refer to Pietsch [9]. We only want to repeat the definition of some special quasi-normed operator ideals.

We say that an operator $T \in L(X, Y)$ is absolutely p-summing for $1 \leqq p < \infty$ if there exists a constant $c > 0$ such that for all elements $x_1, \ldots, x_n \in X$ the inequality

$$\left(\sum_{i=1}^{n} \|Tx_i\|^p\right)^{1/p} \leqq c \sup\left\{\left(\sum_{i=1}^{n} |\langle x_i, a\rangle|^p\right)^{1/p}: a \in X',\ \|a\| \leqq 1\right\}$$

holds. In this case we write $T \in P_p(X, Y)$ and put

$$\|T \mid P_p\| = \inf c$$

where the infimum is taken over all possible constants c of the above definition.

*) Friedrich-Schiller-Universität, Sektion Mathematik, DDR-6900, UHH, 17. o. G.

An operator $T \in L(X, Y)$ is called nuclear if it admits a representation

$$T = \sum_{i=1}^{\infty} a_i \otimes y_i$$

with $a_i \in X'$ and $y_i \in Y$

$$\sum_{i=1}^{\infty} \|a_i\| \, \|y_i\| < \infty .$$

Then we write $T \in N(X, Y)$ and put

$$\|T \mid N\| = \inf \sum_{i=1}^{\infty} \|a_i\| \, \|y_i\|$$

where the infimum is taken over all admissible representations of T.

For a given normed operator ideal A an operator $T \in L(X, Y)$ belongs to the adjoint operator ideal A^* if there is a constant $c > 0$ such that

$$|\text{trace}\,(S_0 S T T_0)| \leqq c \|S_0\| \, \|S \mid A\| \, \|T_0\|$$

for all $S_0 \in L(Y, Y_0)$, $S \in A(Y, X)$, $T_0 \in L(X_0, X)$ and X_0, Y_0 finite dimensional. In this case we put

$$\|T \mid A^*\| = \inf c$$

where the infimum is taken over all admissible constants. Further T belongs to the dual operator ideal A' if $T' \in A(Y', X')$. Then $\|T \mid A'\| = \|T' \mid A\|$.

2. Generalized Banach-Mazur distance

Let A and B two quasi-normed operator ideals. We say that the Banach spaces X and Y are (A, B)-isomorphic if there exist operators $S \in A(X, Y)$ and $T \in B(Y, X)$ with $TS = I_X$ and $ST = I_Y$. In this case we put

$$d_{A,B}(X, Y) = \inf \{\|S \mid A\| \, \|T \mid B\| \colon S, T \text{ as above}\} .$$

Of course we have the following necessary condition. If X and Y are (A, B)-isomorphic then we have

$$I_X \in B \circ A \quad \text{and} \quad I_Y \in A \circ B .$$

Since most of the operator ideals are proper that means that only the identity of finite dimensional spaces belongs to the ideal we restrict ourselves to finite dimensional spaces. First we will list some simple properties. Here and in the following the sign $\leqq$ $\leqq$ means also $\ldots \leqq c \ldots$ for some constant $c > 0$ not depending on the Banach spaces X and Y or their dimension.

(1) $$d_{L,L}(X, Y) = d(X, Y)$$

(2) $$d_{A,B}(X', Y') \leqq d_{A',B'}(Y, X)$$

(3) The inclusions $A \subset C$ and $B \subset D$ imply $d_{C,D}(X, Y) \leqq d_{A,B}(X, Y)$.
It follows equality in (2) if A and B are symmetric.

(4) $$d_{A \circ C, B \circ D}(X, Y) \leqq d_{A,B}(X, Z)\, d_{C,D}(Z, Y).$$

(5) If $A^2 = A$ then $\ln d_{A,A}(X, Y)$ becomes a quasi-metric.
(6) The inequalities of Lewis-type

$$\|S: X_n \to Y_n \mid A\| \leqq n^{\lambda} \|S \mid C\|$$

$$\|T: Y_n \to X_n \mid B\| \leqq n^{\mu} \|T \mid D\|$$

$$\text{imply} \quad d_{A,B}(X_n, Y_n) \leqq n^{\lambda+\mu} d_{C,D}(X_n, Y_n).$$

Directly from the definition we obtain

Lemma 1. *Let A and B two quasi-normed operator ideals. Then*

$$d_{A,B}(X, Y) \geqq \max \{\|I_X \mid B \circ A\|, \|I_Y \mid A \circ B\|\}.$$

In the following we will use a result of Lewis [6].

Lemma 2. *Let A be any normed operator ideal. There exists for any two n-dimensional spaces X_n and Y_n an isomorphism $T: X_n \to Y_n$ with $\|T \mid A\| = 1$ and $\|T^{-1} \mid A^*\| = n$.*

Therefore we have the next result about generalized Banach-Mazur distance.

Proposition 1. *Let A be any normed operator ideal. It holds for every n-dimensional spaces X_n and Y_n the equality*

$$d_{A,A^*}(X_n, Y_n) = n.$$

Proof. The estimate from above follows directly from lemma 2. Otherwise we have

$$n = \operatorname{trace}(I: Y_n \to Y_n) \leqq \|S \mid A\|\, \|S^{-1} \mid A^*\|$$

for every other isomorphism. This implies equality.

Corollary. *For every n-dimensional spaces X_n and Y_n it holds*

$$d(X_n, Y_n) \leqq n.$$

The next result which goes back to F. John goes in the same direction as lemma 2.

Lemma 3. *Let X_n be any n-dimensional space. Then there exists an isomorphism $T: X_n \to l_2^n$ with $\|T \mid P_2\| = n^{1/2}$ and $\|T^{-1}\| = 1$.*

From this we deduce the next result.

Proposition 2. *Let $1 \leqq p < \infty$ and X_n with $\dim X_n = n$. Then*

$$n^{\min(1/p, 1/2)} \leqq d_{P_p, L}(X_n, l_2^n) \leqq n^{\max(1/p, 1/2)}$$

Proof. Let $1 \leqq p \leqq 2$. Take the isomorphism T of lemma 3. Then

$$\|T \mid P_p\| \leqq n^{1/p-1/2} \|T \mid P_2\| = n^{1/p} \quad \text{implies} \quad d_{P_p,L}(X_n, l_2^n) \leqq n^{1/p}.$$

From

$$n^{1/2} = \|I: X_n \to X_n \mid P_2\| \leqq \|I \mid P_p\| \leqq \|S \mid P_p\| \, \|S^{-1}\|$$

for any other isomorphism $S: X_n \to l_2^n$ it follows

$$d_{P_p,L}(X_n, l_2^n) \geqq n^{1/2}.$$

For $2 < p < \infty$ the proof is analogous.

Corollary. *For any n-dimensional space X_n it is*

$$d_{P_2,L}(X_n, l_2^n) = n^{1/2}.$$

Remark. The inequality of proposition 2 cannot be improved for $1 \leqq p < 2$. We have

$$d_{P_p,L}(l_u^n, l_2^n) = \begin{cases} n^{1/2} & \text{for } 1 \leqq u \leqq 2 \\ n^{1/p} & \text{for } p' \leqq u \leqq \infty \end{cases}$$

To show this we have to prove that

$$d_{P_p,L}(l_u^n, l_2^n) \leqq n^{1/2} \quad \text{for} \quad 1 \leqq u \leqq 2 \quad \text{and}$$

$$d_{P_p,L}(l_u^n, l_2^n) \geqq n^{1/p} \quad \text{for} \quad p' \leqq u \leqq \infty.$$

The first inequality follows by considering the identity from l_u^n in l_2^n and from l_2^n in l_u^n, respectively. The second one is implied by Kwapien's [5] result that $P_p \circ P'_{p'} = L_p^*$ and

$$\begin{aligned} n^{1/p+1/u} &= \|I: l_u^n \to l_u^n \mid L_p^*\| \leqq \|S: l_u^n \to l_2^n \mid P_p\| \, \|S^{-1}: l_2^n \to l_u^n \mid P'_{p'}\| = \\ &= \|S \mid P_p\| \, \|S^{-1\prime}: l_{u'}^n \to l_2^n \mid P_{p'}\| \leqq \\ &\leqq \|S \mid P_p\| \, \|S^{-1\prime}: l_1^n \to l_2^n \mid P_{p'}\| \, n^{1-1/u'} \leqq \\ &\leqq c_G \|S \mid P_p\| \, \|S^{-1}: l_2^n \to l_\infty^n\| \, n^{1/u} \leqq \\ &\leqq c_G \|S \mid P_p\| \, \|S^{-1}: l_2^n \to l_u^n\| \, n^{1/u}. \end{aligned}$$

By analogous considerations it follows for $2 \leqq p < \infty$ that

$$d_{P_p,L}(l_u^n, l_2^n) = n^{1/2} \quad \text{for} \quad 2 \leqq u \leqq \infty.$$

So also in the case $2 \leqq p < \infty$ the estimate from above cannot be improved. But we have the following

Problem. Are there for $2 < p < \infty$ a Banach space X_n and an isomorphism $T: X_n \to l_2^n$ with $\|T \mid P_p\| \, \|T^{-1}\| \leqq n^{1/p}$?

If we change the order of P_p and L we get another estimate.

Proposition 3. *Let* $1 \leqq p < \infty$ *and* X_n *with* $\dim X_n = n$. *Then*

$$n^{\min(1/p, 1/2)} \leqq d_{L,P_p}(X_n, l_2^n) \leqq n .$$

Proof. The estimate from below is the same as in proposition 2. For the estimate from above we will use lemma 3 to X_n'. Let $T: X_n' \to l_2^n$ with $\|T \mid P_2\| = n^{1/2}$ and $\|T^{-1}\| = 1$. Putting $S = (T^{-1})'$ we get $\|S\| = 1$ and

$$\|S^{-1} \mid P_p\| = \|T' \mid P_p\| \leqq \|T' \mid N\| \leqq \|T \mid N\| \leqq n^{1/2}\|T \mid P_2\| = n .$$

Remark. The estimate cannot be improved even in the case $p = 2$. We have

$$d_{L,P_2}(l_u^n, l_2^n) = n^{\max(1/2, 1/u')}$$

Taking the identities we get the estimate from above. Otherwise for any isomorphism $S: l_u^n \to l_2^n$ we have

$$n = \operatorname{trace}(I: l_u^n \to l_u^n) \leqq \|S \mid P_2\| \, \|S^{-1} \mid P_2^*\| =$$
$$= \|S \mid P_2\| \, \|S^{-1} \mid P_2\| \leqq \|S: l_\infty^n \to l_2^n \mid P_2\| \, \|S^{-1} \mid P_2\| \leqq$$
$$\leqq c_G \|S\| \, \|S^{-1} \mid P_2\| \leqq c_G n^{1/u} \|S: l_u^n \to l_2^n\| \, \|S^{-1} \mid P_2\| .$$

3. Estimates for quasi-norms generated by s-numbers

Let s be any s-number function [8]. Let $0 < r < \infty$ and $0 < w \leqq \infty$. Then $T \in L(X, Y)$ belongs to $L_{r,w}^s(X, Y)$ if the following quasi-norm is finite

$$\|T \mid L_{r,w}^s\| = \Big(\sum_{n=1}^{\infty} (n^{1/r - 1/w} s_n(T))^w\Big)^{1/w} \quad (w < \infty)$$

$$\|T \mid L_{r,\infty}^s\| = \sup\{n^{1/r} s_n(T): n \in \mathbb{N}\} .$$

Of course we have the inequality

$$\|T: X_n \to Y_n \mid L_{r,w}^s\| \leqq n^{1/r}\|T\|$$

which implies

$$d_{L^s_{r(1),w(1)}, L^s_{r(2),w(2)}}(X_n, Y_n) \leqq n^{1/r(1) + 1/r(2)} d(X_n, Y_n) .$$

Proposition 4. *Let s be any multiplicative s-number function,* $0 < r(1), r(2) < \infty$ *and* $0 < w(1), w(2) \leqq \infty$. *Then*

$$d_{L^s_{r(1),w(1)}, L^s_{r(2),w(2)}}(X, Y) \geqq \max\{\|I_X \mid L_{r,w}^s\|, \|I_Y \mid L_{r,w}^s\|\}$$

for $1/r = 1/r(1) + 1/r(2)$ *and* $1/w = 1/w(1) + 1/w(2)$.

Proof. The assumption follows from lemma 1 and

$$L_{r(1),w(1)}^s \circ L_{r(2),w(2)}^s \subset L_{r,w}^s .$$

For the approximation numbers

$$a_n(T) = \inf\{\|T - L\|: \text{rank } L < n\}$$

the Gelfand numbers

$$c_n(T) = \inf\{\|TJ_M\|: M \subset X, \text{ codim } M < n\}$$

and the Kolmogorov numbers

$$d_n(T) = \inf\{\|Q_N T\|: N \subset Y, \dim N < n\}$$

we get the following

Corollary. *For* $s \in \{a, c, d\}$ *we get*

$$d_{L^s_{r(1),w(1)}, L^s_{r(2)w(2)}}(X_n, Y_n) \geqq n^{1/r}$$

Proof. The assumption follows from proposition 4, $s_k(I_{X_n}) = 1$ for $1 \leqq k \leqq n$ and

$$\left(\sum_{k=1}^{n} k^{w/r-1}\right)^{1/w} \asymp n^{1/r}.$$

Remark. Using Gluskin's result [3] about $c_k(I: l_u^n \to l_v^n)$ we get equality in the preceding corollary for $X_n = l_u^n$ and $Y_n = l_v^n$ in some cases. Namely for

$$1 \leqq v < u \leqq 2, \quad r_2 < 2\frac{1/v - 1/2}{1/v - 1/u}$$

and

$$1 \leqq u < v \leqq 2, \quad r_1 < 2\frac{1/u - 1/2}{1/u - 1/v}.$$

References

[1] BENYAMINI Y. and GÖRDÖN Y., Random factorization of operators between Banach spaces, J. d'Anal. Math. **39** (1981), 45—74.

[2] CARL B.: Inequalities between absolutely (p, q)-summing norms, Studia Math. **69** (1980), 143—148.

[3] GLUSKIN E. D.: On somefinite dimensional problems of the theory of diameters (Russian), Vestnik Leningr. Univ. **13** (1981), 5—10.

[4] GLUSKIN E. D., The diameter of Minkowski compactum is roughly equal to n, Funkcional. Anal. i Prilozhen. **15** (1) (1981), 72—73.

[5] KWAPIEŃ S.: On operators factorable through L_p spaces, Bull. Soc. Math. France, Mém **31/32** (1972), 215—225.

[6] LEWIS D. R., Ellipsoids defined by Banach ideal norms, Mathematika **26** (1979), 18—29.

[7] LINDE R., s-Numbers of diagonal operators and Besov embeddings, Proc. 13-th Winter School, Suppl. Rend. Circ. Mat. Palermo (2) **10** (1985), 83—110.

[8] PIETSCH A., s-Numbers of operators in Banach spaces, Studia Math. **51** (1974), 201—223.

[9] PIETSCH A., Operator ideals, Berlin 1978.

[10] PIETSCH A., Eigenvalues and s-numbers, Leipzig 1987.