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Frame Functions, Signed Measures and Completeness of Inner Product Spaces

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We show that an inner product space is complete iff it possesses at least one nonzero frame function, or, equivalently, when and only when some systems of closed subspaces possess at least one nonzero signed measure.

1. Introduction

Suppose that S is a real or complex inner product space with an inner product $(\cdot, \cdot)$. There are many characterizations of the completeness using topological [4], algebraic [2, 3, 5], or measure-theoretical methods [6–8, 12]. An interesting characterization is due to Gudder and Holland [11]: S is complete iff for any maximal orthonormal system (MONS for short) $\{x_t\}$ we have $x = \sum_t (x_t, x)\, x_t$ for any $x \in S$.

In the present contribution, we generalize this result showing that S is complete iff S possesses at least one nonzero frame function.

2. Frame Functions

Let $\mathscr{S}(S)$ be a unit sphere in S, that is, $\mathscr{S}(S) = \{x \in S\colon \|x\| = 1\}$. A mapping $f\colon \mathscr{S}(S) \to R$ such that there is a constant $W \in R$ called the weight of f and such that, for any MONS $\{x_t\}$ in S, we have

$$\sum_t f(x_t) = W \tag{2.1}$$

is said to be a frame function. It is clear that for a frame function $f(\lambda x) = f(x)$ for all scalars λ, $|\lambda| = 1$, and all $x \in \mathscr{S}(S)$. The notion of a frame function for Hilbert spaces has been introduced by Gleason [9]. We denote by $\mathscr{F}(S)$ the set of all nonzero frame functions on S.

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A mapping $f\colon \mathscr{S}(S) \to R$ such that

(i) $\sum_i f(x_i)$ is finite for any orthonormal system $\{x_i\}$ in S;

(ii) $f \mid \mathscr{S}(K)$, $\dim K < \infty$, is a frame function is said to be a *frame function.*

Lemma 2.1. *If $\mathscr{F}(S) \neq \emptyset$, then, for any unit vector x in S, there exists an $f \in \mathscr{F}(S)$ such that*

$$f(x) > 0\,. \tag{2.2}$$

Proof. Choose a $g \in \mathscr{F}(S)$. For any MONS $\{y_j\}$ in S, there is a y_j such that $g(y_j) \neq 0$. The frame function g may be chosen such that $g(y_j) > 0$.

Let x be an arbitrary unit vector from S. We may define a unitary operator $U\colon S \to \to S$ such that $Uy_j = x$ and $Uy = y$ for any $y \perp x, y_j$. The mapping $f\colon z \mapsto g(U^{-1}z)$, $z \in \mathscr{S}(S)$, is an element of $\mathscr{F}(S)$ with (2.2). Q.E.D.

Lemma 2.2. *For any frame function f on S, $\dim S = \infty$, there exists a unique Hermitian operator $T = T_f\colon S \to S$ such that*

$$f(x) = (Tx, x)\,, \quad x \in \mathscr{S}(S)\,. \tag{2.3}$$

Proof. Suppose that f is a nonzero frame function on S. Then f is bounded on $\mathscr{S}(S)$. If not, then there is a sequence of unit vectors $\{x_n\}$ such that $\lim_n |f(x_n)| = \infty$. Let S_0 denote an infinite-dimensional linear submanifold of S containing all x_n's. Then $f \mid \mathscr{S}(S_0)$ is a frame-type function and, due to Šerstnev [14], $\sup\{|f(x)|\colon x \in S_0, \|x\| = 1\} < \infty$.

Since every two-dimensional subspace $N \subset S$ may be imbedded into a three-dimensional one, for a bounded $f \mid \mathscr{S}(N)$, there exists [4, 15] a unique symmetric bilinear form t_N such that $f(x) = t_N(x, x)$, $x \in \mathscr{S}(N)$. Consequently, there exists a unique Hermitian operator $T_N\colon N \to N$ sech that $t_N(x, x) = (T_N x, x)$, $x \in N$. We shall now define a Hermitian operator $T\colon S \to S$ as follows: let y be a vector from S and let N be a two-dimensional subspace containing y, then $Ty = T_N y$. This T is defined well, since if N_1 and N_2 are two-dimensional subspaces of S containing y, then there exists a three-dimensional subspace M containing N_1 and N_2. There is a Hermitian operator $T_M\colon M \to M$ such that $f(x) = (T_M x, x)$, $x \in \mathscr{S}(M)$. Then $T_{N_1} y = T_M y = T_{N_2} y$. Moreover, T is bounded Hermitian operator on S for which (2.3) holds. Q.E.D.

We note that the assumption on the infinite-dimensionality of S is not superfluous [15].

Now we introduce two important classes of closed subspaces of S. Denote by $E(S)$ the set of all splitting subspaces of S, i.e., of all subspaces M of S for which the condition $M + M^{\perp} = S$ holds, which is an orthocomplemented, orthomodular orthoposet containing $\{0\}$, S and all complete and, therefore, all finite-dimensional subspaces of S.

$L(S)$ is a system of all $\perp$-closed subspaces of S, i.e., of all subspaces M of S for which $M = M^{\perp\perp}$. $L(S)$ is an orthocomplemented, complete lattice.

We recall that $M^{\perp} = \{x \in S: (x, y) = 0 \text{ for all } y \in M\}$.

It is clear that $E(S) \subseteq L(S)$. Due to Amemiya and Araki [2], we have: S is complete iff $L(S)$ is orthomodular (or, equivalently, $E(S) = L(S)$).

For any $f \in \mathscr{F}(S)$, we define

$L_f(S) = \{M \subseteq S: M$ is a closed subspace such that if $\{a_i\}$ and $\{b_i\}$ are two MONSs in M, then $\sum_i f(a_i) = \sum_i f(b_i)\}$,

then $E(S) \subseteq L_f(S)$, and we define a mapping $m_f: L_f(S) \to R$ such that

$$m_f(M) = \sum_i f(a_i), \quad \{a_i\} \text{ is a MONS in } M.$$

The following important lemma has been motivated by similar result in [12]. We recall that for all $v_1, \ldots, v_n \in \mathscr{S}(S)$, $\mathrm{sp}(v_1, \ldots, v_n)$ denotes the finite-dimensional subspace of S generated by $v_1, \ldots, v_n$.

Lemma 2.3. *Let v be a unit vector in the completion $\bar{S}$ of an infinite-dimensional inner product space S. Then, for any $\varepsilon > 0$ and any $K > 0$, there is a $\delta > 0$ such that the following statement holds: If $w \in S$ is a unit vector such that $\|v - w\| < \delta$, then, for any frame function f such that the norm of $T = T_f$ is less than K, and for each finite-dimensional A satisfying the property $v \perp A$, we have the inequality*

$$|m_f(A \vee \mathrm{sp}(w)) - m_f(A) - f(w)| < \varepsilon. \tag{2.4}$$

Proof. Let $\varepsilon > 0$ and $K > 0$ be given. By the continuity of the function $\varrho(t) = (t^2 + (1 - (1 - t^2)^{1/2})^2)^{1/2}$ we can find a $\delta_1 > 0$ such that $\varrho(t) < \varepsilon/2K$ for any $t \in [0, \delta_1]$. The continuity of the projection $P_{\mathrm{sp}(v)^{\perp}}$ allows us to find a $\delta \in (0, 1)$ such that the assumption $\|v - w\| < \delta$ implies $\|P_{\mathrm{sp}(v)^{\perp}}(w)\| < \delta_1$. Fix a $w \in \mathscr{S}(S)$ with $\|v - w\| < \delta$ and suppose that A is any finite-dimensional subspace orthogonal to v. Then, for the projection P_A, we have

$$\|P_A(w)\| = \|P_A P_{\mathrm{sp}(v)^{\perp}}(w)\| \leqq \|P_{\mathrm{sp}(v)^{\perp}}(w)\| < \delta_1.$$

Thus, we obtain

$$\|(1 - P_A)(w)/\|(1 - P_A)(w)\| - w\| = \varrho(\|P_A(w)\|) < \varepsilon/2K.$$

Put $w' = (1 - P_A)(w)/\|(1 - P_A)(w)\|$. Then we have $\|w - w'\| < \varepsilon/2K$, $A \vee \mathrm{sp}(w) = A \vee \mathrm{sp}(w')$ and $w' \perp A$. Calculate

$$|m_f(A \vee \mathrm{sp}(w)) - m_f(A) - f(w)| = |m_f(A) + f(w') - m_f(A) - f(w)| =$$
$$= |f(w') - f(w)| = |(Tw', w') - (Tw, w)| \leqq |(Tw', w') - (Tw', w)| +$$
$$+ |(Tw', w) - (Tw, w)| \leqq 2\|T\| \|w - w'\| < \varepsilon.$$

Q.E.D.

By $\oplus_{L,i} M_i$ we mean the join of mutually orthogonal $\perp$-closed subspaces in $L(S)$.

Lemma 2.4. *Let $\{x_i\}$ be a nonvoid system of orthonormal vectors from S and put $M = \bigoplus_{L\,i} \mathrm{sp}(x_i)$.*

(i) *If $\{y_j\}$ is a* MONS *in $M^\perp$, then $\{x_i\} \cup \{y_j\}$ is a* MONS *in S.*

(ii) *$M^\perp \in L_f(S)$ for any $f \in \mathscr{F}(S)$.*

Proof. (i) is evident. For (ii) suppose that $\{y_j\}$ and $\{z_j\}$ are two MONSs in $M^\perp$. Due to (i), $\{x_i\} \cup \{y_j\}$ and $\{x_i\} \cup \{z_j\}$ are two MONSs in S. Therefore,

$$\sum_i f(x_i) + \sum_j f(y_j) = \sum_i f(x_i) + \sum_j f(z_j)$$

which entails $\sum_j f(y_j) = \sum_j f(z_j)$. Q.E.D.

Lemma 2.5. *Let $\mathscr{F}(S) \neq \emptyset$. Suppose that $\{y_j\}_{j=1}^\infty$ in Lemma 2.4 is a countable* MONS *in $M^\perp$. Then*

$$\bigoplus_{L\,i=1}^{\infty} \mathrm{sp}(y_j) = M^\perp . \tag{2.5}$$

Proof. Put $M_0 = \bigoplus_{L\,j=1}^{\infty} \mathrm{sp}(y_j) \in L(S)$. We assert that $M_0 = M^\perp$. If not, then $\overline{M}_0 \neq \overline{M}^\perp$. Hence, there is a $v \in \overline{M}^\perp$ that is orthogonal to $\overline{M}_0$. According to Lemma 2.1, we may assume that f is a frame function on S such that $f(z) > 0$ for some $z \in M^\perp$, $\|z\| = 1$. Let $\varepsilon = f(z)/3 > 0$. Applying Lemma 2.3, we find a $w \in M^\perp$ with $\|w - v\| < \delta$ for some $\delta > 0$ such that (2.4) holds for any finite-dimensional $A \perp v$, $A \subseteq M^\perp$ and any $s \in \mathscr{F}(S)$ for which $\|T_s\| = \|T_f\|$.

Define a unitary operator $U: S \to S$ such that $Uz = w$ and $Ux = x$ for any $x \perp w, z$ and put $s(u) = f(U^{-1}u)$, $u \in \mathscr{S}(S)$. It is simple to verify that $\|T_s\| = \|T_f\|$.

Put $A_n = \mathrm{sp}(y_1, \ldots, y_n)$, $B_n = A_n \vee \mathrm{sp}(w)$, $n = 1, 2, \ldots$. The vectors $w, y_1, y_2, \ldots$ may be orthogonalized using the Gram-Schmidt orthogonalization process. Thus, we obtain orthonormal vectors $w, z_1, z_2, \ldots$. Then $\{w, z_1, z_2, \ldots\}$ is a MONS in $M^\perp$ and

$$m_s(M^\perp) = s(w) + \sum_k s(z_k) = \lim m_s(B_n) .$$

Moreover,

$$m_s(M^\perp) = \sum_{j=1}^{\infty} s(y_j) = \lim_n m_s(A_n) .$$

Therefore, for any $\varepsilon > 0$, there is an integer n_0 such that for any $n > n_0$

$$m_s(B_n) - \varepsilon < m_s(M^\perp) < m_s(B_n) + \varepsilon$$

and

$$m_s(M^\perp) - \varepsilon < m_s(A_n) < m_s(M^\perp) + \varepsilon .$$

Using these inequalities and (2.4), we get

$$m_s(M^\perp) > m_s(B_n) - \varepsilon = m_s(A_n \vee \mathrm{sp}(w)) - \varepsilon > m_s(A_n) + s(w) - 2\varepsilon >$$
$$> m_s(M^\perp) - 3\varepsilon + s(w) = m_s(M^\perp)$$

which contradicts the beginning of the last inequality, and the lemma is proved.

Q.E.D.

3. The Main Result

In the present section, we prove the frame function completeness criterion for inner product spaces.

Theorem 3.1. *An inner product space S is complete iff S possesses at least one nonzero frame function.*

Proof. If S is complete, then the mapping $f_y: x \mapsto |(x, y)|^2$, $x \in \mathscr{S}(S)$, is a frame function on S, where y is any nonzero vector from S.

Conversely, suppose that f is a nonzero frame function. If S is finite-dimensional, then it is evidently complete. Assume, therefore, S is infinite-dimensional.

Now we claim to show that, for any sequence of orthonormal vectors $\{y_j\}_{j=1}^\infty$ from S, the subspace $M = \bigoplus_{j=1}^{\infty}{}_L\, \mathrm{sp}(y_j)$ is splitting. Complete $\{y_j\}_{j=1}^\infty$ by $\{x_i\}$ to be a MONS in S and put $N = \bigoplus_{i}{}_L\, \mathrm{sp}(x_i)$. Then $\{y_j\}$ is a MONS in $N^\perp$ and, according to Lemma 2.5, $M = N^\perp$. Therefore, if $\{z_j\}_{j=1}^\infty$ is a MONS in M, then $M = \bigoplus_{j=1}^{\infty}{}_L\, \mathrm{sp}(z_j)$.

Define

$$L(0, M) = \{N \in L(S): N \subseteq M,\ (N^{\perp M})^{\perp M} = N\},$$
$$L(M) = \{N \subseteq M: (N^{\perp M})^{\perp M} = N\},$$

and

$$L_M = \{N \subseteq M: N \in L(S)\},$$

where $N^{\perp M} = \{x \in M: (x, y) = 0 \text{ for any } y \in N\} = N^\perp \cap M$. Then

$$L(M) = L(0, M) \subseteq L_M .$$

Indeed, it is evident that $L(0, M) \subseteq L(M)$. Conversely, let $N \in L(M)$, we claim to show that $N^{\perp\perp} = N$. Calculate

$$N = N \cap M \subseteq N^{\perp\perp} \cap M = (N^\perp)^\perp \cap M = (N^\perp)^{\perp M} \subseteq (N^{\perp M})^{\perp M} = N .$$

Here we used the fact that if $A \subseteq M$, then $A^\perp \supseteq A^{\perp M}$. Similarly we may show that $L(M) \subseteq L_M$.

On the other hand, it is simple to verify that L_M is orthomodular with respect to the orthocomplementation ${}^{\perp M}$, that is, if $A, B \in L_M$, $A \subseteq B$, then $B = A \vee$

$\vee (B \wedge A^{\perp M})$. Applying the result of Amemiya and Araki [2], M is complete, and we know that any complete subspace of S is splitting. In view of the criterion[5], S is complete iff, for any sequence of orthonormal vectors $\{y_j\}_{j=1}^{\infty}$ from S, $\bigoplus_{j=1}^{\infty}{}_L \operatorname{sp}(y_j)$ is splitting, which entails our result.

Q.E.D.

The result of Gudder and Holland [11] may be now remarkably improved:

Corollary 3.2. *An inner product space S is complete iff there is a nonzero vector $v \in \bar{S}$ such that, for any* MONS *$\{x_i\}$ in S, we have*

$$\|v\|^2 = \sum_i |(v, x_i)\|^2 .$$

Proof. The assertion follows from Theorem 3.1, if we define a mapping f_v: $x \mapsto |(v, x)|^2$, $x \in \mathscr{S}(S)$. Q.E.D.

4. Signed measures

In this section, we apply the main theorem for signed measure completeness criteria. We introduce the following four families of closed subspaces that show quite different properties from the ordering point of view:

(1) $W(S)$ is the set of all closed subspaces of S which is a weakly orthocomplemented, complete lattice.

(2) $D(S)$ is the set of all Foulis-Randal subspaces of S, i.e., of all subspaces M for which there exists an orthonormal system $\{u_i\}$ such that $M = \bigoplus_{i}{}_L \operatorname{sp}(u_i)$, which is a complete orthoposet.

(3) $R(S)$ is the set of all subspaces M of S such that $M = \bigoplus_{i}{}_L \operatorname{sp}(u_i)$ for all MONSs $\{u_i\}$ in M, which is a poset.

(4) $V(S)$ is the set of all subspaces M of S such that $M = \bigoplus_{i}{}_L \operatorname{sp}(u_i)$ and $M^{\perp} = \bigoplus_{j}{}_L \operatorname{sp}(v_j)$ for every MONS $\{u_i\}$ and $\{v_j\}$ in M and $M^{\perp}$, respectively, which is an orthocomplemented poset.

We may verify that

$$E(S) \subseteq V(S) \subseteq R(S) \subseteq D(S) \subseteq L(S) \subseteq W(S) . \tag{4.1}$$

Let K be a capital from $\{E, V, R, D, L, W\}$. A mapping m from $K(S)$ into the real line R such that

$$m\Big(\bigoplus_{t \in T}{}_K M_t\Big) = \sum_{t \in T} m(M_t) \tag{4.2}$$

whenever $\{M_t\colon t \in T\}$ is a system of mutually orthogonal subspaces from $K(S)$ for which the join $\bigoplus_{t \in T}{}_K M_t$ exists in $K(S)$, and if $K = W$, then

(4.2) $$m(M \vee_W M^\perp) = m(S) \quad \text{for any} \quad M \in W(S)\,,$$

is said to be a signed measure on $K(S)$.

Hamhalter and Pták [12] proved an interesting criterion: A separable inner product space S is complete iff $L(S)$ possesses at least one state, that is, a positive signed measure m such that $m(S) = 1$. This result has been generalized in [6–8]. In the below, we present the more general form.

Theorem 4.1. *An inner product space S is complete iff $K(S)$, where K is an arbitrary capital from $\{E, V, R, D, L, W\}$, possesses at least one nonzero signed measure.*

Proof. If S is complete, then, in (4.1), we have only equalities, and the mapping

$$m_x(M) = \|P_M x\|^2\,, \quad M \in K(S)\,,$$

where P_M is the orthoprojector onto M, and x is a unit vector, is a nonzero signed measure.

Conversely, let m be a nonzero signed measure on $K(S)$. We assert that $\overline{m} := := m \mid E(S)$ is a nonzero signed measure on $E(S)$. Really, if $K \in \{E, V, R, D, L\}$ then, due to Lemma 2 of [6], we have: let, for a system of mutually orthogonal splitting subspaces $\{M_t : t \in T\}$, the join $\bigoplus_{E\, t\in T} M_t$ exists in $E(S)$, then $\bigoplus_{K\, t\in T} M_t = = \bigoplus_{L\, t\in T} M_t$. This implies that $\overline{m}$ is a nonzero signed measure.

Now let $K = W$. In view of (4.2),

(4.3) $$m(M^\perp) = m(S) - m(M)\,.$$

Applying (4.3) to $M^\perp$ and $M^{\perp\perp}$, we conclude that

(4.4) $$m(M) = m(M^{\perp\perp}) \quad \text{for any} \quad M \in W(S)\,.$$

The property (4.4) enables us to show that $\tilde{m} := m \mid L(S)$ is a nonzero signed measure on $L(S)$. Indeed, let $\{M_t : t \in T\}$ be a system of mutually orthogonal $\perp$-closed subspaces of S. Then using the de Morgan laws, we have

$$m(\bigoplus_{W\,t} M_t) = m((\bigoplus_{W\,t} M_t)^{\perp\perp}) = m((\bigwedge_{W\,t} M_t^\perp)^\perp) = m((\bigcap_t M_t^\perp)^\perp) =$$
$$= m((\bigwedge_{L\,t} M_t^\perp)^\perp) = m(\bigvee_{L\,t} M_t^{\perp\perp}) = m(\bigoplus_{L\,t} M_t) = \tilde{m}(\bigoplus_{L\,t} M_t)\,.$$

On the other hand,

$$m(\bigoplus_{W\,t} M_t) = \sum_t m(M_t) = \sum_i \tilde{m}(M_t) = \tilde{m}(\bigoplus_{L\,t} M_t)\,,$$

so that $\overline{m} := m \mid E(S)$ is a signed measure on $E(S)$.

We have seen that we may limit ourselves to the case that m is a nonzero signed measure on $E(S)$. Define a mapping $f\colon \mathscr{S}(S) \to R$ via $f(x) = m(\mathrm{sp}(x))$, $x \in \mathscr{S}(S)$.

It is clear that f is a nonzero frame function on S. The criterion of Theorem 3.1 entails that S is complete. Q.E.D.

Finally, we note that due to [6], $E(S)$ possesses plenty of finitely additive states, but we know nothing on the existence of finitely additive states on $K(S)$, where $K \in \{V, R, D, L, W\}$. In the below, we present only a particular assertion which generalizes the result from [13, 1] known for $L(S)$ and complete S, respectively; however, the analoguous assertion is still unknown for $K \in \{E, V, R\}$.

Proposition 4.2. *Let $K \in \{D, L, W\}$ and $\dim S \geqq 3$. Then there is no two-valued finitely additive state on $K(S)$.*

Proof. Due to the Gleason theorem [9], we may assume only $\dim S = \infty$. Suppose that $K = D$. Then $\bigoplus_{D\,t} M_t = \bigoplus_{L\,t} M_t$ whenever $\{M_t: t \in T\}$ is a system of mutually orthogonal Foulis-Randal subspace of S. Indeed, let $\{x_t^i\}$ be an orthonormal system in M_t such that $M_t = \bigoplus_{L\,i} \mathrm{sp}(x_t^i)$, then $M_t \subseteq M := \bigoplus_{L\,i,t} \mathrm{sp}(x_t^i)$ for any t. Now, let $M_t \subseteq N \in D(S)$ for any t. Then $\{x_t^i\} \subseteq N$ and, therefore, $\bigcup_i \{x_t^i\} \subseteq N$ which gives $M \subseteq N$.

Let $\{x_i: i \in I\}$ be a MONS in S. We exptess the index set I in the form of disjoint union of three-elements sets I_β, $I = \bigcup_\beta I_\beta$. Let S_3 be a three-dimensional Hilbert space. Choose a unitary operator U_β from S_3 onto $\mathrm{sp}(e_i: e_i \in I_\beta)$ and define a mapping $U: D(S_3) \to D(S)$ via

$$U(M) = \bigoplus_D U_\beta(M)\,, \quad M \in D(S_3)\,.$$

Then $U(M \vee N) = U(M) \vee_D U(N)$, and $M \perp N$ iff $U(M) \perp U(N)$.

If m is a two-valued finitely additive state on $D(S)$, then the mapping $m_U: D(S_3) \to \{0, 1\}$ defined via

$$m_U(M) = m(U(M))\,, \quad M \in D(S_3)\,,$$

is a two-valued state on $D(S_3)$ which contradicts the Gleason theorem [9].

For the general case of K, analogically as in the last theorem, we conclude that $\bar{m} := m \mid D(S)$ is a two-valued finitely additive state on $D(S)$. Applying the first part of the present proof, we get the contradiction. Q.E.D.

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