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NATURAL TRANSFORMATIONS Of FOLIATIONS INTO FOLIATIONS ON THE COTANGENT BUNDLE

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In this paper a classification of natural transformations of foliations into foliations on the cotangent bundle is given. All manifolds, foliations and maps are assumed to be of class C^∞. Foliations are assumed to be without singularities.

1. Let n be a natural number. Let M be an n-dimensional manifold. The vector bundle $(\pi_M : T^*M \to M) = T^*M := (T^*M)$ (dual to the tangent bundle TM of M) is called the cotangent bundle of M. Every embedding $f : M \to N$ of n-manifolds induces a vector bundle embedding $T^*f := (T(f^{-1}))^* : T^*M \to T^*N$ covering f. where Tf denotes the differential of f. One can verifies easily that the rule $M \to T^*M$, $f \to T^*f$, is a natural bundle in the sense of [4].

From now on we fix two natural numbers n and p such that $1 \leq p \leq n-1$. We identify a foliation with its tangent distribution (see [5]). A *natural transformation of foliations into foliations on the cotangent bundle* is a system of foliations $Q(M, F)$ on T^*M, for every n-manifold M and every p-dimensional foliation F on M, satisfying the following naturality condition: for any n-manifolds M, N, p-dimensional foliations F_1 on M and F_2 on N and every embedding $f : M \to N$ the assumption $Tf \circ F_1 = F_2 \circ f$ implies $TT^*f \circ Q(M, F_1) = Q(N, F_2) \circ T^*f$. (This definition is similar to the definition of natural base-extending operators (see [2]).

We have the following five natural transformations of foliations into foliations on the cotangent bundle. Let F be a p-dimensional foliation on an n-manifold M. Then we define the following distributions on T^*M :

$$^1Q(M,F)_\omega = \{0\},$$
$$^2Q(M,F)_\omega = \{\frac{d}{dt}(\omega + t\sigma)_{t=0} \in T_\omega T^*M : \sigma \in \mathrm{Anih}(F_{\pi_M(\omega)})\},$$

[0]This paper is in final form and no version of it will be submitted for publication elsewhere.

$${}^3Q(M,F)_\omega = \ker(T_\omega \pi_M),$$
$${}^4Q(M,F)_\omega = \{T^*X|\omega : X \text{ is a } F\text{-vector field}\} + \ker(T_\omega \pi_M),$$
$${}^5Q(M,F)_\omega = T_\omega T^* M,$$

where $\omega \in T^*M$, T^*X is the complete lift of X to the cotangent bundle (see [1], [6]) and $\mathrm{Anih}(F_y) = \{\sigma \in T^*_y M : \sigma(v) = 0 \text{ for all } v \in F_y\}$. If $(x^1, ..., x^n)$ are F-adapted coordinates on M and $(x^1, ..., x^n, v^1, ..., v^n)$ are the induced coordinates on T^*M, then

$${}^2Q(M,F) \text{ is spaned by } \frac{\partial}{\partial v^{p+1}}, ..., \frac{\partial}{\partial v^n},$$
$${}^3Q(M,F) \text{ is spaned by } \frac{\partial}{\partial v^1}, ..., \frac{\partial}{\partial v^n},$$
$${}^4Q(M,F) \text{ is spaned by } \frac{\partial}{\partial v^1}, ..., \frac{\partial}{\partial v^n}, \frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^p}.$$

Therefore ${}^iQ(M,F)$ is of class C^∞ and involutive. It is easy to verify that the system ${}^iQ = \{{}^iQ(M,F)\}$ is a natural transformation of foliations into foliations on the cotangent bundle.

The main result in this paper is the following theorem.

Theorem 1.1. *Any natural transformation of foliations into foliations on the cotangent bundle belongs to the set* $\{{}^1Q, {}^2Q, {}^3Q, {}^4Q, {}^5Q\}$ *defined above.*

The proof of this theorem will occupy the rest of the paper.

2. From now on we denote by $\mathcal{F}^p$ the standard p-dimensional foliation on $\mathbf{R}^n$ spaned by $\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^p}$. By $dx^1, ..., dx^n$ we denote the canonical forms on $\mathbf{R}^n$ dual to $\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}$.

The following lemma plays an essential role in the proof of the main theorem.

Lemma 2.1. *Let* Q_1 *and* Q_2 *be two natural transformations of foliations into foliations to the cotangent bundle. Let us assume that* $Q_1(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} \subset Q_2(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$. *Then* $Q_1(M,F)_\omega \subset Q_2(M,F)_\omega$ *for any p-dimensional foliation* F *on an n-manifold* M. *In particular, the equality* $Q_1(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} = Q_2(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$ *implies* $Q_1 = Q_2$.

Proof. . Consider $\omega \in T^*_y M - \mathrm{Anih}(F_y)$. By the Frobenius theorem there exists an embedding $f : \mathbf{R}^n \to M$ such that $Tf \circ \mathcal{F}^p = F \circ f$ on some open neighbourhood V of $0 \in \mathbf{R}^n$ and $T^*f(dx^1|0) = \omega$. Let $\tilde{\mathcal{F}}^p$ be a foliation on V such that $Tj \circ \tilde{\mathcal{F}}^p = \mathcal{F}^p \circ j$, where $j : V \to \mathbf{R}^n$ is the inclusion. Let $\omega_o \in T^*V$ be such that $T^*j(\omega_o) = dx^1|0$. Then by the naturality condition we obtain $Q_i(M,F)_\omega = Q_i(M,F) \circ T^*(f \circ j)(\omega_o) = TT^*(f \circ j)(Q_i(V, \tilde{\mathcal{F}}^p)_{\omega_o} = TT^*f(Q_i(\mathbf{R}^n, \mathcal{F}^p)_{dx^i|0})$ for $i = 1, 2$. Hence $Q_1(M,F)_\omega \subset$

$Q_2(M,F)_\omega$. Since $T_y^*M - \mathrm{Anih}(F_y)$ is dense in T_y^*M, we obtain the inclusion for all $\omega \in T^*M$.□

3. Let $\omega \in T_0^*\mathbf{R}^n$. A diffeomorphism $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ is called *ω-admissible* iff $T^*\varphi(\omega) = \omega$ and $T\varphi \circ \mathcal{F}^p = \mathcal{F}^p \circ \varphi$. A subspace $W \subset T_\omega T^*\mathbf{R}^n$ is called *ω-admissible* iff $TT^*\varphi(W) = W$ for any ω-admissible diffeomorphism φ.

We have the following corollary of the naturality condition.

Corollary 3.1. *If Q is a natural transformation of foliations into foliations on the cotangent bundle, then $Q(\mathbf{R}^n, \mathcal{F}^p)_\omega$ is ω-admissible for any $\omega \in T_0^*\mathbf{R}^n$.*

In particular, ${}^1Q(\mathbf{R}^n,\mathcal{F}^p)_\omega, \dots, {}^5Q(\mathbf{R}^n,\mathcal{F}^p)_\omega$ are ω-admissible, where ${}^iQ$ is defined in Item 1.

We have also the following corollary.

Corollary 3.2. *The vector spaces*

$${}^3Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} + \mathrm{span}\{T^*(\frac{\partial}{\partial x^i})_{dx^1|0} : i = 2, \dots, n\}$$

and $\mathrm{span}\{\frac{d}{dt}[(dx^1|0) + t(dx^1|0)]_{t=0}\}$ *are $(dx^1|0)$-admissible.*

Proof. . It is easy to verify that the second space is $(dx^1|0)$-admissible. Of course, the first space is equal to $\ker\{(dx^1|0) \circ T_{(dx^1|0)}\pi_{\mathbf{R}^n}\}$ i.e. $(dx^1|0)$-admissible. □

4. In the proof of Theorem 1.1 we use the following lemmas.

Lemma 4.1. *If $W \subset {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_0$ is a 0-admissible subspace such that $W \neq \{0\}$ and $W \neq {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_0$, then $W = {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_0$.*

Lemma 4.2. *Let $W \subset {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$ be a $(dx^1|0)$-admissible subspace such that* $\dim W = n - p$. *Then we have the following implications:*

(a) If $n - p \geq 2$, then $W = {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$.

(b) If $n - p = 1$, then $W = {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$ or $W = \mathrm{span}\{\frac{d}{dt}[(dx^1|0) + t(dx^1|0)]_{t=0}\}$.

Lemma 4.3. *Let W be a $(dx^1|0)$-admissible subspace. If $W - {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} \neq \emptyset$, then ${}^3Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} \subset W$.*

Lemma 4.4. *Let W be a 0-admissible subspace. Assume that $W \neq {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_0$, ${}^3Q(\mathbf{R}^n, \mathcal{F}^p)_0 \subset W$ and $W \neq {}^5Q(\mathbf{R}^n, \mathcal{F}^p)_0$. Then $W = {}^4Q(\mathbf{R}^n, \mathcal{F}^p)_0$.*

Lemma 4.5. *Let W be a $(dx^1|0)$-admissible subspace such that* $\dim W = n + p$ *and* ${}^3Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} \subset W$. *Then we have the following implications:*

(a) If $n + p < 2n - 1$, then $W = {}^4Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$.

(b) If $n+p = 2n-1$, then $W = {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} + \mathrm{span}\{T^*(\frac{\partial}{\partial x^i})_{dx^1|0} : i = 2, ..., n\}$ or $W = {}^4Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$.

Proof of Lemma 4.1. Consider two cases:

(I) $W \not\subset {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_0$. Then there exists $\sigma \in T_0^*\mathbf{R}^n - \mathrm{Anih}(\mathcal{F}_0^p)$ such that $\frac{d}{dt}[t\sigma]_{t=0} \in W$. Consider $\mu \in T_0^*\mathbf{R}^n - \mathrm{Anih}(\mathcal{F}_0^p)$. There exists an 0-admissible linear isomorphism φ such that $T^*\varphi(\sigma) = \mu$. Then $\frac{d}{dt}[t\mu]_{t=0} = TT^*\varphi(\frac{d}{dt}[t\sigma]_{t=0}) \in W$. Therefore $W = {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_0$. Contradiction.

(II) $W \subset {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_0$. Let $\sigma \in \mathrm{Anih}(\mathcal{F}_0^p) - \{0\}$ be such that $\frac{d}{dt}[t\sigma]_{t=0} \in W$. Consider $\mu \in \mathrm{Anih}(\mathcal{F}_0^p) - \{0\}$. There exists an 0-admissible linear isomorphism φ such that $T^*\varphi(\sigma) = \mu$. Then $\frac{d}{dt}[t\mu]_{t=0} \in W$. That is why $W = {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_0$. □

Proof of Lemma 4.2. Consider two cases:

(I) $W \subset {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$. Then $W = {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$ because of the dimension argument.

(II) $W \not\subset {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$. We can assume that $\mathrm{span}\{\frac{d}{dt}[(dx^1|0) + t(dx^1|0)]_{t=0}\} \not\subset W$. Consider two subcases;

(a) $W \not\subset {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} \oplus \mathrm{span}\{\frac{d}{dt}[(dx^1|0) + t(dx^1|0)]_{t=0}\}$. Then there exists $\sigma \in T_0^*\mathbf{R}^n - \mathrm{Anih}(\mathcal{F}_0^p) \oplus \mathrm{span}\{dx^1|0\}$ such that $\frac{d}{dt}[(dx^1|0) + t\sigma]_{t=0} \in W$. It is clear that $p \geq 2$. Consider two subsubcases:

(1) $\sigma(\frac{\partial}{\partial x^1}|0) \neq 0$. If $\mu \in T_0^*\mathbf{R}^n - \mathrm{Anih}(\mathcal{F}_0^p) \oplus \mathrm{span}\{dx^1|0\}$ and $\mu(\frac{\partial}{\partial x^1}|0) \neq 0$, then there exist $\lambda \in \mathbf{R}$ and a $(dx^1|0)$-admissible linear isomorphism φ such that $T^*\varphi(\lambda\sigma) = \mu$, and then $\frac{d}{dt}[(dx^1|0) + t\mu]_{t=0} = \lambda TT^*\varphi(\frac{d}{dt}[(dx^1|0) + t\sigma]_{t=0}) \in W$. Therefore $W = {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$ i.e. $\dim W = n > n - p$. Contradiction.

(2) $\sigma(\frac{\partial}{\partial x^1}|0) = 0$. If $\mu \in T_0^*\mathbf{R}^n - \mathrm{Anih}(\mathcal{F}_0^p) \oplus \mathrm{span}\{dx^1|0\}$ and $\mu(\frac{\partial}{\partial x^1}|0) = 0$, then there exists a $(dx^1|0)$-admissible linear isomorphism φ such that $T^*\varphi(\sigma) = \mu$, and then $\frac{d}{dt}[(dx^1|0) + t\mu]_{t=0} = TT^*\varphi(\frac{d}{dt}[(dx^1|0) + t\sigma]_{t=0}) \in W$. Therefore $\mathrm{span}\{\frac{d}{dt}[(dx^1|0) + t(dx^i|0)]_{t=0} : i = 2, ..., n\} \subset W$ i.e. $\dim W \geq n - 1 > n - p$. Contradiction.

(b) $W \subset {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} \oplus \mathrm{span}\{\frac{d}{dt}[(dx^1|0) + t(dx^1|0)]_{t=0}\}$. Then there exists $\sigma \in \mathrm{Anih}(\mathcal{F}_0^p) \oplus \mathrm{span}\{dx^1|0\} - \mathrm{Anih}(\mathcal{F}_0^p) \cup \mathrm{span}\{dx^1|0\}$ such that $\frac{d}{dt}[(dx^1|0) + t\sigma]_{t=0} \in W$. If $\mu \in \mathrm{Anih}(\mathcal{F}_0^p) \oplus \mathrm{span}\{dx^1|0\} - \mathrm{Anih}(\mathcal{F}_0^p) \cup \mathrm{span}\{dx^1|0\}$, then there exist $\lambda \in \mathbf{R}$ and a $(dx^1|0)$-admissible linear isomorphism φ such that $T^*\varphi(\lambda\sigma) = \mu$, and then $\frac{d}{dt}[(dx^1|0) + t\mu]_{t=0} \in W$. Therefore $W = {}^2Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} \oplus \mathrm{span}\{\frac{d}{dt}[(dx^1|0) + t(dx^1|0)]_{t=0}\}$ i.e. $\dim W = n - p + 1$. Contradiction. □

Proof of Lemma 4.3. There exist real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbf{R}$ such that $Y := a_1 \frac{d}{dt}[(dx^1|0) + t(dx^1|0)]_{t=0} + \ldots + a_n \frac{d}{dt}[(dx^1|0) + t(dx^n|0)]_{t=0} + b_1 T^*(\frac{\partial}{\partial x^1})_{dx^1|0} + \ldots + b_n T^*(\frac{\partial}{\partial x^n})_{dx^1|0} \in W$ and $b_q \neq 0$ for some $q \in \{1, \ldots, n\}$.

Consider $k \in \{1, \ldots, n\}$. Let $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ be a diffeomorphism such that $\varphi^{-1}(y^1, \ldots, y^n) = (y^1 + y^q y^k, y^2, \ldots, y^n)$ on some open neighbourhood of $0 \in \mathbf{R}^n$. Then φ is $(dx^1|0)$-admissible. By a standard verification (see [6]) one can show that $TT^*\varphi(Y) = Y + b_q \frac{d}{dt}[(dx^1|0) + t(dx^k|0)]_{t=0} + b_k \frac{d}{dt}[(dx^1|0) + t(dx^q|0)]_{t=0}$. Since W is $(dx^1|0)$-admissible and $Y \in W$, then $TT^*\varphi(Y) \in W$, and then $b_q \frac{d}{dt}[(dx^1|0) + t(dx^k|0)]_{t=0} + b_k \frac{d}{dt}[(dx^1|0) + t(dx^q|0)]_{t=0} \in W$. Putting $k = q$ we find $\frac{d}{dt}[(dx^1|0) + t(dx^q|0)]_{t=0} \in W$, and then $\frac{d}{dt}[(dx^1|0) + t(dx^k|0)]_{t=0} \in W$. □

Proof of Lemma 4.4. Consider two cases:

(I) $W \not\subset {}^4Q(\mathbf{R}^n, \mathcal{F}^p)_0$. Then there exists real numbers $a_1, \ldots, a_n \in \mathbf{R}$ such that $T^*(a_1 \frac{\partial}{\partial x^1} + \ldots + a_n \frac{\partial}{\partial x^n})_0 \in W$ and $a_i \neq 0$ for some $i = p+1, \ldots, n$. Consider $b_1, \ldots, b_n \in \mathbf{R}$ such that $b_j \neq 0$ for some $j = p+1, \ldots, n$. There exists an 0-admissible linear isomorphism φ such that $T\varphi(a_1(\frac{\partial}{\partial x^1})_0 + \ldots + a_n(\frac{\partial}{\partial x^n})_0) = b_1(\frac{\partial}{\partial x^1})_0 + \ldots + b_n(\frac{\partial}{\partial x^n})_0$. Then $T^*(b_1 \frac{\partial}{\partial x^1} + \ldots + b_n \frac{\partial}{\partial x^n})_0 = TT^*\varphi(T^*(a_1 \frac{\partial}{\partial x^1} + \ldots + a_n \frac{\partial}{\partial x^n})_0) \in W$. Therefore $W = {}^5Q(\mathbf{R}^n, \mathcal{F}^p)_0$. Contradiction.

(II) $W \subset {}^4Q(\mathbf{R}^n, \mathcal{F}^p)_0$. Let $a_1, \ldots, a_p \in \mathbf{R}$ be such that $T^*(a_1 \frac{\partial}{\partial x^1} + \ldots + a_p \frac{\partial}{\partial x^p})_0 \in W$ and $a_i \neq 0$ for some $i = 1, \ldots, p$. Consider $b_1, \ldots, b_p \in \mathbf{R}$ such that $b_j \neq 0$ for some $j = 1, \ldots, p$. There exists an 0-admissible linear isomorphism φ such that $T\varphi(a_1(\frac{\partial}{\partial x^1})_0 + \ldots + a_p(\frac{\partial}{\partial x^p})_0) = b_1(\frac{\partial}{\partial x^1})_0 + \ldots + b_p(\frac{\partial}{\partial x^p})_0$. Then $T^*(b_1 \frac{\partial}{\partial x^1} + \ldots + b_p \frac{\partial}{\partial x^p})_0 = TT^*\varphi(T^*(a_1 \frac{\partial}{\partial x^1} + \ldots + a_p \frac{\partial}{\partial x^p})_0) \in W$. That is why $W = {}^4Q(\mathbf{R}^n, \mathcal{F}^p)_0$. □

Proof of Lemma 4.5. Consider two cases:

(I) $W \subset {}^4Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$. Then $W = {}^4Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$ because of the dimension argument.

(II) $W \not\subset {}^4Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0}$. Consider two subcases;

(a) At first we assume that there exist $a_2, \ldots, a_n \in \mathbf{R}$ such that $T^*(a_2 \frac{\partial}{\partial x^2} + \ldots + a_n \frac{\partial}{\partial x^n})_0 \in W$ and $a_j \neq 0$ for some $j = p+1, \ldots, n$. Consider $b_2, \ldots, b_n \in \mathbf{R}$ such that $b_q \neq 0$ for some $q = p+1, \ldots, n$. There exists a $(dx^1|0)$-admissible linear isomorphism φ such that $T\varphi(a_2(\frac{\partial}{\partial x^2})_0 + \ldots + a_n(\frac{\partial}{\partial x^n})_0) = b_2(\frac{\partial}{\partial x^2})_0 + \ldots + b_n(\frac{\partial}{\partial x^n})_0$. Then $T^*(b_2 \frac{\partial}{\partial x^2} + \ldots + b_n \frac{\partial}{\partial x^n})_{dx^1|0} = TT^*\varphi(T^*(a_2 \frac{\partial}{\partial x^2} + \ldots + a_n \frac{\partial}{\partial x^n})_{dx^1|0}) \in W$. Hence $\operatorname{span}\{T^*(\frac{\partial}{\partial x^i})_{dx^1|0} : i = 2, \ldots, n\} \subset W$ i.e. $\dim W \geq 2n-1$. Therefore $W = {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_{dx^1|0} + \operatorname{span}\{T^*(\frac{\partial}{\partial x^i})_{dx^1|0} : i = 2, \ldots, n\}$, provided $\dim W = n + p = 2n - 1$.

(b) Now, we suppose that there exist $a_1, \ldots, a_n \in \mathbf{R}$ such that $T^*(a_1 \frac{\partial}{\partial x^1} + \ldots + a_n \frac{\partial}{\partial x^n})_0 \in W$, $a_1 \neq 0$ and $a_j \neq 0$ for some $j = p+1, \ldots, n$. Consider $b_1, \ldots, b_n \in \mathbf{R}$ such that $b_1 \neq 0$ and $b_q \neq 0$ for some $q = p+1, \ldots, n$. Then there exists a $(dx^1|0)$-admissible linear isomorphism φ such that $T\varphi(a_1(\frac{\partial}{\partial x^1})_0+...+a_n(\frac{\partial}{\partial x^n})_0) = \frac{a_1}{b_1}\{b_1(\frac{\partial}{\partial x^1})_0+...+b_n(\frac{\partial}{\partial x^n})_0\}$. Then $T^*(b_1\frac{\partial}{\partial x^1}+...+b_n\frac{\partial}{\partial x^n})_{dx^1|0} = \frac{a_1}{b_1}TT^*\varphi(T^*(a_1\frac{\partial}{\partial x^1}+...+a_n\frac{\partial}{\partial x^n})_{dx^1|0}) \in W$. We have proved that $\mathrm{span}\{T^*(\frac{\partial}{\partial x^i})_{dx^1|0} : i = 1,...,n\} \subset W$. Hence $\dim W = 2n$. Contradiction. □

5. We are now in position to prove Theorem 1.1. Let Q be a natural transformation of foliations into foliations on the cotangent bundle such that $Q \neq {}^1Q$, $Q \neq {}^3Q$ and $Q \neq {}^5Q$. We want to show that $Q = {}^2Q$ or $Q = {}^4Q$.

It follows from Lemma 2.1 that $Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0} \neq {}^iQ(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0}$ for $i = 1,3,5$. Of course, ${}^1Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0} \subset Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0} \subset {}^5Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0}$.

Consider two cases:

(I) $Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0} \subset {}^3Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0}$. Then then it follows from Lemma 2.1 that $Q(\mathbf{R}^n,\mathcal{F}^p)_0 \subset {}^3Q(\mathbf{R}^n,\mathcal{F}^p)_0$. Of course, $Q(\mathbf{R}^n,\mathcal{F}^p)_0 \neq {}^iQ(\mathbf{R}^n,\mathcal{F}^p)_0$ for $i = 1,3$ because of the dimension argument. Then Lemma 4.1 implies $Q(\mathbf{R}^n,\mathcal{F}^p)_0 = {}^2Q(\mathbf{R}^n,\mathcal{F}^p)_0$. Hence $\dim Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0} = n-p$. Consider two subcases:

(a) $n-p \geq 2$. Then by Lemma 4.2(a) $Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0} = {}^2Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0}$, and then $Q = {}^2Q$ because of Lemma 2.1.

(b) $n-p = 1$. Then by Lemma 4.2(b), $Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0} = {}^2Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0}$ (i.e. $Q = {}^2Q$ because of Lemma 2.1) or $Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0} = \mathrm{span}\{\frac{d}{dt}[(dx^1|0)+t(dx^1|0)]_{t=0}\}$. So, we suppose that $Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0} = \mathrm{span}\{\frac{d}{dt}[(dx^1|0)+t(dx^1|0)]_{t=0}\}$. Then from the naturality condition with respect to the homotheties $\tau\,\mathrm{id}_{\mathbf{R}^n}$, $\tau \neq 0$, it follows that $Q(\mathbf{R}^n,\mathcal{F}^p)_{\tau(dx^1|0)} = \mathrm{span}\{\frac{d}{dt}[\tau(dx^1|0)+t(dx^1|0)]_{t=0}\}$ for any $\tau \in \mathbf{R}-\{0\}$. On the other hand, $Q(\mathbf{R}^n,\mathcal{F}^p)_0 = \mathrm{span}\{\frac{d}{dt}[t(dx^n|0)]_{t=0}\} \neq \mathrm{span}\{\frac{d}{dt}[t(dx^1|0)]_{t=0}\}$. Contradiction.

(II) $Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0} - {}^3Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0} \neq \emptyset$. Then it follows from Lemma 4.3 that ${}^3Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0} \subset Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0}$. Then by Lemma 2.1. ${}^3Q(\mathbf{R}^n,\mathcal{F}^p)_0 \subset Q(\mathbf{R}^n,\mathcal{F}^p)_0$. Of course, $Q(\mathbf{R}^n,\mathcal{F}^p)_0 \neq {}^iQ(\mathbf{R}^n,\mathcal{F}^p)_0$ for $i = 3,5$ because of the dimension argument. Then $Q(\mathbf{R}^n,\mathcal{F}^p)_0 = {}^4Q(\mathbf{R}^n,\mathcal{F}^p)_0$ because of Lemma 4.4. Hence $\dim Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0} = \dim Q(\mathbf{R}^n,\mathcal{F}^p)_0 = n+p$. Consider two subcases:

(a) $n+p < 2n-1$. Then it follows from Lemma 4.5(a) that $Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0} = {}^4Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0}$, and then $Q = {}^4Q$ because of Lemma 2.1.

(b) $n+p = 2n-1$. Then by Lemma 4.5(b), $Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0} = {}^4Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0}$ (i.e. $Q = {}^4Q$) or $Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0} = {}^3Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0}+\mathrm{span}\{T^*(\frac{\partial}{\partial x^i})_{dx^1|0} : i = 2,...,n\}$. So, suppose that $Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0} = {}^3Q(\mathbf{R}^n,\mathcal{F}^p)_{dx^1|0} + \mathrm{span}\{T^*(\frac{\partial}{\partial x^i})_{dx^1|0} : i = 2,...,n\}$. Then by the naturality condition with respect to the homotheties $\tau\,\mathrm{id}_{\mathbf{R}^n}$, $\tau \neq 0$, we obtain that $Q(\mathbf{R}^n,\mathcal{F}^p)_{\tau(dx^1|0)} = {}^3Q(\mathbf{R}^n,\mathcal{F}^p)_{\tau(dx^1|0)} + \mathrm{span}\{T^*(\frac{\partial}{\partial x^i})_{\tau(dx^1|0)} : i =$

$2,\dots,n\}$ for any $\tau \in \mathbf{R} - \{0\}$. On the other hand, since $n \geq 2$, then $Q(\mathbf{R}^n, \mathcal{F}^p)_0 = {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_0 + \mathrm{span}\{T^*(\frac{\partial}{\partial x^i})_0 : i = 1,\dots,n-1\} \neq {}^3Q(\mathbf{R}^n, \mathcal{F}^p)_0 + \mathrm{span}\{T^*(\frac{\partial}{\partial x^i})_0 : i = 2,\dots,n\}$. Contradiction. □

6. Similarly as in [3], we introduce the following definition. A *natural lifting of foliations to the cotangent bundle* is a system of foliations $Q(M, F)$ on T^*M projecting (by the cotangent bundle projection) onto F, for every n-manifold M and every p-dimensional foliation F on M, satisfying the following naturality condition: for any n-manifolds M, N, p-dimensional foliations F_1 on M and F_2 on N and every embedding $f : M \to N$ the assumption $Tf \circ F_1 = F_2 \circ f$ implies $TT^*f \circ Q(M, F_1) = Q(N, F_2) \circ T^*f$.

We have the following obvious corollary of Theorem 1.1.

Corollary 6.1. *Any natural lifting of foliations to the cotangent bundle is equal to* ${}^4Q$, *where* ${}^4Q$ *is defined in Item 1.*

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