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PARABOLIC GEOMETRIES DETERMINED BY FILTRATIONS OF THE TANGENT BUNDLE

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ABSTRACT. Let $\mathfrak{g}$ be a real semisimple $|k|$-graded Lie algebra such that the Lie algebra cohomology group $H^1(\mathfrak{g}_-, \mathfrak{g})$ is contained in negative homogeneous degrees. We show that if we choose $G = \mathrm{Aut}(\mathfrak{g})$ and denote by P the parabolic subgroup determined by the grading, there is an equivalence between regular, normal parabolic geometries of type (G, P) and filtrations of the tangent bundle, such that each symbol algebra $\mathrm{gr}(T_xM)$ is isomorphic to the graded Lie algebra $\mathfrak{g}_-$. Examples of parabolic geometries determined by filtrations of the tangent bundle are discussed.

1. SOME BACKGROUND ON PARABOLIC GEOMETRIES

We begin by reviewing some facts about parabolic geometries. Details on the subject can be found for example in [3], [4], [5]. The basic idea that goes back to Elie Cartan is to associate to any homogeneous space G/P a geometric structure, called Cartan geometry. Manifolds together with this structure may be thought of as curved analogs of the homogeneous space G/P. To be more precise, a Cartan geometry of type (G, P) on a manifold M is given by a principal P-bundle $\mathcal{G} \to M$ and a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, i.e. a smooth one-form with values in $\mathfrak{g}$, which is P-equivariant, reproduces generators of fundamental vector fields and satisfies the condition that $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$. The P-principal bundle generalizes the natural projection $G \to G/P$ and the Cartan connection generalizes the Maurer Cartan form on G. Parabolic geometries are Cartan geometries of type (G, P), where G is a real semisimple Lie group and $P \subset G$ a parabolic subgroup.

Recall that parabolic subalgebras are in bijective correspondence with $|k|$-gradings on semisimple Lie algebras, i.e. vector space decompositions $\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ and $\mathfrak{g}_- = \mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_{-k}$ is generated by $\mathfrak{g}_{-1}$. The parabolic subalgebra corresponding to such a $|k|$-grading is given by $\mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$. The parabolic subgroup P can be defined as the subgroup of those element g in G such that $\mathrm{Ad}(g)$ preserves the filtration corresponding to the $|k|$-grading.

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From any parabolic geometry an underlying geometric structure can be constructed, which is in many cases easier to interpret geometrically. For example conformal structures and partially integrable CR structures are underlying structures to parabolic geometries of some type. Under certain restrictions, even an equivalence of categories between parabolic geometries and underlying structures can be established.

First, we restrict our consideration to regular parabolic geometries, which means that we require that a certain condition on the Cartan connection is satisfied. One can construct from any regular parabolic geometry a filtration $TM = T^{-k}M \supset \cdots \supset T^{-1}M$ of the tangent bundle by smooth subbundles that is compatible with the Lie bracket, meaning that for $\xi \in \Gamma(T^iM)$ and $\eta \in \Gamma(T^jM)$ the bracket $[\xi, \eta]$ is contained in $\Gamma(T^{i+j}M)$. Given such a filtration, one gets an induced tensorial bracket $\mathcal{L}$ on the graded vector bundle $\mathrm{gr}(TM) = T^{-k}M/T^{-k+1}M \oplus \cdots \oplus T^{-1}M$, which is called Levi bracket. The associated graded vector bundle $\mathrm{gr}(TM)$ together with the Levi bracket is a bundle of Lie algebras. The graded Lie algebra $\mathrm{gr}(T_xM)$ is called symbol algebra of the filtration at the point x. In case of a filtration underlying a regular parabolic geometry of type (G, P), each symbol algebra is isomorphic to $\mathfrak{g}_-$. Hence, such a filtration gives rise to a natural frame bundle $\mathcal{P}$, where the fibre $\mathcal{P}_x$ is the set of all isomorphisms $\phi : \mathfrak{g}_- \to \mathrm{gr}(T_xM)$ such that ϕ is homogeneous of degree zero and satisfies $\phi([X, Y]) = \mathcal{L}(\phi(X), \phi(Y))$. This is a principal bundle with structure group the group $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$ of Lie algebra automorphisms on $\mathfrak{g}_-$ that are homogeneous of degree zero. Let us denote by G_0 the subgroup of all elements $g \in G$ such that $\mathrm{Ad}(g)$ preserves the grading on $\mathfrak{g}$. Then it can be shown that the Cartan connection of the regular parabolic geometry induces a reduction of the structure group of the principal $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$-bundle $\mathcal{P}$ to the group G_0 with respect to $\mathrm{Ad} : G_0 \to \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$. Summarizing, every regular parabolic geometry gives rise to a filtration of the tangent bundle compatible with the Lie bracket such that each symbol algebra is isomorphic to $\mathfrak{g}_-$ and a reduction of the structure group of the corresponding frame bundle $\mathcal{P}$ to the group G_0. Such a structure is called regular infinitesimal flag structure.

Next, one may ask whether a regular parabolic geometry is determined by its underlying regular infinitesimal flag structure. It turns out, that an additional condition on the Cartan connection needs to be introduced, which leads to the notion of normal parabolic geometries. Making the (technical) assumption that no simple ideal of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$ and a cohomological assumption that we will discuss later, one can prove that there is an equivalence of categories between regular, normal parabolic geometries of type (G, P) and regular infinitesimal flag structures of the same type, see [4].

2. Parabolic geometries determined by the filtration

2.1. In this article, we will be concerned with parabolic geometries that correspond via the above equivalence to filtrations of the tangent bundle. Note, that in case the subgroup G_0 is isomorphic to $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$, there is no reduction of the structure group. Thus, a regular infinitesimal structure is given only by a filtration such that each symbol algebra is isomorphic to $\mathfrak{g}_-$. Our aim is to determine a computable condition on the type (G, P) such that $G_0 \cong \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$ is satisfied.

On Lie algebra level there is such a condition. To make this observation look at the part of the complex one needs to compute the Lie algebra cohomology group $H^1(\mathfrak{g}_-, \mathfrak{g})$:

$$\dots \to \mathfrak{g} \xrightarrow{\partial_0} \mathfrak{g}_-^* \otimes \mathfrak{g} \xrightarrow{\partial_1} \Lambda^2\mathfrak{g}_-^* \otimes \mathfrak{g} \to \dots$$

Here ∂_0 is given by $\partial_0(X)(Y) = [Y, X] = -\mathrm{ad}\,(X)(Y)$ for $X \in \mathfrak{g}$ and $Y \in \mathfrak{g}_-$ and we have $\partial_1(\phi)(X_0, X_1) = [X_0, \phi(X_1)] + [\phi(X_0), X_1] - \phi([X_0, X_1])$ for $\phi \in \mathfrak{g}_-^* \otimes \mathfrak{g}$ and $X_0, X_1 \in \mathfrak{g}_-$. The gradings on $\mathfrak{g}_-$ and $\mathfrak{g}$ give rise to gradings on the spaces $\Lambda^k\mathfrak{g}_-^* \otimes \mathfrak{g}$, that is one gets a decomposition $\Lambda^k\mathfrak{g}_-^* \otimes \mathfrak{g} = \bigoplus_l (\mathfrak{g}_-^* \otimes \mathfrak{g})_l$ according to homogeneous degrees. In particular, the graded component $(\mathfrak{g}_-^* \otimes \mathfrak{g})_0$ consists of all linear maps on $\mathfrak{g}_-$, that preserve the grading. Thus, for $\phi \in (\mathfrak{g}_-^* \otimes \mathfrak{g})_0$, the condition $\partial_1(\phi) = 0$ means that ϕ is a derivation on $\mathfrak{g}_-$ that preserves the grading. Since ∂_0 is homogeneous of degree zero, it follows that the component of $H^1(\mathfrak{g}_-, \mathfrak{g})$ of degree zero is isomorphic to $\mathrm{Der}_{\mathrm{gr}}(\mathfrak{g}_-)/\mathfrak{g}_0$. Therefore, the cohomological condition $H^1(\mathfrak{g}_-, \mathfrak{g})_0 = \{0\}$ is equivalent to the fact that $\mathfrak{g}_0$ is isomorphic to the Lie algebra of derivations on $\mathfrak{g}_-$ that preserve the grading.

Remark. We can now also formulate the cohomological condition one needs in order to prove the equivalence between regular, normal parabolic geometries and regular infinitesimal flag structures. Namely, one needs to require that $H^1(\mathfrak{g}_-, \mathfrak{g})$ is contained in non-positive homogeneous degrees, that is $H^1(\mathfrak{g}_-, \mathfrak{g})_l = \{0\}$ for all $l \geq 1$.

Next we assume that $\mathfrak{g}$ satisfies the cohomological condition $H^1(\mathfrak{g}_-, \mathfrak{g})_0 = \{0\}$. We show that choosing $G = \mathrm{Aut}(\mathfrak{g})$, one always obtains an isomorphism $G_0 \cong \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$. For $G = \mathrm{Aut}(\mathfrak{g})$, the subgroup G_0 equals the group of automorphism on $\mathfrak{g}$ that preserve the grading. Thus, in order to get the result it remains to verify the following proposition.

Proposition. *Let $\mathfrak{g}$ be a $|k|$-graded semisimple Lie algebra such that $\mathfrak{g}_0 \cong \mathrm{Der}_{\mathrm{gr}}(\mathfrak{g}_-)$. Then one obtains an isomorphism $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}) \cong \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$, which is given by restriction.*

Proof. Restricting an automorphism of the Lie algebra $\mathfrak{g}$ to the subalgebra $\mathfrak{g}_-$ gives an homomorphism $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}) \to \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$. We show that it is indeed an isomorphism.

To show injectivity one needs to check that every element $\phi \in \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g})$ is uniquely determined by its restriction to $\mathfrak{g}_-$. The Killing form is nondegenerate on $\mathfrak{g}_i \times \mathfrak{g}_{-i}$ and thus induces an isomorphism $\mathfrak{g}_i \cong (\mathfrak{g}_{-i})^*$. By invariance of the Killing form under automorphisms, the restriction of ϕ to $\mathfrak{g}_i$ corresponds via this isomorphism to the restriction of ϕ^{-1} to $\mathfrak{g}_{-i}$. Since ϕ is an automorphism we have $\mathrm{ad}\,(\phi(A)) = \phi \circ \mathrm{ad}\,(A) \circ \phi^{-1}$ and by assumption $\mathrm{ad} : \mathfrak{g}_0 \to \mathrm{Der}_{\mathrm{gr}}(\mathfrak{g}_-)$ is an isomorphism. Hence, the restriction of ϕ to $\mathfrak{g}_0$ is uniquely determined by its restriction to $\mathfrak{g}_-$ as well.

It remains to prove surjectivity. Given $\phi \in \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$ we show that it can be extended to $\tilde{\phi} \in \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g})$. We put $\tilde{\phi}(X) := \phi(X)$ for $X \in \mathfrak{g}_-$. On $\mathfrak{p}_+$ we define $\tilde{\phi}$ to be the map induced from ϕ^{-1} via the Killing form, that is for $U \in \mathfrak{p}_+$ we have $B(\tilde{\phi}(U), X) := B(U, \phi^{-1}(X))$ for all $X \in \mathfrak{g}_-$. Finally, for $A \in \mathfrak{g}_0$ we put $\mathrm{ad}\,(\tilde{\phi}(A)) := \phi \circ \mathrm{ad}\,(A) \circ \phi^{-1} \in \mathrm{Der}_{\mathrm{gr}}(\mathfrak{g}_-)$. Since $\mathfrak{g}_0$ is isomorphic to $\mathrm{Der}_{\mathrm{gr}}(\mathfrak{g}_-)$, this defines $\tilde{\phi}$ on $\mathfrak{g}_0$. Obviously, $\tilde{\phi}$ is a linear isomorphism on $\mathfrak{g}$ and preserves the grading. To show that it is contained in $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g})$ we need to check that it is a Lie algebra homomorphism. This is certainly true for its restriction to $\mathfrak{g}_-$ since it coincides with ϕ there. For $A \in \mathfrak{g}_0$

and $X \in \mathfrak{g}_-$ we have

$$[\tilde{\phi}(A), \tilde{\phi}(X)] = \mathrm{ad}\,\big(\tilde{\phi}(A)\big)\big(\tilde{\phi}(X)\big) = \phi \circ \mathrm{ad}\,(A) \circ \phi^{-1} \circ \phi(X) = \tilde{\phi}([A, X])\,.$$

For $A, B \in \mathfrak{g}_0$ we make the following computation

$$\begin{aligned}\mathrm{ad}\,([\tilde{\phi}(A), \tilde{\phi}(B)]) &= \big[\mathrm{ad}\,\big(\tilde{\phi}(A)\big), \mathrm{ad}\,\big(\tilde{\phi}(B)\big)\big] = \big[\phi \circ \mathrm{ad}\,(A) \circ \phi^{-1}, \phi \circ \mathrm{ad}\,(B) \circ \phi^{-1}\big] \\ &= \phi \circ \big[\mathrm{ad}\,(A), \mathrm{ad}\,(B)\big] \circ \phi^{-1} = \phi \circ \mathrm{ad}\,([A, B]) \circ \phi^{-1} = \mathrm{ad}\,\big(\tilde{\phi}([A, B])\big)\,.\end{aligned}$$

Hence $[\tilde{\phi}(A), \tilde{\phi}(B)] = \tilde{\phi}([A, B])$. Next we take $X \in \mathfrak{g}_{-1}$, $U \in \mathfrak{g}_i$, $i \geq 2$ and $Y \in \mathfrak{g}_{1-i}$ and compute

$$B(\tilde{\phi}([U, X]), Y) = B([U, X], \phi^{-1}(Y)) = B(U, [X, \phi^{-1}(Y)])\,.$$

Since $X, Y \in \mathfrak{g}_-$ we know that $[X, \phi^{-1}(Y)] = \phi^{-1}\big([\tilde{\phi}(X), Y]\big)$, hence

$$B\big(U, [X, \tilde{\phi}^{-1}(Y)]\big) = B\big(\tilde{\phi}(U), [\tilde{\phi}(X), Y]\big) = B\big([\tilde{\phi}(U), \tilde{\phi}(X)], Y\big)\,.$$

It follows that $[\tilde{\phi}(U), \tilde{\phi}(X)] = \tilde{\phi}([U, X])$ as well. For the Lie bracket $\mathfrak{g}_1 \times \mathfrak{g}_{-1} \to \mathfrak{g}_0$ we apply a similar argument. We take $X \in \mathfrak{g}_{-1}$, $U \in \mathfrak{g}_1$ and $A \in \mathfrak{g}_0$. This time we need to verify that $B(\tilde{\phi}([U, X]), A)$ equals $B([U, X], \tilde{\phi}^{-1}(A))$:

$$\begin{aligned}B\big(\tilde{\phi}([U, X]), A\big) &= \mathrm{tr}\,\big(\mathrm{ad}\,(\tilde{\phi}([U, X])) \circ \mathrm{ad}\,(A)\big) \\ &= \mathrm{tr}\,\big(\phi \circ \mathrm{ad}\,([U, X]) \circ \phi^{-1} \circ \mathrm{ad}\,(A)\big) \\ &= \mathrm{tr}\,\big(\mathrm{ad}\,([U, X]) \circ \phi^{-1} \circ \mathrm{ad}\,(A) \circ \phi\big) \\ &= \mathrm{tr}\,(\mathrm{ad}\,([U, X]) \circ \mathrm{ad}\,(\tilde{\phi}^{-1}(A))) \\ &= B\big([U, X], \tilde{\phi}^{-1}(A)\big)\,.\end{aligned}$$

The rest of the argument works exactly as in the previous case. Thus, $\tilde{\phi}$ is a Lie algebra homomorphism with respect to all brackets $\mathfrak{g}_{-1} \times \mathfrak{g} \to \mathfrak{g}$. Since $\mathfrak{g}_-$ is generated by $\mathfrak{g}_{-1}$, the case $\mathfrak{g}_- \times \mathfrak{g} \to \mathfrak{g}$ follows by induction: Assume that we already know $[\tilde{\phi}(X), \tilde{\phi}(B)] = \tilde{\phi}([X, B])$ for all $X \in \mathfrak{g}_i$, $-l < i < 0$, and $B \in \mathfrak{g}$. Take $X \in \mathfrak{g}_{-l}$, then $X = [Y, Z]$, where $Y \in \mathfrak{g}_{-1}$ and $Z \in \mathfrak{g}_{-l+1}$, and

$$\tilde{\phi}([[Y, Z], B]) = \tilde{\phi}\big(-[[B, Y], Z] - [[Z, B], Y]\big) = \big[\tilde{\phi}([Y, Z]), \tilde{\phi}(B)\big]\,.$$

The only remaining case is $\mathfrak{p}_+ \times (\mathfrak{g}_0 \oplus \mathfrak{p}_+) \to \mathfrak{p}_+$. A computation similar to the $\mathfrak{g}_{-1} \times \mathfrak{p}_+ \to \mathfrak{p}_+$ case shows that $\tilde{\phi}$ is a Lie algebra homomorphism with respect to these brackets as well. □

We get the following corollary:

Corollary. *Let $\mathfrak{g}$ be a real $|k|$-graded semisimple Lie algebra such that no simple ideal of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$ and $H^1(\mathfrak{g}_-, \mathfrak{g})_l = \{0\}$, for all $l \geq 0$. Put $G = \mathrm{Aut}(\mathfrak{g})$ and denote by P the parabolic subgroup of G corresponding to the grading. Then there is an eqivalence of categories between regular, normal parabolic geometries of type (G, P) and filtrations of the tangent bundle such that each symbol algebra is isomorphic to $\mathfrak{g}_-$.*

2.2. It can be shown that the condition $H^1(\mathfrak{g}_-, \mathfrak{g})_l = \{0\}$, $l \geq 0$, is satisfied for most semisimple $|k|$-graded Lie algebras $\mathfrak{g}$. A complete list of $|k|$-graded semisimple Lie algebras such that the condition is not satisfied can be found in [8].

We briefly discuss how to verify that $H^1(\mathfrak{g}_-, \mathfrak{g})$ is contained in negative homogeneous degrees. It is not difficult to see that the cohomology groups $H^n(\mathfrak{g}_-, \mathfrak{g})$ and $H^n(\mathfrak{p}_+, \mathfrak{g})$, where $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, are $\mathfrak{g}_0$-modules. Moreover, there is an isomorphism $H^n(\mathfrak{g}_-, \mathfrak{g}) \cong (H^n(\mathfrak{p}_+, \mathfrak{g}))^*$. Kostant's version of the Bott-Borel-Weil theorem [7] gives a description of the $\mathfrak{g}_0$-module $H^*(\mathfrak{p}_+, \mathfrak{g})$ in terms of the Hasse diagram. More precisely, it says that the set of all irreducible components of $H^*(\mathfrak{p}_+, \mathfrak{g})$ is in bijective correspondence with the Hasse diagram and that the irreducible component corresponding to an element w in the Hasse diagram is contained in $H^{l(w)}(\mathfrak{p}_+, \mathfrak{g})$, where $l(w)$ denotes the length of w. A highest weight vector for the irreducible component corresponding to an element w in the Hasse diagram is given by the cohomology class of $F \otimes s_\alpha(A)$, where F is the wedge product of one nonzero element from each of the root spaces $\mathfrak{g}_\alpha$, α a positive root such that $w^{-1}\alpha$ is negative, and A is contained in the root space corresponding to the maximal root. Using this theorem, one obtains an algorithm to compute the cohomologies $H^*(\mathfrak{p}_+, \mathfrak{g})$ explicitly, see for example [1].

To prepare for the examples we will discuss later, we deduce a criterion whose verification is sufficient to see that in those cases $H^1(\mathfrak{g}_-, \mathfrak{g})$ is contained in negative homogeneous degrees. Let $\mathfrak{g}$ be a $|k|$-graded semisimple Lie algebra, the grading given by the Σ-height, where Σ is a subset of simple roots. We assume that each $\alpha \in \Sigma$ is orthogonal to the maximal root. Kostant's version of the Bott-Borel-Weil theorem says that irreducible components of $H^1(\mathfrak{p}_+, \mathfrak{g})$ are in bijective correspondence with reflections s_α, $\alpha \in \Sigma$. A highest weight vector for the irreducible component corresponding to a reflection s_α is given by the cohomology class of $F \otimes s_\alpha(A)$, where F is contained in $\mathfrak{g}_\alpha$ and A is contained in the root space corresponding to the maximal root β. Since $\mathfrak{g}_{-\alpha} \subset \mathfrak{g}_{-1}$, the homogeneous degree of such a weight vector is given by $l - 1$, the integer l denoting the Σ-height of $s_\alpha(\beta)$. By assumption, α is orthogonal to β which implies that $s_\alpha(\beta) = \beta$. Hence, the highest weight vector has homogeneous degree $k-1$. Thus, $H^1(\mathfrak{p}_+, \mathfrak{g})$ is contained in degree $k-1$ and consequently $H^1(\mathfrak{g}_-, \mathfrak{g}) \cong (H^1(\mathfrak{p}_+, \mathfrak{g}))^*$ is contained in homogeneous degree $1 - k$. In case k is greater or equal to 2, it follows that $H^1(\mathfrak{g}_-, \mathfrak{g})_l = \{0\}$ for all $l \geq 0$.

3. Examples

Finally, we apply the result to give examples of parabolic geometries that are uniquely determined by filtrations on the tangent bundle.

Example 1. The first example goes back to Elie Cartan [6]. In that paper, he considered distributions $\mathcal{H}$ of rank two on five-dimensional manifolds M, that satisfy the generic conditions that

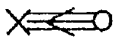
$$\mathcal{H}^1 := \mathcal{H} + \{[\xi, \eta] : \xi, \eta \in \Gamma(\mathcal{H})\}$$

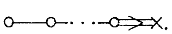
is a distribution of rank three and

$$\mathcal{H}^1 + \{[\xi, \eta] : \xi \in \Gamma(\mathcal{H}), \eta \in \Gamma(\mathcal{H}^1)\} = TM.$$

Then $\mathcal{H} \subset \mathcal{H}^1 \subset TM$, is a filtration such that $\mathcal{L} : \Lambda^2\mathcal{H} \to \mathcal{H}^1/\mathcal{H}$ is an isomorphism and $\mathcal{L} : \mathcal{H} \otimes \mathcal{H}^1/\mathcal{H} \to TM/\mathcal{H}^1$ is an isomorphism.

In order to determine the parabolic geometry corresponding to such a filtration, we look at the split real form of the exceptional Lie algebra G_2, which we denote by $\mathfrak{g}$. We choose a simple system $\Delta^0 = \{\alpha_1, \alpha_2\}$, then $\Delta^+ = \{3\alpha_1 + 2\alpha_2, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_1, \alpha_2\}$. Next, we consider the grading on $\mathfrak{g}$ given by the Σ-height, $\Sigma = \{\alpha_1\}$. Then $\mathfrak{g}_- = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, where $\mathfrak{g}_{-3}$ is the direct sum of the root spaces corresponding to $-(3\alpha_1 + 2\alpha_2)$ and $-(3\alpha_1 + \alpha_2)$, $\mathfrak{g}_{-2}$ is the root space corresponding to $-(2\alpha_1 + \alpha_2)$ and $\mathfrak{g}_{-1}$ is the direct sum of the root spaces corresponding to $-(\alpha_1 + \alpha_2)$ and $-\alpha_1$. Since $-(\alpha_1 + \alpha_2) - \alpha_1 = -(2\alpha_1 + \alpha_2)$ the Lie bracket $[\ ,\ ] : \Lambda^2 \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is surjective and by equality of dimensions it is an isomorphism. Similarily, we see that $[\ ,\ ] : \mathfrak{g}_{-1} \otimes \mathfrak{g}_{-2} \to \mathfrak{g}_{-3}$ is an isomorphism.

Hence, $\mathrm{gr}(T_xM) = \mathcal{H}_x \oplus \mathcal{H}_x^1/\mathcal{H}_x \oplus T_xM/\mathcal{H}_x^1$ corresponding to a filtration as above, is isomorphic to $\mathfrak{g}_-$ for all $x \in M$. Using the criterion explained in 2.2 one shows that $H^1(\mathfrak{g}_-, \mathfrak{g})$ is contained in negative homogeneous degree. One only needs to verify that α_1 is orthogonal to the maximal root $3\alpha_1 + 2\alpha_2$: Since $2\frac{\langle \alpha_1, \alpha_2 \rangle}{\langle \alpha_1, \alpha_1 \rangle} = -3$, we see that $\langle \alpha_1, 3\alpha_1 + 2\alpha_2 \rangle = 3\langle \alpha_1, \alpha_1 \rangle + 2\langle \alpha_1, \alpha_2 \rangle = 0$. It follows, that generic distributions of rank two on five-dimensional manifolds are equivalent to normal regular parabolic geometries of type (G, P), where G is the split real form of the exceptional Lie group G_2 and P the parabolic subgroup corresponding to .

Example 2. We consider $\mathfrak{g} = \mathfrak{so}(n+1, n)$, $n \geq 3$, and the grading corresponding to $\Sigma = \{\alpha_n\}$. One verifies that $\mathfrak{g}_- = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$, where $\mathfrak{g}_{-1}$ is n-dimensional, $\mathfrak{g}_{-2}$ is $n(n-1)/2$-dimensional and $[\ ,\ ] : \Lambda^2 \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is an isomorphism. It is known that the adjoint representation of $\mathfrak{so}(2n+1, \mathbb{C})$ is isomorphic to $\Lambda^2 \mathbb{C}^{2n+1}$. Thus, the maximal root is the second fundamental weight, which is orthogonal to α_n, and we can apply the criterion explained in 2.2 to see that $H^1(\mathfrak{g}_-, \mathfrak{g})_l = \{0\}$, for all $l \geq 0$. Hence distributions $\mathcal{H} \subset TM$ of rank n on $n(n+1)/2$-dimensional manifolds such that $\mathcal{L} : \Lambda^2 \mathcal{H} \to TM/\mathcal{H}$ is an isomorphism, are equivalent to parabolic geometries of type (G, P), where $G = \mathrm{Aut}(\mathfrak{so}(n+1, 1))$, which is isomorphic to the connected component of the identity in $SO(n+1, 1)$, and P is the parabolic subgroup corresponding to .

Example 3. Finally, we consider $\mathfrak{g} = \mathfrak{sp}(p+1, q+1)$ with the grading corresponding to the second simple root. It turns out that $\mathfrak{g}_-$ is in this case isomorphic to $\mathbb{H}^n \oplus \mathrm{Im}\mathbb{H}$ with the Lie bracket given by the imaginary part of a quaternionic hermitian form of signature (p, q). The adjoint representation of $\mathfrak{sp}(2n, \mathbb{C})$ is isomorphic to $S^2\mathbb{C}^{2n}$. Hence the maximal root is twice the first fundamental weight. Thus, $H^1(\mathfrak{g}_-, \mathfrak{g})_l = \{0\}$, for all $l \geq 0$, also in this case. It follows, that distributions of rank $4n$ on $(4n+3)$-dimensional manifolds such that for all x the Levi bracket $\mathcal{L}_x$ is the imaginary part of a quaternionic hermitian form of signature (p, q) are equivalent to regular normal parabolic geometries of type (G, P), where $G = \mathrm{Aut}(\mathfrak{g}) = PSp(p+1, q+1)$ and P is the parabolic subgroup corresponding to the second simple root.

In dimension seven these distributions form an open subset in the set of all distributions of rank four. This is not true, however, in higher dimensions, though in higher dimensions there are lots of examples as well. Olivier Biquard considered such distributions where $\mathrm{gr}(T_xM)$ is isomorphic to the quaternionic Heisenberg algebra, i.e. the case where the Levi bracket has signature $(n, 0)$, and called them quaternionic contact structures. Using a twistorial construction, he showed that there are many examples of such structures, see [2].

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