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THE POISSON TRANSFORM FOR HIGHER ORDER DIFFERENTIAL OPERATORS

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ABSTRACT. The Poisson transform maps sections of vector bundles over a parabolic geometry G/P to sections of vector bundles over a symmetric space G/K. The parabolic geometry can be understood as a (part of the) boundary of the symmetric space. We show that the Poisson transform intertwines invariant differential operators on these spaces, generalizing the results of Ørsted and Korányi and Reimann.

Invariant differential operators in parabolic geometries were studied in recent years by several authors. The articles [12, 15, 10] provide various proofs of the classification of invariant differential operators of the first order, the article [3] contains a construction of some strongly invariant operators of higher order and gives a simple algorithm for expressing them in terms of a Weyl structure. These operators were first defined on vector bundles over homogeneous spaces of the type G/P where G is semisimple and P parabolic, but they generalize to arbitrary curved parabolic Cartan geometries, as shown in [4, 2]. These operators form sequences that are in fact complexes in the flat case, the so called BGG complexes. The BGG operators and the spaces between them they operate have several interesting interpretations from both the mathematical and physical point of view, see for example [5, 1, 6].

The classification of differential operators on symmetric spaces is simpler and some information concerning the first order case can be found in [14]. In this article we deal with the relation between invariant differential operators on a symmetric space G/K and on a parabolic geometry that may be viewed as its boundary. A typical example we have in mind is the hyperbolic space $\mathrm{SO}(n+1,1)/\mathrm{SO}(n+1)$ considered as a subspace in $\mathbb{R}^{n+1,1}$ and of the projectivization of the null cone in $\mathbb{R}^{n+1,1}$ that can be identified with the conformal sphere $\mathrm{SO}(n+1,1)/P$ where $P = \mathrm{CO}(n) \ltimes (\mathbb{R}^*)^n$. There is an integral operator analogous to the classical Poisson transform [13, 11] between vector bundles over the boundary parabolic geometry and the interior symmetric space. It was shown in [9] and [12] that the Poisson transform intertwines the first order differential operators over the interior and over the boundary, together with some applications of this fact, namely for extensions of quasiconformal maps. In fact, the authors of [9] use also an explicitly defined second order operator on the boundary that is also

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intertwined with a composition of two first order operators over the interior. This suggests that the result of [12] can be generalized with the help of the Ricci corrected derivatives of [3] to a more general result that Poisson transform intertwines operators of arbitrary order. This really shows up to be the case.

As we noted before, the operators on the boundary (and on any parabolic geometry) form a rich structure of BGG sequences. It would be interesting to check whether the Poisson transform makes possible to extend some of these data to the interior, for example allows to construct there complexes of differential operators or, on the other hand, to use the properties of the operators in the interior to obtain some information about operators on the boundary similarly as in the programme of ambient metric construction [7, 8].

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1. Preliminaries and notation

We shall consider a symmetric space where G is a semisimple Lie group and K its maximal compact subgroup. Then Cartan involution θ gives the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ onto ± 1 eigenspaces. The tangent bundle can be written as $T(G/K) = G \times_K \mathfrak{q}$.

G has the Iwasawa decomposition KAN which translates into the Lie algebra level as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{q}$. If we denote the centralizer of A in K by M then $P := MAN$ is a minimal parabolic subgroup of G, i.e. G/P is the maximal boundary of the symmetric space G/K.

The Lie algebra $\mathfrak{g}$ then splits as $\bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ where $\bar{\mathfrak{n}} = \theta\mathfrak{n}$. The tangent bundle of the boundary is then $T(G/P) = G \times_P \bar{\mathfrak{n}}$. Sometimes $\mathfrak{m} \oplus \mathfrak{a}$ is denoted also by $\mathfrak{g}_0$, $\mathfrak{n}$ by $\mathfrak{g}_+$ and $\bar{\mathfrak{n}}$ by $\mathfrak{g}_-$. In fact there is an element $E \in \mathfrak{a}$ that has integral eigenvalues acting on $\mathfrak{g}$ such that $[E, X] = jX$, j is positive for $X \in \mathfrak{g}_+$ and negative for $X \in \mathfrak{g}_-$. This element induces for certain $k \in \mathbb{N}$ a $|k|$-grading of $\mathfrak{g}$, i.e. a splitting

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$$

that satisfies $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, hence E is called the grading element.

2. Invariant operators in parabolic geometries

Definition 1 (Cartan connection). A Cartan connection of type $(\mathfrak{g}, P)$ on a manifold N is a 1-form $\eta : T\mathcal{G} \to \mathfrak{g}$ defined on a principal P-bundle $\pi : \mathcal{G} \to N$ such that

(1) $(\forall y \in \mathcal{G}) \quad \eta_y : T_y\mathcal{G} \to \mathfrak{g}$ is an isomorphism
(2) $(\forall y \in \mathcal{G}, X \in \mathfrak{p}) \quad \eta_y^{-1}(X) = \xi_{X,y}$ where ξ_X is the left invariant vector field generated by X
(3) $(\forall p \in P) \quad Ad(p) \cdot r_p^* \eta = \eta$ where r_p stands for the right P-action on $\mathcal{G}$

Let $\mathbb{V}$ be a finite dimensional P-module carrying a representation λ and $V := \mathcal{G} \times_P \mathbb{V}$ the corresponding associated vector bundle. As usual, we will identify the space of sections $C^\infty(M,V)$ with the space of P-equivariant maps $C^\infty(\mathcal{G},\mathbb{V})^P$, i.e. satisfying $f(y) := \lambda(p)(f\circ r_p)(y)$.

Definition 2 (Invariant derivative). Let $f \in C^\infty(\mathcal{G},\mathbb{V})^P$, $X \in \bar{\mathfrak{n}}$. Then

$$\nabla^\eta : C^\infty(\mathcal{G},\mathbb{V})^P \to C^\infty(\mathcal{G},\bar{\mathfrak{n}}^*\otimes\mathbb{V})$$
$$\nabla^\eta_X f := df(\eta^{-1}(X))$$

is called the *invariant derivative on V*.

Our interest will be focused on the flat (homogeneous) case where η is the Maurer-Cartan form and the invariant derivative is given by the expression

$$\nabla^\eta_X f = \frac{d}{dt}\Big|_0 (f\circ r_{\exp tX})$$

with the action $r_{\exp tX}$ on G given by the group multiplication from right.

The problem with the invariant derivative is that it does not map to $C^\infty(\mathcal{G},\bar{\mathfrak{n}}^*\otimes\mathbb{V})^P$. To remedy this, we start to work with semiholonomic jet modules. First we define a P-module structure on $\bar{J}^1\mathbb{V} = \mathbb{V}\oplus(\bar{\mathfrak{n}}\otimes\mathbb{V})$ such that for $(\phi_0,\phi_1)\in\bar{J}^1\mathbb{V}$

$$[Z\cdot(\phi_0,\phi_1)](X) = (Z\cdot\phi_0, Z\cdot\phi_1(X) - \phi_1([Z,X]_{\bar{\mathfrak{n}}}) - \lambda([Z,X]_{\mathfrak{p}})\phi_0)$$

where $Z\in\mathfrak{p}$. It is easy to show that the map $j^1_\eta : f \to (f,\nabla^\eta f)$ then maps $C^\infty(\mathcal{G},\mathbb{V})^P$ to $C^\infty(\mathcal{G},\mathbb{V}\oplus(\bar{\mathfrak{n}}^*\otimes\mathbb{V}))^P$. This can be a beginning of an inductive definition

Definition 3 (Semiholonomic jet modules). The first order semiholonomic jet module $\bar{J}^1\mathbb{V}$ is a set of pairs $(\phi_0,\phi_1)\in\mathbb{V}\oplus(\bar{\mathfrak{n}}\otimes\mathbb{V})$ with the action of $\mathfrak{p}$ given by the formula above. Let us suppose that $\bar{J}^{k-1}$ is well defined and consider the action of $\mathfrak{p}$ given by the first order action on

$$((\phi_0,\phi_1,\dots,\phi_{k-1}),(\phi'_1,\phi'_2,\dots,\phi'_{k-1},\phi_k))\in\bar{J}^1(\bar{J}^{k-1}\mathbb{V})$$

The k-th order semiholonomic jet module $\bar{J}^k\mathbb{V} := \oplus_0^k(\bar{\mathfrak{n}}^*)^i\otimes\mathbb{V}$ is a set of vectors $(\phi_0,\phi_1,\dots,\phi_k)$ identified with a P-submodule of $\bar{J}^1\bar{J}^{k-1}\mathbb{V}$ of elements satisfying $\phi_i = \phi'_i$ for $1\le i\le k-1$.

The inclusion j^k_η of J^kV into $\mathcal{G}\times_P\bar{J}^k\mathbb{V}$ maps j^kf to $(f,\ \nabla^\eta f,\dots,\ (\nabla^\eta)^k f)$ and the image of $f\in C^\infty(\mathcal{G},\mathbb{V})^P$ is in $C^\infty(\mathcal{G},\bar{J}^k\mathbb{V})^P$, thus it is a well defined map of sections of bundles.

Definition 4 (Strongly invariant operators). Let $\mathbb{V}$ and $\mathbb{W}$ be P-modules and $\Phi : J^k\mathbb{V}\to\mathbb{W}$ be a P-homomorphism. Then Φ induces a bundle map $\Phi : \mathcal{G}\times_P\bar{J}^k\mathbb{V}\to\mathcal{G}\times_P\mathbb{W}$ and thus an invariant differential operator $\Phi\circ j^k_\eta$ from V to W.

The usual strategy for constructing strongly invariant operators ([4],[3]) is to consider Φ as a G_0-morphism first and then to find algebraic conditions for it being also a P-morphism. $\mathbb{W}$ is an irreducible P-module, so the action of the nilpotent part of P is trivial. On the other hand $\bar{J}^k\mathbb{V}$ is G_0-reducible and the P-action maps between its G_0-components. The image of the P-action is a G_0-submodule of $\bar{J}^k\mathbb{V}$ that must be anihilated by Φ to obtain P-equivariancy. This leads in [15] to the following lemma for first order operators:

Lemma 1. *Let $\mathbb{V}$, $\mathbb{W}$ be irreducible P-modules. Then a G_0-module homomorphism $\Phi : \bar{J}^1\mathbb{V} \to \mathbb{W}$ is a P-module homomorphism iff Φ factors through $\mathbb{V} \oplus (\mathfrak{g}_1^* \otimes \mathbb{V})$ and for all $Z \in \mathfrak{g}_1$, $v \in \mathbb{V}$*

$$\Phi\left(\sum_\alpha Y^\alpha \otimes [Z, X_\alpha] \cdot v\right) = 0\,,$$

where Y^α is a basis of $\mathfrak{g}_1$ and X_α is the dual basis of $\mathfrak{g}_{-1}$.

Further considerations in [15] consisting mainly in computing Casimir operator eigenvalues on irreducible G_0-components of the so called restricted jet module $\mathbb{V} \oplus (\mathfrak{g}_1^* \otimes \mathbb{V})$ give us the characterization of invariant first order operators:

Theorem 1. *Let $\mathfrak{g}$ be a graded semisimple Lie algebra and $\mathfrak{g}^{\mathbb{C}}$ its graded complexification. Then $\mathfrak{g}_j = \mathfrak{g} \cap \mathfrak{g}_j^{\mathbb{C}}$. Let $\mathbb{V}_\lambda$ be a complex irreducible representation of $\mathfrak{g}_0$ with highest weight λ and let $\mathfrak{g}_1^{\mathbb{C}} = \sum_j \mathfrak{g}_1^j$ be a decomposition of $\mathfrak{g}_1^{\mathbb{C}}$ into irreducible $\mathfrak{g}_0$-submodules and let α_j be highest weights of $\mathfrak{g}_1^j$. Suppose that*

$$\mathfrak{g}_1 \otimes_{\mathbb{R}} \mathbb{V}_\lambda = \mathfrak{g}_1^{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{V}_\lambda = \sum_j \sum_{\mu_j} \mathbb{V}_{\mu_j}^j$$

be a decomposition of the product into irreducible $\mathfrak{g}_0$-modules and let π_{λ,μ_j} be the corresponding projections. Let us denote by ρ_0 the half-sum of positive roots for $\mathfrak{g}_0$ and let us define constants c_{λ,μ_j} by

$$c_{\lambda,\mu_j} = \frac{1}{2}[(\mu_j, \mu_j + 2\rho_0) - (\lambda, \lambda + 2\rho_0) - (\alpha_j, \alpha_j + 2\rho_0)]$$

Then the operator $D_{j,\mu_j} : \pi_{\lambda,\mu_j} \circ \nabla^\eta$ is an invariant first order differential operator iff $c_{\lambda,\mu_j} = 0$. Moreover, all first order invariant differential operators acting on sections of V_λ are obtained (modulo a scalar multiple and curvature terms) in such a way.

This result was generalized in [3] for a certain class of operators of higher order. Before we state it, we shall define $\mathbb{V}_b$ for a k-tuple $b = (b_1, \dots, b_k)$. It is simply the b_k-th irreducible component of $\otimes^k \mathfrak{g}_1^* \otimes \mathbb{V}_{b_{k-1}}$ where $\mathbb{V}_{b_{k-1}}$ is a b_{k-1}-th component of $\otimes^{k-1}\mathfrak{g}_1^* \otimes \mathbb{V}_{b_{k-2}}$ and so on, $\mathbb{V}_1$ is a b_1-th component of $\mathfrak{g}_1^* \otimes \mathbb{V}$ for an arbitrarily chosen numbering of components.

Theorem 2. *Let α be a positive root in $\mathfrak{g}_1^*$. In the case that $\mathfrak{g}$ has roots of different lengths, we shall suppose that α is a long root. Let λ, μ be two dominant integral weights of $\mathfrak{g}_0$ satisfying*

$$\mu + \rho_0 = \sigma_\alpha(\lambda + \rho_0) = \lambda + \rho_0 - \langle \lambda + \rho_0, \alpha \rangle \alpha\,.$$

Interchanging λ and μ if necessary we can suppose that the integer $k := -\langle \lambda + \rho_0, \alpha \rangle$ is positive. Then:

1. *There is a unique irreducible component $\mathbb{V}_\mu$ with highest weight μ in $(\otimes^k \bar{\mathfrak{n}}^*) \otimes \mathbb{V}_\lambda$. Furthermore, this component belongs to $S^k\bar{\mathfrak{n}}^* \otimes \mathbb{V}_\lambda$ and is of the form $\mathbb{V}_b$ where $\mathbb{V}_{b_j} = \mathbb{V}_{\lambda + j\alpha}$ for $b = (b_1, \dots, b_k)$.*
2. *If $\pi : \bar{J}^k\mathbb{V}_\lambda \to \mathbb{V}_\mu$ is the corresponding G_0-invariant projection, then π is in fact a P-homomorphism and the operator $\pi \circ (\nabla^\eta)^k$ is an invariant differential operator of order k from sections of V_λ to sections of V_μ.*

Remark 1. This theorem is in fact in [3] formulated in terms of the differential operators $D^{(j)}$ called there Ricci corrected derivatives. This involves defining an isomorphism of vector bundles mapping $(f, \nabla^{\eta} f, \ldots, (\nabla^{\eta})^k f) \in \mathcal{G} \times_P \bar{J}^k \mathbb{V}$ to $(f, D^{(1)}f, \ldots, D^{(k)}f) \in \oplus_0^k (\otimes^j T^*M) \otimes V$. This mapping depends on the Weyl structure that we choose but the operator $\pi \circ D^{(k)}$ that we obtain from this is an invariant operator independent of the Weyl structure.

3. Invariant operators on symmetric spaces

Next we turn to the invariant differential operator on the interior symmetric space G/K. For our purposes it would be convenient to consider a certain simple subclass of them, which nevertheless contains all first order invariant differential operators between any two vector bundles $G \times_K E_1$ and $G \times_K E_2$, for $\mathbb{E}_1$, $\mathbb{E}_2$ irreducible K-modules, as shown in [14].

Definition 5 (Generalized Stein-Weiss gradient). Let $\mathbb{E}$ be a K-module, $f \in C^\infty(G, \mathbb{E})^K$, K be acting on $\mathfrak{q}$ by Ad. Then there is a first order differential operator

$$\nabla f(g)(X) := Xf(g) = \frac{d}{dt}\Big|_0 f(g \exp(tX))$$

where $X \in \mathfrak{q}$. Moreover $\nabla f \in C^\infty(G, \mathfrak{q}^* \otimes \mathbb{E})^K$.

We can simply check that ∇f is really K-equivariant. If we denote by ρ the representation of K on $\mathbb{E}$, then for $g \in G$, $k \in K$, $f \in C^\infty(G, \mathbb{E})^K$ we have

$$\begin{aligned}
\nabla_X f(gk) &= \frac{d}{dt}\Big|_0 f(gk\, e^{tX}) \\
&= \frac{d}{dt}\Big|_0 f(g \exp^{\mathrm{Ad}(k)X}\, k) \\
&= \rho(k^{-1}) \frac{d}{dt}\Big|_0 f(g\, e^{\mathrm{Ad}(k)X}) \\
&= \rho(k^{-1}) \nabla_{\mathrm{Ad}(k)X} f(g)
\end{aligned}$$

All first order differential operators between vector bundles associated to irreducible K-modules are then obtained by projection onto irreducible components of $\mathfrak{q}^* \otimes \mathbb{E}$. Moreover, the operator $\mathrm{proj}_j \circ (\nabla)^k$, where proj_j is a projection onto the j-th component of $(\otimes^k \mathfrak{q}^*) \otimes \mathbb{E}$ is an invariant differential operator.

4. Poisson transform

Definition 6 (Poisson transform). Let $\mathbb{V}_{\lambda,\nu}$ be P-module, $\mathbb{E}_\sigma$ a K-module and $I_\lambda : \mathbb{V}_{\lambda,\nu} \to \mathbb{E}_\sigma$ be an M-invariant map. Then there is a map $\mathcal{P} : C^\infty(G, \mathbb{V}_{\lambda,\nu})^P \to C^\infty(G, \mathbb{E}_\sigma)^K$ given by

$$(\mathcal{P}f)(g) = \int_K \sigma(k) I_\lambda(f(gk))\, dk$$

Taking into account invariance of the Haar measure, we can readily check that the Poisson transform maps into K-equivariant maps:

$$((\mathcal{P}f) \circ r_{k'})(g) = \int_K \sigma(k'^{-1}) \sigma(k'k) I_\lambda(f(gk'k)) dk = \sigma(k'^{-1})(\mathcal{P}f)(g),$$

moreover regardless of the P-equivariance of f.

To deal with objects like $\mathcal{P}((\nabla^\eta)^i f)$ we must define the map I on $\otimes^i\mathfrak{n}\otimes\mathbb{V}_{\lambda,\nu}$. The convenient definition is $(I_\theta)^i\times I_\lambda$ where $I_\theta=\mathrm{Id}-\theta$. This really maps $X\in\bar{\mathfrak{n}}$ into $\mathfrak{q}$, since $\theta(I_\theta(X))=-I_\theta(X)$ and $\mathfrak{q}$ is exactly the (-1)-eigenspace of θ.

Let us now consider a representation (λ',ν') such that there exists an invariant differential operator of order j between $C^\infty(G/P,\mathbb{V}_{\lambda,\nu})^P$ and $C^\infty(G/P,\mathbb{V}_{\lambda',\nu'})^P$. This means in particular that $\mathrm{Hom}_M(\mathbb{V}_{\lambda',\nu'},\otimes^j\mathfrak{n}\otimes\mathbb{V}_{\lambda,\nu})$ is nontrivial and since we have a map $(I_\theta)^j\otimes I_\lambda:\otimes^j\mathfrak{n}\otimes\mathbb{V}_{\lambda,\nu}\to\otimes^j\mathfrak{q}^*\otimes\mathbb{E}_\sigma$, also $\mathrm{Hom}_M(\mathbb{V}_{\lambda',\nu'},\otimes^j\mathfrak{q}^*\otimes\mathbb{E}_\sigma)$ is nonzero. Thus there is an irreducible K-submodule $\mathbb{E}_{\sigma'}$ of $\otimes^i\mathfrak{q}^*\otimes\mathbb{E}_\sigma$ such that there exists a M-invariant $I_{\lambda'}$ such that the diagram

$$\begin{array}{ccc}\mathbb{V}_{\lambda',\nu'} & \longrightarrow & \otimes^i\mathfrak{n}\otimes\mathbb{V}_{\lambda,\nu}\\ {\scriptstyle I_{\lambda'}}\downarrow & & \downarrow{\scriptstyle (I_\theta)^i\otimes I_\lambda}\\ \mathbb{E}_{\sigma'} & \longrightarrow & \otimes^i\mathfrak{q}^*\otimes\mathbb{E}_\sigma\end{array}$$

where the horizontal maps are injections of irreducible constituents, is commutative.

Theorem 3. *The diagram*

$$\begin{array}{ccccc}C^\infty(G,\mathbb{V}_{\lambda,\nu})^P & \xrightarrow{j^i_\eta} & C^\infty(G,\bar{J}^i\mathbb{V}_{\lambda,\nu})^P & \longrightarrow & C^\infty(G,\mathbb{V}_{\lambda',\nu'})^P\\ {\scriptstyle\mathcal{P}}\downarrow & & {\scriptstyle\mathcal{P}}\downarrow & & \downarrow{\scriptstyle\mathcal{P}}\\ C^\infty(G,\mathbb{E}_\sigma)^K & \xrightarrow[j^i]{} & C^\infty(G,\bar{J}^i\mathbb{E}_\sigma)^K & \longrightarrow & C^\infty(G,\mathbb{E}_{\sigma'})^K\end{array}$$

is commutative. In particular its outer square expresses the fact that Poisson transform intertwines the two invariant operators $\mathrm{proj}_{\lambda',\nu'}(\nabla^\eta)^i$ *and* $\mathrm{proj}_\sigma(\nabla)^i$.

Proof. Let us denote by $\sigma_{(\mathfrak{n})^i\mathbb{E}}$ the tensor product representation $\otimes^i\mathrm{Ad}^*\otimes\sigma$ on $\otimes^i\mathfrak{q}^*\otimes\mathbb{E}_\sigma$ and suppose $g\in G$, $X_1,\dots X_i\in\bar{\mathfrak{n}}$. Then

$$\begin{aligned}&\mathcal{P}((\nabla^\eta)^i f)(g,X_1,\dots,X_i)\\ &=\int_K\sigma_{(\mathfrak{n})^i\mathbb{E}}(k)(I_\theta)^i\otimes I_\lambda((\nabla^\eta)^i f)(gk)dk(X_1,\dots,X_i)\\ &=\int_K\sigma_{(\mathfrak{n})^i\mathbb{E}}(k)\left[(I_\theta)^i\otimes I_\lambda\right]\frac{d}{dt}\Big|_0 f(gk\,\mathrm{e}^{t_1X_1}k^{-1}k\dots k^{-1}k\,\mathrm{e}^{t_iX_i}k^{-1}k)dk\\ &=\int_K\sigma(k)I_\lambda\frac{d}{dt}\Big|_0 f(g\,\mathrm{e}^{t_1\mathrm{Ad}(k^{-1})I_\theta\mathrm{Ad}(k)X_1}\dots\mathrm{e}^{t_i\mathrm{Ad}(k^{-1})I_\theta\mathrm{Ad}(k)X_i}k)dk\\ &=\int_K\sigma(k)I_\lambda\frac{d}{dt}\Big|_0 f(g\,\mathrm{e}^{t_1I_\theta X_1}\dots\mathrm{e}^{t_iI_\theta X_i}k)dk\\ &=((\nabla)^i(\mathcal{P}f))(g,I_\theta X_1,\dots,I_\theta X_i)\end{aligned}$$

Here we write shortly $\frac{d}{dt}|_0$ for $\frac{d}{dt_1}|_0\dots\frac{d}{dt_1}|_0$ and use $(\mathrm{Ad}^*(k)\omega)(X)=\omega(\mathrm{Ad}(k^{-1})X)$ for a 1-form ω and commutativity of θ and $\mathrm{Ad}(k)$. This gives the commutativity of the left square and the right one is commutative already on the algebraic level. □

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