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LOWER BOUNDS FOR THE EIGENVALUES OF THE BASIC DIRAC OPERATOR

SEOUNG DAL JUNG

ABSTRACT. This talk is a survay on the eigenvalue estimates of the basic Dirac operator on the Riemannian manifold with the transverse spin foliation, which is based on the works of the author[9, 10, 11].

1. INTRODUCTION

In 1963, A. Lichnerowicz [18] proved that on a Riemannian spin manifold the square of the Dirac operator D is given by

$$D^2 = \Delta + \frac{\sigma}{4}, \tag{1.1}$$

where Δ is the positive spinor Laplacian and σ the scalar curvature. In 1980, Th. Friedrich [5] gave a lower bound for the square for the eigenvalues of the Dirac operator D. In fact, by using a suitable Riemannian spin connection, he proved the inequality

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M \sigma \tag{1.2}$$

on manifolds (M^n, g) with positive scalar curvature $\sigma > 0$. He also proved, in the limiting case, that the manifold is an Einstein. The inequality (1.2) has been improved in several directions by many authors [2, 3, 7, 8, 14, 15, 16].

In this talk, we estimate the lower bound of the eigenvalues for the basic Dirac operator D_b on the foliated Riemannian manifold, which are defined by J. Brüning and F. W. Kamber [4, 6]. They obtained the Lichnerowicz type formula on the transverse spin foliation with the basic-harmonic mean curvature form κ;

$$D_b^2 = \nabla_{\rm tr}^* \nabla_{\rm tr} + \frac{1}{4} K_\sigma, \tag{1.3}$$

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where $K_\sigma = \sigma^\nabla + |\kappa|^2$, σ^∇ the transversal scalar curvature of $\mathcal{F}$ and κ the mean curvature form of $\mathcal{F}$. By using the similar method to ordinary case, we obtain the following theorem which is corresponding to (1.2).

Theorem 1.1 ([9]). *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with the transverse spin foliation $\mathcal{F}$ of codimension $q > 1$ and bundle-like metric g_M such that κ is basic-harmonic. Assume $K_\sigma \geq 0$. Then the eigenvalue λ of the basic Dirac operator D_b satisfies*

$$\lambda^2 \geq \frac{1}{4}\frac{q}{q-1}K_\sigma^0, \tag{1.4}$$

where $K_\sigma^0 = \inf_M K_\sigma$.

By transversally conformal change of the metric g_M, we have the following sharp estimation, which is corresponding to the result of Hijazi [7] in ordinary manifold.

Theorem 1.2 ([11]). *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and bundle-like metric g_M such that κ is basic-harmonic. If the transversal scalar curvature satisfies $\sigma^\nabla \geq 0$, then we have*

$$\lambda^2 \geq \frac{q}{4(q-1)}(\mu_1 + \inf_M |\kappa|^2). \tag{1.5}$$

On the Kähler spin foliation, if we use the basic Kähler form Ω acting on the basic spinor field, we have the following theorem (see [14] for ordinary case).

Theorem 1.3 ([10]). *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a Kähler spin foliation $\mathcal{F}$ of codimension $q = 2n$ and a bundle-like metric g_M such that κ is basic-harmonic and transversally holomorphic. If $K_\sigma \geq 0$, then the eigenvalue λ of D_b satisfies*

$$\lambda^2 \geq \frac{q+2}{4q}K_\sigma^0, \tag{1.6}$$

where $K_\sigma^0 = \inf_M K_\sigma$.

In the limiting case, the foliation is minimal, transversally Einsteinian with positive constant transversal scalar curvature σ^∇. In particular, the limiting foliation in (1.6) is minimal, transversally Einsteinian with odd complex codimension. This implies that when complex codimension of $\mathcal{F}$ is even, there exists a shaper estimate than (1.6).

2. Preliminaries and known facts

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$. We recall the exact sequence

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0$$

determined by the tangent bundle L and the normal bundle $Q = TM/L$ of $\mathcal{F}$. The assumption of g_M to be a bundle-like metric means that the induced metric g_Q on the normal bundle $Q \cong L^\perp$ satisfies the holonomy invariance condition $\overset{\circ}{\nabla} g_Q = 0$, where $\overset{\circ}{\nabla}$ is the Bott connection in Q ([12]).

For a distinguished chart $\mathcal{U} \subset M$ the leaves of $\mathcal{F}$ in $\mathcal{U}$ are given as the fibers of a Riemannian submersion $f : \mathcal{U} \to \mathcal{V} \subset N$ onto an open subset $\mathcal{V}$ of a model Riemannian manifold N.

For overlapping charts $U_\alpha \cap U_\beta$, the corresponding local transition functions $\gamma_{\alpha\beta} = f_\alpha \circ f_\beta^{-1}$ on N are isometries.

Further, we denote by ∇ the canonical connection of the normal bundle $Q = TM/L$ of $\mathcal{F}$. It is defined by

$$\begin{cases} \nabla_X s = \pi([X, Y_s]) & \text{for} \quad X \in \Gamma L\,, \\ \nabla_X s = \pi(\nabla^M_X Y_s) & \text{for} \quad X \in \Gamma L^\perp\,, \end{cases} \tag{2.1}$$

where $s \in \Gamma Q$, and $Y_s \in \Gamma L^\perp$ corresponding to s under the canonical isomorphism $L^\perp \cong Q$. The connection ∇ is metric and torsion free. It corresponds to the Riemannian connection of the model space N^q, [12]. The curvature R^∇ of ∇ is defined by

$$R^\nabla_{XY} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \quad \text{for} \quad X,\ Y \in TM\,.$$

Since $i(X)R^\nabla = 0$ for any $X \in \Gamma L$ ([12, 13, 20]), we can define the (transversal) Ricci curvature $\rho^\nabla : \Gamma Q \to \Gamma Q$ and the (transversal) scalar curvature σ^∇ of $\mathcal{F}$ by

$$\rho^\nabla(s) = \sum_a R^\nabla_{sE_a} E_a\,, \quad \sigma^\nabla = \sum_a g_Q(\rho^\nabla(E_a), E_a)\,,$$

where $\{E_a\}_{a=1,\cdots,q}$ is an orthonormal basis for Q. The foliation $\mathcal{F}$ is said to be (transversally) *Einsteinian* if the model space N is Einsteinian, that is,

$$\rho^\nabla = \frac{1}{q}\sigma^\nabla \cdot \mathrm{id} \tag{2.2}$$

with constant transversal scalar curvature σ^∇.

The *mean curvature vector field* of $\mathcal{F}$ is then defined by

$$\tau = \sum_i \pi(\nabla^M_{E_i} E_i)\,, \tag{2.3}$$

where $\{E_i\}_{i=1,\cdots,p}$ is an orthonormal basis of L. The dual form κ, the *mean curvature form* for L, is then given by

$$\kappa(X) = g_Q(\tau, X) \quad \text{for} \quad X \in \Gamma Q\,. \tag{2.4}$$

The foliation $\mathcal{F}$ is said to be *minimal* (or *harmonic*) if $\kappa = 0$.

Let $\Omega^r_B(\mathcal{F})$ be the space of all *basic r-forms*, i.e.,

$$\Omega^r_B(\mathcal{F}) = \{\phi \in \Omega^r(M) \mid i(X)\phi = 0,\ \theta(X)\phi = 0,\ \text{for } X \in \Gamma L\}\,.$$

Since the exterior derivative preserves the basic forms (that is, $\theta(X)d\phi = 0$ and $i(X)d\phi = 0$ for $\phi \in \Omega^r_B(\mathcal{F})$), the restriction $d_B = d|_{\Omega^*_B(\mathcal{F})}$ is well defined. Let δ_B the adjoint operator of d_B. Then it is well-known ([1, 9]) that

$$d_B = \sum_a \theta_a \wedge \nabla_{E_a}\,, \quad \delta_B = -\sum_a i(E_a)\nabla_{E_a} + i(\kappa_B)\,, \tag{2.5}$$

where κ_B is the basic component of κ, $\{E_a\}$ is a local orthonormal basic frame in Q and $\{\theta_a\}$ its g_Q-dual 1-form.

The *basic Laplacian* acting on $\Omega_B^*(\mathcal{F})$ is defined by

$$\Delta_B = d_B\delta_B + \delta_B d_B\,. \tag{2.6}$$

If $\mathcal{F}$ is the foliation by points of M, the basic Laplacian is the ordinary Laplacian.

3. The basic Dirac operator

Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transversally oriented Riemannian foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$. Let $SO(q) \to P \to M$ be the principal bundle of (oriented) transverse orthonormal framings. Then a *transverse spin structure* is a principal Spin(q)-bundle $\tilde{P}$ together with two sheeted covering $\xi : \tilde{P} \to P$ such that $\xi(p\cdot g) = \xi(p)\xi_0(g)$ for all $p \in \tilde{P}$, $g \in \mathrm{Spin}(q)$, where $\xi_0 : \mathrm{Spin}(q) \to SO(q)$ is a covering. In this case, the foliation $\mathcal{F}$ is called a *transverse spin foliation*. We then define the vector bundle S associated with $\tilde{P}$ by

$$S(\mathcal{F}) = \tilde{P} \times_{\mathrm{Spin}(q)} S_q\,, \tag{3.1}$$

where S_q is the irreducible spinor space associated to Q. The Hermitian metric on $S(\mathcal{F})$ is induced from g_Q, and the Riemannian connection ∇ on P defined by (2.1) can be lifted to one on $\tilde{P}$, in particular, to one on $S(\mathcal{F})$, which will be denoted by the same letter. $S(\mathcal{F})$ is called the *foliated spinor bundle*. It is well known that the curvature transform R^S ([17]) is given as

$$R^S_{XY}\Phi = \frac{1}{4}\sum_{a,b} g_Q(R^\nabla_{XY}E_a, E_b)E_a\cdot E_b\cdot\Phi \quad \text{for } X, Y \in TM\,. \tag{3.2}$$

On the foliated spinor bundle $S(\mathcal{F})$, we have

$$\sum_a E_a\cdot R^S_{XE_a}\Phi = -\frac{1}{2}\rho^\nabla(\pi(X))\cdot\Phi\,, \tag{3.3}$$

$$\sum_{a<b} E_a\cdot E_b\cdot R^S_{E_aE_b}\Phi = \frac{1}{4}\sigma^\nabla\Phi \tag{3.4}$$

for $X \in TM$, [9, 11]. Taking $\hat{\pi}$ to denote the projection

$$\hat{\pi} : C^\infty\big(T^*M \otimes S(\mathcal{F})\big) \to C^\infty\big(Q^* \otimes S(\mathcal{F})\big) \cong C^\infty\big(Q \otimes S(\mathcal{F})\big)$$

we define the *transversal Dirac operator* D'_{tr} ([4, 6]) by

$$D'_{\mathrm{tr}} = \cdot \circ \hat{\pi} \circ \nabla\,.$$

If $\{E_a\}_{a=1,\cdots,q}$ is taken to be a local orthonormal basic frame in Q, then

$$D'_{\mathrm{tr}} = \sum_a E_a \cdot \nabla_{E_a}\,.$$

In [4, 6] it was shown that the formal adjoint ${D'_{\mathrm{tr}}}^*$ is given by ${D'_{\mathrm{tr}}}^* = D'_{\mathrm{tr}} - \kappa\cdot$ and that therefore

$$D_{\mathrm{tr}} = D'_{\mathrm{tr}} - \frac{1}{2}\kappa\cdot \tag{3.5}$$

is a symmetric, transversally elliptic differential operator, with symbol $\sigma_{D_{\rm tr}}$ satisfying $\sigma_{D_{\rm tr}}(x,\xi)=\xi$ for $\xi\in Q_x^*$ and $\sigma_{D_{\rm tr}}(x,\xi)=0$ for $\xi\in L_x^*$. We define the subspce $\Gamma_B S(\mathcal{F})$ of *basic* or *holonomy invariant* sections of $S(\mathcal{F})$ by

$$\Gamma_B S(\mathcal{F})=\{\Phi\in\Gamma S(\mathcal{F})|\ \nabla_X\Phi=0 \text{ for } X\in\Gamma L\}\,. \tag{3.6}$$

From (3.5), we see that $D_{\rm tr}$ leaves $\Gamma_B S(\mathcal{F})$ invariant if and only if the foliation $\mathcal{F}$ is isoparametric, i.e., $\kappa\in\Omega_B^1(\mathcal{F})$. Let $D_b=D_{\rm tr}|_{\Gamma_B S(\mathcal{F})}:\Gamma_B S(\mathcal{F})\to\Gamma_B S(\mathcal{F})$. This operator D_b is called the *basic Dirac operator* on (smooth) basic sections $\Gamma_B S(\mathcal{F})$. We now define $\nabla_{\rm tr}^*\nabla_{\rm tr}:\Gamma S(\mathcal{F})\to\Gamma S(\mathcal{F})$ as

$$\nabla_{\rm tr}^*\nabla_{\rm tr}\Phi=-\sum_a\nabla^2_{E_a,E_a}\Phi+\nabla_\kappa\Phi\,, \tag{3.7}$$

where $\nabla^2_{V,W}=\nabla_V\nabla_W-\nabla_{\nabla_V W}$ for any $V,W\in TM$.

Proposition 3.1 ([9]). *Let $(M,g_M,\mathcal{F},S(\mathcal{F}))$ be a compact Riemannian manifold with the transverse spin foliation $\mathcal{F}$ and a bundle-like metric g_M with respect to $\mathcal{F}$. Then*

$$\langle\langle\nabla_{\rm tr}^*\nabla_{\rm tr}\Phi,\Psi\rangle\rangle=\langle\langle\nabla_{\rm tr}\Phi,\nabla_{\rm tr}\Psi\rangle\rangle$$

for all $\Phi,\Psi\in\Gamma E$, where $\langle\langle\Phi,\Psi\rangle\rangle=\int_M\langle\Phi,\Psi\rangle$ is the inner product on $S(\mathcal{F})$.

Proposition 3.2 ([9]). *Let $(M,g_M,\mathcal{F},S(\mathcal{F}))$ be the same as in Proposition 3.1. Assume that κ is basic-harmonic. Then the basic Dirac operator D_b satisfies*

$$D_b^2=\nabla_{\rm tr}^*\nabla_{\rm tr}+\frac{1}{4}K_\sigma\,, \tag{3.8}$$

where $K_\sigma=\sigma^\nabla+|\kappa|^2$.

4. An estimation of the eigenvalues on Riemannian spin foliation

Let $(M,g_M,\mathcal{F},S(\mathcal{F}))$ be a compact Riemannian manifold with the transverse spin foliation $\mathcal{F}$ of codimension q, a bundle-like metric g_M with respect to $\mathcal{F}$ and a foliated spinor bundle $S(\mathcal{F})$. Now, we introduce a new connection $\overset{f}{\nabla}$ on $S(\mathcal{F})$ as

$$\overset{f}{\nabla}_X\,\Phi=\nabla_X\Phi+f\pi(X)\cdot\Phi\quad\text{for } X\in TM\,, \tag{4.1}$$

where f is a real valued basic function on M. Trivially, this connection $\overset{f}{\nabla}$ is a metric connection on Q. By similar calculation to proposition 3.1, we have

$$\langle\langle\overset{f}{\nabla}{}_{\rm tr}^*\overset{f}{\nabla}_{\rm tr}\,\Phi,\Psi\rangle\rangle\ =\ \langle\langle\overset{f}{\nabla}_{\rm tr}\,\Phi,\overset{f}{\nabla}_{\rm tr}\,\Psi\rangle\rangle \tag{4.2}$$

for all $\Phi,\Psi\in\Gamma S(\mathcal{F})$. Let $D_b\Phi=\lambda\Phi$. From (3.8), (4.1) and (4.2) we have

$$\|\overset{f}{\nabla}_{\rm tr}\,\Phi\|^2=\int_M((\frac{q-1}{q}\lambda^2-\frac{1}{4}K_\sigma)|\Phi|^2\,, \tag{4.3}$$

where $K_\sigma=\sigma^\nabla+|\kappa|^2$. From (4.3), we have the following theorem.

Theorem 4.1 ([9]). *Let $(M,g_M,\mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q>1$ and bundle-like metric g_M such that κ is* *basic-harmonic. Assume $K_\sigma \geq 0$. Then the eigenvalue λ of the basic Dirac operator D_b satisfies*

$$\lambda^2 \geq \frac{1}{4}\frac{q}{q-1}\inf_M K_\sigma, \tag{4.4}$$

where $K_\sigma = \sigma^\nabla + |\kappa|^2$.

Remark. If $\mathcal{F}$ is a point foliation, then the transversal (basic) Dirac operator is just a Dirac operator on an ordinary manifold. Therefore Theorem 4.1 is a generalization of the result on an ordinary manifold (cf.[5]).

Theorem 4.2 ([9]). *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q > 1$ and a bundle-like metric g_M such that κ is basic-harmonic. Assume $K_\sigma > 0$. If there exists an eigenspinor field Ψ_1 of the basic Dirac operator D_b for the eigenvalue $\lambda_1^2 = \frac{q}{4(q-1)}K_\sigma^0$, then $\mathcal{F}$ is a minimal, transversally Einsteinian with constant transversal scalar curvature.*

Remark. Theorem 4.2 implies that if the foliation $\mathcal{F}$ is not minimal, then $\lambda^2 > \frac{q}{4(q-1)}K_\sigma^0$. So when $\mathcal{F}$ is not minimal, there exists a sharper estimate than (4.4).

5. An estimation of the eigenvalues by the conformal change

Now, we consider, for any real basic function u on M, the transversally conformal metric $\bar{g}_Q = e^{2u}g_Q$. Let $\bar{P}_{so}(\mathcal{F})$ be the principal bundle of $\bar{g}_Q$-orthogonal frames. Locally, the section $\bar{s}$ of $\bar{P}_{so}(\mathcal{F})$ corresponding a section $s = (E_1, \cdots, E_q)$ of $P_{so}(\mathcal{F})$ is $\bar{s} = (\bar{E}_1, \cdots, \bar{E}_q)$, where $\bar{E}_a = e^{-u}E_a$ $(a = 1, \cdots, q)$. This isometry will be denoted by I_u. Thanks to the isomorphism I_u one can define a transverse spin structure $\bar{P}_{\rm spin}(\mathcal{F})$ on $\mathcal{F}$ in such a way that the diagram

$$\begin{array}{ccc} P_{\rm spin}(\mathcal{F}) & \xrightarrow{\tilde{I}_u} & \bar{P}_{\rm spin}(\mathcal{F}) \\ \downarrow & & \downarrow \\ P_{so}(\mathcal{F}) & \xrightarrow{I_u} & \bar{P}_{so}(\mathcal{F}) \end{array}$$

commutes. Let $\bar{S}(\mathcal{F})$ be the foliated spinor bundle associated with $\bar{P}_{\rm spin}(\mathcal{F})$. For any section Ψ of $S(\mathcal{F})$, we write $\bar{\Psi} \equiv I_u\Psi$. If $\langle\ ,\ \rangle_{g_Q}$ and $\langle\ ,\ \rangle_{\bar{g}_Q}$ denote respectively the natural Hermitian metrics on $S(\mathcal{F})$ and $\bar{S}(\mathcal{F})$, then for any $\Phi,\ \Psi \in \Gamma S(\mathcal{F})$

$$\langle\Phi, \Psi\rangle_{g_Q} = \langle\bar{\Phi}, \bar{\Psi}\rangle_{\bar{g}_Q}, \tag{5.1}$$

and the Clifford multiplication in $\bar{S}(\mathcal{F})$ is given by

$$\bar{X} \bar{\cdot} \bar{\Psi} = \overline{X \cdot \Psi} \quad \text{for } X \in \Gamma Q. \tag{5.2}$$

Let $\bar{\nabla}$ be the metric and torsion free connection corresponding to $\bar{g}_Q$. Then we have for $X, Y \in \Gamma TM$,

$$\bar{\nabla}_X\pi(Y) = \nabla_X\pi(Y) + X(u)\pi(Y) + Y(u)\pi(X) - g_Q(\pi(X), \pi(Y))\,\mathrm{grad}_\nabla(u), \tag{5.3}$$

where $\mathrm{grad}_\nabla(u) = \sum_a E_a(u)E_a$ is a transversal gradient of u and $X(u)$ is the Lie derivative of the function u in the direction of X. The formula (5.3) follows from that $\bar{\nabla}$ is the metric and torsion free connection with respect to $\bar{g}_Q$. The connection ∇ and

$\bar{\nabla}$ acting repectively on the sections of $S(\mathcal{F})$ and $\bar{S}(\mathcal{F})$, are related, for any vector field X and any spinor field Ψ by

$$\bar{\nabla}_X\bar{\Psi} = \overline{\nabla_X\Psi} - \frac{1}{2}\overline{\pi(X)\cdot \mathrm{grad}_\nabla(u)\cdot\Psi} - \frac{1}{2}g_Q(\mathrm{grad}_\nabla(u),\pi(X))\bar{\Psi}. \tag{5.4}$$

Now, we introduce a new connection $\overset{f}{\bar{\nabla}}$ on $\bar{S}(\mathcal{F})$ as

$$\overset{f}{\bar{\nabla}}_X \bar{\Psi} = \bar{\nabla}_X\bar{\Psi} + f\pi(X)\,\bar{\cdot}\,\bar{\Psi} \quad \text{for } X\in TM, \tag{5.5}$$

where f is a real-valued basic function on M. Trivially, this connection $\overset{f}{\bar{\nabla}}$ is a metric connection.

Lemma 5.1. *On the foliated spinor bundle $\bar{S}(\mathcal{F})$, we have*

$$\langle\langle \overset{f}{\bar{\nabla}}{}^*_{\mathrm{tr}}\overset{f}{\bar{\nabla}}_{\mathrm{tr}}\,\bar{\Psi},\bar{\Phi}\rangle\rangle_{\bar{g}_Q} = \langle\langle \overset{f}{\bar{\nabla}}_{\mathrm{tr}}\,\bar{\Psi},\overset{f}{\bar{\nabla}}_{\mathrm{tr}}\,\bar{\Phi}\rangle\rangle_{\bar{g}_Q}$$

for all $\Psi,\Phi\in\Gamma S(\mathcal{F})$, where $\langle \overset{f}{\bar{\nabla}}_{\mathrm{tr}}\,\bar{\Psi},\overset{f}{\bar{\nabla}}_{\mathrm{tr}}\,\bar{\Phi}\rangle_{\bar{g}_Q} = \sum_a\langle \overset{f}{\bar{\nabla}}_{\bar{E}_a}\,\bar{\Psi},\overset{f}{\bar{\nabla}}_{\bar{E}_a}\,\bar{\Phi}\rangle_{\bar{g}_Q}$.

On the other hand, from (3.7) and (5.5) we have

$$\overset{f}{\bar{\nabla}}{}^*_{\mathrm{tr}}\overset{f}{\bar{\nabla}}_{\mathrm{tr}}\,\bar{\Psi} = \bar{\nabla}^*_{\mathrm{tr}}\bar{\nabla}_{\mathrm{tr}}\bar{\Psi} - 2f\bar{D}_{\mathrm{tr}}\bar{\Psi} + qf^2\bar{\Psi} - e^{-u}\overline{\mathrm{grad}_\nabla(f)\cdot\Psi}. \tag{5.6}$$

Let $D_b\Phi = \lambda\Phi(\Phi\neq 0)$. If we put $f=\frac{\lambda}{q}e^{-u}$, then we have

$$\int|\overset{f}{\bar{\nabla}}_{\mathrm{tr}}\,\bar{\Psi}|^2_{\bar{g}_Q} = \frac{q-1}{q}\int e^{-2u}(\lambda^2 - \frac{q}{4(q-1)}e^{2u}K_\sigma^{\bar{\nabla}})|\bar{\Psi}|^2_{\bar{g}_Q}, \tag{5.7}$$

where $K_\sigma^{\bar{\nabla}} = h^{-1}Y_b h + |\kappa|^2$, Y_b is a basic Yamabe operator of $\mathcal{F}$, which is defined by

$$Y_b = 4\frac{q-1}{q-2}\Delta_B + \sigma^\nabla. \tag{5.8}$$

From (5.7), we have the following theorem ([11]).

Theorem 5.2. *Let $(M,g_M,\mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q\geq 3$ and bundle-like metric g_M such that $\kappa\in\Omega^1_B(\mathcal{F})$ and $\delta\kappa=0$. If the transversal scalar curvature is non-negative, then we have*

$$\lambda^2 \geq \frac{q}{4(q-1)}(\mu_1 + \inf|\kappa|^2), \tag{5.9}$$

where μ_1 is the smallest eigenvalue of the basic Yamabe operator.

Remark. Since $\mu_1\geq \inf\sigma^\nabla$, the inequality (5.9) is a sharper estimate than (4.4).

6. An estimation of the eigenvalues on Kähler spin foliation

Let $\mathcal{F}$ be a Kähler foliation. Namely, by a *Kähler foliation* $\mathcal{F}$([19]) we mean a foliation satisfying the following conditions; (i) $\mathcal{F}$ is Riemannian, with a bundle-like metric g_M on M inducing the holonomy invariant metric g_Q on $Q \equiv L^\perp$, (ii) there is a holonomy invariant almost complex structure $J : Q \to Q$, where $dimQ = q(= 2n)$ (real dimension), with respect to which g_Q is Hermitian, i.e.,

$$g_Q(JX, JY) = g_Q(X, Y) \tag{6.1}$$

for $X, Y \in \Gamma Q$, and (iii) if ∇ is almost complex, i.e., $\nabla J = 0$. Note that

$$\Omega(X, Y) = g_Q(X, JY) \tag{6.2}$$

defines a basic 2-form Ω, which is closed as a consequence of $\nabla g_Q = 0$ and $\nabla J = 0$. Then we can express the basic 2-form Ω by

$$\Omega = \sum_{k=1}^{n} \theta^{2k-1} \wedge \theta^{2k}\,, \tag{6.3}$$

where $\{\theta^a\}$ is a g_Q-dual 1-form on M. For a Kähler foliation, we have the following identities ([19]):

$$R^\nabla_{XY} J = J R^\nabla_{XY}\,, \quad R^\nabla_{JXJY} = R^\nabla_{XY}\,, \quad R^\nabla_{XY} Z + R^\nabla_{YZ} X + R^\nabla_{ZX} Y = 0\,, \tag{6.4}$$

where X,Y and Z are elements of ΓQ.

Let $\mathcal{F}$ be a Kähler spin foliation on a compact oriented Riemannian manifold M. From (6.3), we know that

$$\Omega = -\frac{1}{2} \sum_a E_a \cdot JE_a = \frac{1}{2} \sum_a JE_a \cdot E_a\,, \tag{6.5}$$

where $\{E_a\}$ is a local orthonormal basic frame in Q.

Note that the foliated spinor bundle $S(\mathcal{F})$ of a Kähler spin foliation $\mathcal{F}$ splits into the orthogonal direct sum

$$S(\mathcal{F}) = S_0 \oplus S_1 \oplus \cdots \oplus S_n\,, \tag{6.6}$$

where the fiber $(S_r)_x$ of the subbundle S_r is just defined as the eigenspace corresponding to the eigenvalue $i(n - 2r)(r = 0, \cdots, n)$ of $\Omega_x : S_x(\mathcal{F}) \to S_x(\mathcal{F})$. If $p_r : S(\mathcal{F}) \to S_r$ is the projection, then we have

$$\Omega = \sum_{r=0}^{n} i\mu_r p_r\,, \quad \mu_r = n - 2r\,. \tag{6.7}$$

The decomposition (6.6) is compatible with ∇, i.e., if Ψ is a section of S_r, then $\nabla_X \Psi$ is also a section of S_r for any vector field X.

Let $\tilde{D}_{\rm tr}$ be the operator which is locally defined by

$$\tilde{D}_{\rm tr}\Phi = \sum_a JE_a \cdot \nabla_{E_a}\Phi - \frac{1}{2} J\kappa \cdot \Phi \quad \text{for} \quad \Phi \in \Gamma S(\mathcal{F})\,. \tag{6.8}$$

Using Green's theorem on the foliated Riemannian manifold ([21]), we know for any $\Phi, \Psi \in \Gamma S(\mathcal{F})$

$$\int_M \langle \tilde{D}_{\mathrm{tr}}\Phi, \Psi\rangle = \int_M \langle \Phi, \tilde{D}_{\mathrm{tr}}\Psi\rangle , \tag{6.9}$$

i.e., $\tilde{D}_{\mathrm{tr}}$ is self-adjoint transversally elliptic operator.

Proposition 6.1 ([10]). *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a Kähler spin foliation $\mathcal{F}$ and a bundle-like metric g_M with $\kappa \in \Omega^1_B(\mathcal{F})$. Suppose the mean curvature of $\mathcal{F}$ is a transversally holomorphic. Then we have*

$$D^2_{\mathrm{tr}} = \tilde{D}^2_{\mathrm{tr}}, \quad D_{\mathrm{tr}}\tilde{D}_{\mathrm{tr}} + \tilde{D}_{\mathrm{tr}}D_{tr} = 0 .$$

On the foliated spinor bundle $S(\mathcal{F})$, we introduce a new connection of the form

$$\overset{fg}{\nabla}_X\phi = \nabla_X\phi + f\pi(X)\cdot\phi + igJ\pi(X)\cdot\iota^2\phi \quad \text{for } X \in TM , \tag{6.10}$$

where f, g are real valued basic functions on M and $\iota : S(\mathcal{F}) \to S(\mathcal{F})$ is a bundle map (see [10]). By similar method to section 5, if we put $f = \frac{\lambda}{q+2}$ and $g = \frac{(-1)^{\ell}\lambda}{q+2}$, then we have takes the form

$$\|\overset{fg}{\nabla}_{\mathrm{tr}}\phi\|^2 = \int_M (\frac{q}{q+2}\lambda^2 - \frac{1}{4}K_\sigma)|\phi|^2 , \tag{6.11}$$

where $K_\sigma = \sigma^\nabla + |\kappa|^2$. From (6.11), we have the following theorem ([10]).

Theorem 6.2. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a Kähler spin foliation $\mathcal{F}$ of codimension $q = 2n$ and a bundle-like metric g_M such that κ is basic-harmonic and transversally holomorphic. If $K_\sigma \geq 0$, then the eigenvalue λ of D_b satisfies*

$$\lambda^2 \geq \frac{q+2}{4q}\inf_M K_\sigma , \tag{6.12}$$

where $K_\sigma = \sigma^\nabla + |\kappa|^2$.

Remark. The estimation of the eigenvalue of the transversal Dirac operator on a Kähler spin foliation is a shaper estimate than the one in Theorem 4.1.

Theorem 6.3 ([10]). *Let $(M, g_M, \mathcal{F})$ be the same as in Theorem 6.2. If there exists an eigenspinor field $\phi(\neq 0)$ of the basic Dirac operator D_b for the eigenvalue $\lambda^2 = \frac{q+2}{4q}K^0_\sigma$, then $\mathcal{F}$ is a minimal, transversally Einsteinian of odd complex codimension n with nonnegative constant transversal scalar curvature σ^∇.*

Question. In Theorem 6.3, the limiting foliation is odd complex codimension. This implies that if the codimension of $\mathcal{F}$ is even, then there exists a sharper estimate than (6.12) in Theorem 6.2. What is the estimate?

REFERENCES

[1] J. A. Alvarez López, *The basic component of the mean curvature of Riemannian foliations*, Ann. Global Anal. Geom. **10** (1992), 179–194.

[2] C. Bär, *Real Killing spinors and holonomy*, Comm. Math. Phys. **154** (1993), 509–521.

[3] H. Baum, Th. Friedrich, R. Grunewald and I. Kath, *Twistors and Killing spinors on Riemannian manifolds*, Teubner, Leipzig/Stuttgart 1991.

[4] J. Brüning and F. W. Kamber, *Vanishing theorems and index formulas for transversal Dirac operators*, A.M.S Meeting 845, Special Session on operator theory and applications to Geometry, Lawrence, KA; A.M.S. Abstracts, October 1988.

[5] Th. Friedrich, *Der erste Eigenwert des Dirac operators einer kompakten,Riemannschen Mannigfaltigkeit nichtnegative skalarkrümmung*, Math. Nachr. **97** (1980), 117–146.

[6] J. F. Glazebrook and F. W. Kamber, *Transversal Dirac families in Riemannian foliations*, Comm. Math. Phys. **140** (1991), 217–240.

[7] O. Hijazi, *A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors*, Comm. Math. Phys. **104** (1986), 151–162.

[8] O. Hijazi, *Lower bounds for the eigenvalues of the Dirac operator*, J. Geom. Phys. **16** (1995), 27–38.

[9] S. D. Jung, *The first eigenvalue of the transversal Dirac operator*, J. Geom. Phys. **39** (2001), 253–264.

[10] S. D. Jung and T. H. Kang, *Lower bounds for the eigenvalue of the transversal Dirac operator on a Kähler foliation*, J. Geom. Phys. **45** (2003), 75–90.

[11] S. D. Jung, B. H. Kim and J. S. Pak, *Lower bounds for the eigenvalues of the basic Dirac operator on a Riemannian foliation*, to appear in J. Geom. Phys.

[12] F. W. Kamber and Ph. Tondeur, *Harmonic foliations*, Proc. National Science Foundation Conference on Harmonic Maps, Tulane, Dec. 1980, Lecture Notes in Math. **949**, Springer-Verlag, New York 1982, 87–121.

[13] F. W. Kamber and Ph. Tondeur, *Foliated bundles and Characteristic classes*, Lecture Notes in Math. **493**, Springer-Verlag, Berlin 1975.

[14] K.-D. Kirchberg, *An estimation for the first eigenvalue of the Dirac operator on closed Kähler manifolds of positive scalar curvature*, Ann. Glob. Anal. Geom. **4** (1986), 291–326.

[15] K.-D. Kirchberg, *The first eigenvalue of the Dirac operator on Kähler manifolds*, J. Geom. Phys. **7** (1990), 449–468.

[16] W. Kramer, U. Semmelmann and G. Weingart, *Eigenvalue estimates for Dirac operator on quaternionic Kähler manifolds*, Math. Z. **230** (1999), 727–751.

[17] H. B. Lawson, Jr. and M. L. Michelsohn, *Spin geometry*, Princeton Univ. Press, Princeton, New Jersey 1989.

[18] A. Lichnerowicz, *Spineurs harmoniques*, C. R. Acad. Sci. Paris Ser. A-B **257** (1963).

[19] S. Nishikawa and Ph. Tondeur, *Transversal infinitesimal automorphisms for harmonic Kähler foliations*, Tohoku Math. J. **40** (1988), 599–611.

[20] Ph. Tondeur, *Foliations on Riemannian manifolds*, Springer-Verlag, New-York 1988.

[21] S. Yorozu and T. Tanemura *Green's theorem on a foliated Riemannian manifold and its applications*, Acta Math. Hungar. **56** (1990), 239–245.

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