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In: Jarolím Bureš (ed.): Proceedings of the 22nd Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 2003. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 71. pp. [143]--150.

Persistent URL: http://dml.cz/dmlcz/701713

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 71 (2003), pp. 143-150

MORE ON DEFORMED OSCILLATOR ALGEBRAS AND EXTENDED UMBRAL CALCULUS

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ABSTRACT. $\psi(q)$-calculus is an almost unavoidable extension of finite operator calculus of Rota [1]. Main results of Rotas' finite operator calculus might be quite easily given their ψ-extensions. The specific $\psi_n(q) = [n_q!]^{-1}$ case is known to be relevant for quantum groups investigation [2]–[5]. In general $\psi(q)$-calculus is as a matter of fact Ward's *"..calculus of sequences"* [6] in Rotas' finite operator calculus form [7]. This we owe to Viskov and other distinguished authors (see for example [8]–[13]). Here we show that such $\psi(q)$-umbral calculus leads to infinitely many new ψ-deformed "quantum-like" oscillator algebras representations. Among others one may formulate q-extended finite operator calculus with help of the "quantum q-plane" q-commuting variables $A, B : AB - qBA \equiv [A, B]_q = 0$ as done in [11], [12]. This presentation is mostly an editorial actualization and enrichment of [14] based on [15] (see also [1]) and is intended to be further extension of last years talks given at Srní.

1. FEW BASIC NOTIONS OF $\psi(q)$-EXTENDED UMBRAL CALCULUS

$\psi(q)$-extended umbral calculus is arrived at [8], [9] by considering not only polynomial sequences of binomial type but also of $\{s_n\}_{n\geq 1}$-binomial type where $\{s_n\}_{n\geq 1}$-binomial coefficients are defined with help of the generalized factorial $n_S! = s_1 s_2 s_3 \cdots s_n$; $S = \{s_n\}_{n\geq 1}$ is an arbitrary sequence with the condition $s_n \neq 0$, $n \in N$. Then the extension relies on the notion of ∂_ψ-shift invariance of ∂_ψ-delta operators. Here the linear operator ∂_ψ acting on the algebra of polynomials denotes the ψ-derivative i.e. $\partial_\psi x^n = n_\psi x^{n-1}$; $n \geq 0$ and n_ψ denotes the ψ-deformed number (see also [6] and [13]) where in conformity with Viskov notation we put

$n_\psi \equiv \psi_{n-1}(q)\,\psi_n^{-1}(q)$ hence $(0_\psi! = 1)$
$n_\psi! \equiv \psi_n^{-1}(q) \equiv n_\psi\,(n-1)_\psi\,(n-2)_\psi\,(n-3)_\psi \cdots 2_\psi 1_\psi$ and
$n_\psi^{\underline{k}} = n_\psi\,(n-1)_\psi \cdots (n-k+1)_\psi$.

We choose to work with $\Im$ — the family of functions sequences such that: $\Im = \{\psi : R \supset [a,b]; q \in [a,b]; \psi(q) : Z \to F; \psi_0(q) = 1; \psi_n(q) \neq 0; \psi_{-n}(q) = 0; n \in N\}$. With the choice $\psi_n(q) = [R(q^n)!]^{-1}$ and $R(x) = \frac{1-x}{1-q}$ we get the well known

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2000 *Mathematics Subject Classification.* 05A40, 81S99.
Key words and phrases. Extended umbral calculus, deformed quantum oscillators.
The paper is in final form and no version of it will be submitted elsewhere.

q-factorial $n_q! = n_q\,(n-1)_q!$; $1_q! = 0_q! = 1$ while the ψ-derivative ∂_ψ becomes now the Jackson's derivative (see [16])

$$\partial_q : (\partial_q \varphi)(x) = \frac{\varphi(x) - \varphi(qx)}{(1-q)\,x}.$$

A polynomial sequence $\{p_n\}_0^\infty$ is called to be of ψ-binomial type if it satisfies the recurrence

$E^y(\partial_\psi)\, p_n(x) \equiv p_n(x +_\psi y) \equiv \sum_{k\geq 0} \binom{n}{k}_\psi p_k(x)\, p_{n-k}(y)$; where $\binom{n}{k}_\psi \equiv \frac{n_\psi^{\underline{k}}}{k_\psi!}$.

$E^y(\partial_\psi) \equiv \exp_\psi\{y\partial_\psi\} = \sum_{k=0}^{\infty} \frac{y^k \partial_\psi^k}{n_\psi!}$ denotes a generalized translation operator [6] and ∂_ψ-shift invariance is defined accordingly. The algebra $\sum_\psi$ is the algebra of all F-linear ∂_ψ-shift invariant operators T acting on the algebra P of polynomials. We assume that $\operatorname{char} F = 0$ for any field F chosen. In another words

$\forall\, \alpha \in F \quad [T, E^\alpha(\partial_\psi)] = 0$; $\operatorname{char} F = 0$.

One then introduces the notion of ∂_ψ-*delta* operator according to Definition 1.1.

Definition 1.1. *Let $Q(\partial_\psi) : P \to P$; the linear operator $Q(\partial_\psi)$ is a ∂_ψ-delta operator iff*

(1) *$Q(\partial_\psi)$ is ∂_ψ-shift invariant;*
(2) *$Q(\partial_\psi)(\mathrm{id}) = const \neq 0$.*

As in unextended case [7] — one may construct [1] the bijective correspondence between ∂_ψ-*delta* operators with their ∂_ψ-basic polynomial sequences.

Definition 1.2. *Let $Q(\partial_\psi) : P \to P$ be the ∂_ψ-delta operator. A polynomial sequence $\{p_n\}_{n\geq 0}$; $\deg p_n = n$ such that:*

(1) *$p_o(x) = 1$;*
(2) *$p_n(0) = 0$; $n > 0$;*
(3) *$Q(\partial_\psi)\, p_n = n_\psi p_{n-1}$ is called the ∂_ψ-basic polynomial sequence of the ∂_ψ-delta operator $Q(\partial\psi)$.*

Now using the fact that $\forall Q(\partial_\psi)\, \exists!$ invertible $S_{\partial_\psi} \in \Sigma_\psi$ such that $Q(\partial_\psi) = \partial_\psi S_{\partial_\psi}$ one may prove (analogously to special cases [7], [12]) the crucial Theorem 1.1 (see [1], [10]).

Theorem 1.1. *Let $\{p_n(x)\}_{n=0}^\infty$ be ∂_ψ-basic polynomial sequence of the ∂_ψ-delta operator $Q(\partial_\psi)$:*

$Q(\partial_\psi) \;= \partial_\psi S_{\partial_\psi}$. Then for $n > 0$:

(1) *$p_n(x) = Q(\partial_\psi)'\, S_{\partial_\psi}^{-n-1} x^n$;*
(2) *$p_n(x) = S_{\partial_\psi}^{-n} x^n - \frac{n_\psi}{n}\,(S_{\partial_\psi}^{-n})' x^{n-1}$;*
(3) *$p_n(x) = \frac{n_\psi}{n} \hat{x}_\psi S_{\partial_\psi}^{-n} x^{n-1}$;*
(4) *$p_n(x) = \frac{n_\psi}{n} \hat{x}_\psi (Q(\partial_\psi)')^{-1} p_{n-1}(x)$.*

In order to prove this one uses the properties of the Pincherle ψ-derivative.

Definition 1.3. *The Pincherle ψ-derivative i.e. the linear map $'$: $\Sigma_\psi \to \Sigma_\psi$;*

$$T' = T\,\hat{x}_\psi - \hat{x}_\psi T \;\equiv\; [T, \hat{x}_\psi]$$

where the linear map $\hat{x}_\psi : P \to P$; is defined in the basis $\{x^n\}_{n\geq 0}$ as follows

$$\hat{x}_\psi x^n = \frac{\psi_{n+1}(q)(n+1)}{\psi_n(q)} x^{n+1} = \frac{(n+1)}{(n+1)_\psi} x^{n+1}; \quad n \geq 0.$$

One may also define Sheffer ∂_ψ-polynomials which constitute the more general class of polynomial sequences than the class of ∂_ψ-basic polynomial sequences.

Definition 1.4. *A polynomial sequence $\{s_n(x)\}_{n=0}^\infty$ is called the sequence of Sheffer ∂_ψ-polynomials of the ∂_ψ-delta operator $Q(\partial_\psi)$ iff*

(1) $s_0(x) = c \neq 0$;
(2) $Q(\partial_\psi) s_n(x) = n_\psi s_{n-1}(x)$.

The following proposition relates Sheffer ∂_ψ-polynomials of the ∂_ψ-delta operator $Q(\partial_\psi)$ to the unique ∂_ψ-basic polynomial sequence of the ∂_ψ-delta operator $Q(\partial_\psi)$:

Proposition 1.1. *Let $Q(\partial_\psi)$ be a ∂_ψ-delta operator with ∂_ψ-basic polynomial sequence $\{q_n(x)\}_{n=0}^\infty$. Then $\{s_n(x)\}_{n=0}^\infty$ is a sequence of Sheffer q-polynomials of the ∂_ψ-delta operator $Q(\partial_\psi)$ iff there exists a ∂_ψ-shift invariant operator S_{∂_ψ} such that $s_n(x) = S_{\partial_\psi}^{-1} q_n(x)$.*

Examples: According to Proposition 1.1 with $Q(\partial_q) = \partial_q$ and $S = \exp_\psi\{\frac{1}{2}\alpha\partial_q^2\}$ we get q-Hermite polynomials while with choice $Q(\partial_q) = \frac{\partial_q}{\partial_q - 1}$ and $S = (1-\partial_q)^{-\alpha-1}$ we obtain q-Laguerre polynomials $L_{n,q}^{(\alpha)}(x)$ of order α. ψ-extensions include of course q-Hermite, q-Laguerre polynomials $L_{n,q}^{(\alpha)}(x)$ of order α with their ψ-correspondents. These are already well known q-Sheffer polynomials [17], [11], [12]. Specifically q-Laguerre polynomials $L_{n,q}^{(-1)}(x) \equiv L_{n,q}(x)$ form the ∂_q-basic polynomial sequence $\{L_{n,q}(x)\}_{n\geq 0}$ of the ∂_q operator $Q(\partial_q) = -\sum_{k=0}^{\infty} \partial_q^{k+1} \equiv \frac{\partial_q}{\partial_q - 1} \equiv -\left[\partial_q + \partial_q^2 + \partial_q^3 + \partial_q^4 + \partial_q^5 + \cdots\right]$. Using then Theorem 1.1 one arrives at the explicit form of $L_{n,q}(x)$. Namely:

$$L_{n,q}(x) = \frac{n_q}{n}\hat{x}_q[\frac{1}{\partial_q - 1}]^{-n} x^{n-1} = \frac{n_q}{n}\hat{x}_q(\partial_q - 1)^n x^{n-1} =$$

$$= \frac{n_q}{n}\hat{x}_q \sum_{k=1}^{n} (-1)^k \binom{n}{k}_q \partial_q^{n-k} x^{n-1} = \frac{n_q}{n}\sum_{k=1}^{n} (-1)^k \binom{n}{k}_q (n-1)_q^{\underline{n-k}} \frac{k}{k_q} x^k =$$

$$= \frac{n_q}{n}\sum_{k=1}^{n} (-1)^k \frac{n_q!}{k_q!} \frac{(n-1)_q^{\underline{n-k}}}{(n-k)_q!} \frac{k}{k_q} x^k.$$

So finally

$$L_{n,q}(x) = \frac{n_q}{n}\sum_{k=1}^{n} (-1)^k \frac{n_q!}{k_q!} \binom{n-1}{k-1}_q \frac{k}{k_q} x^k. \tag{1.1}$$

Note: ψ-extended case is covered in this example just by replacement $q \to \psi$.

With the choice $\psi_n(q) = [R(q^n)!]^{-1}$ we arrive at interesting R-Laguerre polynomials.

Let us also stress here again that q-deformed quantum oscillator algebra provides a natural setting for q-Laguerre polynomials and q-Hermite polynomials [18], [19], [20].

$sl_q(2)$ and the q-oscillator algebra give rise to basic geometric functions as matrix elements of certain operators in analogy with Lie theory [18], [19]. Also automorphisms of the q-oscillator algebra lead to Sheffer q-polynomials for example to q-generalization of the Charlier polynomials [18], [19].

2. Extended umbral calculus and ψ-deformed "quantum oscillator" algebras

∂_q-delta operators and their duals and similarly ∂_ψ-delta operators with their duals provide us with pairs of generators of ψ-deformed quantum oscillator-like algebras (see Remark 2.2). Namely as we shall see: $\left[Q\left(\partial_\psi\right), \hat{x}_{Q(\partial_\psi)}\right]_{\hat{q}_{\psi,Q}} = \mathrm{id}$. With the choice $\psi_n(q) = \left[R\left(q^n\right)!\right]^{-1}$ and $R(x) = \frac{1-x}{1-q}$ we get the well known q-deformed oscillator dual pair of operators — generators of the well known q-Heisenberg-Weyl algebra. These oscillator-like algebras generators and q-oscillator-like algebras generators are encountered explicitly or implicitly in [2], [3] and in many other subsequent references — see [26], [5] and references therein. In many such references [18], [19] q-Laguerre and q-Hermite or q-Charlier polynomials appear which are just either Sheffer ψ-polynomials or just ∂_ψ-basic polynomial sequences of the ∂_ψ-delta operators $Q\left(\partial_\psi\right)$ for $\psi_n\left(q\right) = \frac{1}{R(q^n)!}$; $R\left(x\right) = \frac{1-x}{1-q}$ and corresponding choice of $Q\left(\partial_\psi\right)$ functions of ∂_ψ (for example $Q = \mathrm{id}$). The case $\psi_n\left(q\right) = \frac{1}{R(q^n)!}$: $n_\psi = n_R$; $\partial_\psi = \partial_R$ and $n_{\psi(q)} = n_{R(q)} = R\left(q^n\right)$ appears implicitly in [21] where advanced theory of general quantum coherent states is being developed. However there is no mention of $R\left(q^n\right)$-umbral calculus in [21] neither in "q-references" quoted in this note. In the q-case it was noticed among others also in [22] that commutation relations for the q-oscillator-like algebras generators from [2, 3] and others (see [5]) might be chosen in appropriate operator variables to be of the form [22]:

$$AA^+ - \mu A^+A = 1; \quad \mu = q^2 \tag{2.1}$$

As for the Fock space representation of normalized eigenstates $|n>$ of excitation number operator N various q-deformations of the natural number n are used in literature on quantum groups and at least some families of quantum groups may be constructed from q-analogues of Heisenberg algebra [2], [3], [22], [4]. Our q-oscillator algebras generators are just the ∂_q-delta operators $Q\left(\partial_q\right)$ and their duals i.e. basic objects of the q-extended finite operator calculus of Rota. (An elementary example: $\partial_q\hat{x} - q\hat{x}\partial_q = \mathrm{id}$.)

Here in below we shall propose a ψ-extension of the q-oscillator model algebra using basic concepts of Viskov's ψ-extension of calculus of Rota.

Definition 2.1. *Let $\left\{p_n\right\}_{n\geq 0}$ be the ∂_q-basic polynomial sequence of the ∂_q-delta operator $Q\left(\partial_q\right)$. A linear map $\hat{x}_{Q(\partial_q)} : P \to P$; $\hat{x}_{Q(\partial_q)}p_n = p_{n+1}$; $n \geq 0$ is called the operator dual to $Q\left(\partial_q\right)$.*

For $Q = \mathrm{id}$ we have : $\hat{x}_{Q(\partial_q)} \equiv \hat{x}_{\partial_q} \equiv \hat{x}$.

Definition 2.2. *Let $\left\{p_n\right\}_{n\geq 0}$ be the ∂_ψ-basic polynomial sequence of the ∂_ψ-delta operator $Q\left(\partial_\psi\right) = Q$. Then the $\hat{q}_{\psi,Q}$-operator is a liner map;*

$$\hat{q}_{\psi,Q} : P \to P; \quad \hat{q}_{\psi,Q}p_n = \frac{(n+1)_\psi - 1}{n_\psi}p_n; \quad n \geq 0.$$

We call the $\hat{q}_{\psi,Q}$ operator the $\hat{q}_{\psi,Q}$-*mutator operator.*

Note: For $Q = \mathrm{id}\, Q(\partial_\psi) = \partial_\psi$ the natural notation is $\hat{q}_{\psi,\mathrm{id}} \equiv \hat{q}_\psi$. For $Q = \mathrm{id}$ and $\psi_n(q) = \frac{1}{R(q^n)!}$ and $R(x) = \frac{1-x}{1-q}$ $\hat{q}_{\psi,Q} \equiv \hat{q}_{R,\mathrm{id}} \equiv \hat{q}_R \equiv \hat{q}_{q,\mathrm{id}} \equiv \hat{q}_q \equiv \hat{q}$ and $\hat{q}_{\psi,Q} x^n = qx^n$.

Definition 2.3. *Let A and B be linear operators acting on P;* $A : P \to P$; $B : P \to P$. *Then* $AB - \hat{q}_{\psi,Q} BA \equiv [A, B]_{\hat{q}_{\psi,Q}}$ *is called* $\hat{q}_{\psi,Q}$*-mutator of A and B operators.*

Note: $Q(\partial_\psi)\, \hat{x}_{Q(\partial_\psi)} - \hat{q}_{\psi,Q} \hat{x}_{Q(\partial_\psi)} Q(\partial_\psi) \equiv \left[Q(\partial_\psi), \hat{x}_{Q(\partial_\psi)}\right]_{\hat{q}_{\psi,Q}} = \mathrm{id}.$

This is easily verified in the ∂_ψ-basic $\{p_n\}_{n\geq 0}$ of the ∂_ψ-delta operator $Q(\partial_\psi)$.

Equipped with pair of operators $(Q(\partial_\psi),\ \hat{x}_{Q(\partial_\psi)})$ and $\hat{q}_{\psi,Q}$-mutator we have at our disposal all possible representants of "canonical pairs" of differential operators on the P algebra. For historical reasons let us however at first quote a suitable remark [1].

Remark 2.1. The ψ-derivative is a particular example of a linear operator that reduces by one the degree of any polynomial. In 1901 it was proved [23] by Pincherle and Amaldi that every linear operator T mapping P into P may be represented as infinite series in operators $\hat{x}$ and D. In 1986 Kurbanov and Maximov [24] supplied the explicit expression for such series in most general case of polynomials in one variable; namely according to Proposition 1 from [24] one has: " *Let $\mathcal{D}$ be a linear operator that reduces by one each polynomial.* Let $\{q_n(\hat{x})\}_{n\geq 0}$ *be an arbitrary sequence of polynomials in the operator $\hat{x}$. Then $T = \sum_{n\geq 0} q_n(\hat{x})\mathcal{D}^n$ defines a linear operator that maps polynomials into polynomials. Conversely, if T is linear operator that maps polynomials into polynomials then there exists a unique expansion of the form*

$$T = \sum_{n\geq 0} q_n(\hat{x})\mathcal{D}^n\text{"}.$$

Note: In 1996 this was extended to algebra of many variables polynomials [25].

Remark 2.2. The importance of the pair of dual operators: $Q(\partial_\psi)$ and $\hat{x}_{Q(\partial_\psi)}$ is reflected by the facts:

a) $Q(\partial_\psi)\, \hat{x}_{Q(\partial_\psi)} - \hat{q}_{\psi,Q} \hat{x}_{Q(\partial_\psi)} Q(\partial_\psi) \equiv \left[Q(\partial_\psi), \hat{x}_{Q(\partial_\psi)}\right]_{\hat{q}_{\psi,Q}} = \mathrm{id}.$

b) *Let* $\{q_n\left(\hat{x}_{Q(\partial_\psi)}\right)\}_{n\geq 0}$ *be an arbitrary sequence of polynomials in the operator* $\hat{x}_{Q(\partial_\psi)}$. *Then* $T = \sum_{n\geq 0} q_n\left(\hat{x}_{Q(\partial_\psi)}\right) Q(\partial_\psi)^n$ *defines a linear operator that maps polynomials into polynomials. Conversely, if T is linear operator that maps polynomials into polynomials then there exists a unique expansion of the form*

$$T = \sum_{n\geq 0} q_n\left(\hat{x}_{Q(\partial_\psi)}\right) Q(\partial_\psi)^n. \tag{2.2}$$

Equipped with pair of operators $(Q(\partial_\psi),\ \hat{x}_{Q(\partial_\psi)})$ and $\hat{q}_{\psi,Q}$-mutator we have at our disposal all possible representants of "canonical pairs" of differential operators on the P algebra such that:

a) the above unique expansion $T = \sum_{n\geq 0} q_n\left(\hat{x}_{Q(\partial_\psi)}\right) Q(\partial_\psi)^n$ holds

b) we have the structure of ψ-umbral or ψ-extended finite operator calculus - coworking.

3. Looking for ψ-analogue of quantum q-plane formulation

Cigler and Kirschenhofer defined in [11, 12] the polynomial sequence $\{p_n\}_o^\infty$ of q-binomial type equivalently by

$$p_n(A+B) \equiv \sum_{k\geq 0} \binom{n}{k}_q p_k(A) p_{n-k}(B) \quad \text{where} \quad [B,A]_q \equiv BA - qAB = 0. \tag{3.1}$$

A and B might be interpreted then as coordinates on quantum q-plane. For example $A = \hat{x}$ and $B = y\hat{Q}$ where $\hat{Q}\varphi(x) = \varphi(qx)$. With this being adopted the following identification holds:

$$p_n(x +_q y) \equiv E^y(\partial_q) p_n(x) = \sum_{k\geq 0} \binom{n}{k}_q p_k(x) p_{n-k}(y) = p_n\left(\hat{x} + y\hat{Q}\right) 1$$

Also q-Sheffer polynomials $\{s_n(x)\}_{n=0}^\infty$ are defined equivalently (see 2.1.1. Kirschenhofer in [12]) by

$$s_n(A+B) \equiv \sum_{k\geq 0} \binom{n}{k}_q s_k(A) p_{n-k}(B) \tag{3.2}$$

where $[B,A]_q \equiv BA - qAB = 0$ and $\{p_n(x)\}_{n=0}^\infty$ of q-binomial type. For example $A = \hat{x}$ and $B = y\hat{Q}$ where $\hat{Q}\varphi(x) = \varphi(qx)$. Then the following identification takes place:

$$s_n(x +_q y) \equiv E^y(\partial_q) s_n(x) = \sum_{k\geq 0} \binom{n}{k}_q s_k(x) p_{n-k}(y) = s_n\left(\hat{x} + y\hat{Q}\right) 1 \tag{3.3}$$

This means that one may formulate q-extended finite operator calculus with help of the "quantum q-plane" q-commuting variables A, B:
$AB - qBA \equiv [A,B]_q = 0$.

Let us now try to formulate — perhaps in vain — the basic notions of ψ-extended finite operator calculus with help of the "quantum ψ-plane" $\hat{q}_{\psi,Q}$-commuting variables $A, B : [A,B]_{\hat{q}_{\psi,Q}} = 0$ exactly in the same way as it was done by Cigler and Kirschenhofer in [11], [12].

For that to do let us consider appropriate generalization of $A = \hat{x}$ and $B = y\hat{Q}$ where this time the action of $\hat{Q}$ on $\{x^n\}_0^\infty$ is to be found from the condition

$$AB - \hat{q}_\psi BA \equiv [A,B]_{\hat{q}_\psi} = 0.$$

Acting with $[A,B]_{\hat{q}_\psi}$ on $\{x^n\}_0^\infty$ one easily sees that due to $\hat{q}_\psi x^n = \frac{(n+1)_\psi - 1}{n_\psi} x^n$; $n \geq 0$, $\hat{Q}x^n = b_n x^n$ where $b_0 = 0$ and $b_n = \prod_{k=1}^{n} \frac{(k+1)_\psi - 1}{k_\psi}$ for $n > 0$ is the solution of the difference equation: $b_n - b_{n-1}\frac{(n+1)_\psi - 1}{n_\psi} = 0$; $n > 0$.

With all above taken into account one immediately verifies that for our A and B $\hat{q}_\psi$-commuting variables already

$$(A+B)^n \neq \sum_{k\geq 0} \binom{n}{k}_\psi A^k B^{n-k} \tag{3.4}$$

unless $\psi_n(q) = \frac{1}{R(q^n)!}$; $R(x) = \frac{1-x}{1-q}$ hence $\hat{q}_{\psi,Q} \equiv \hat{q}_{R,\mathrm{id}} \equiv \hat{q}_R \equiv \hat{q}_{q,\mathrm{id}} \equiv \hat{q}_q \equiv \hat{q}$ and $\hat{q}_{\psi,Q} x^n = q^n x^n$ i.e. unless we are back to the q-case.

In conclusion one sees that the above identifications of polynomial sequence $\{p_n\}_o^\infty$ of q-binomial type and Sheffer q-polynomials $\{s_n(x)\}_{n=0}^\infty$ fail to be extended to the more general ψ-case. This means that we cannot formulate *that way* the ψ-extended finite operator calculus with help of the "quantum ψ-plane" $\hat{q}_{\psi,Q}$-commuting variables $A, B : AB - \hat{q}_{\psi,Q} BA \equiv [A,B]_{\hat{q}_{\psi,Q}} = 0$ while considering algebra of polynomials P over the field F.

Nevertheless — already the q-case is already reach enough in abundant applications to various "q-quantum mechanical models" — $q \equiv \omega \equiv \exp\left\{\frac{2\pi i}{n}\right\}$ case included. One may expect the natural use of q-umbral calculus in these applications to be advantageous. Models using $\hat{q}_{\psi,Q}$-mutator $\left[Q(\partial_\psi), \hat{x}_{Q(\partial_\psi)}\right]_{\hat{q}_{R,Q}} = \mathrm{id}$ relations are suitable play-ground for ψ-umbral calculus (leading perhaps to ψ-lasers ? — see the q-footnote in [2, p. 1887]).

For the most general cases and for further links to further readings the reader is referred to [27] and [28].

For very recent and qualitatively new applications of q-umbral and $\psi(q)$-calculus one is referred to [29], [30], [31] and [32]. There — due to the invention of a specific $*\psi$ product of formal series — new families of $\psi(q)$-extensions of Poisson processes and q-Bernoulli-Taylor formula with the rest q-term of the Cauchy type are derived among others.

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