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BIHAMILTONIAN SYSTEMS IN THE QUANTUM-CLASSICAL TRANSITION

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ABSTRACT. Mathematical structures appearing in the various Pictures of Quantum Mechanics are considered. In particular Weyl-Wigner Picture is considered for a better comparison with the classical limit on phase space in relation to alternative Hamiltonian descriptions for dynamical systems.

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1. INTRODUCTION

At the beginning of last century, several problems (black-body radiation, photoelectric effect, absorption and emission spectra) required a radical departure from the classical formalization (description) of physical phenomena, i.e. particles (matter) and radiation (waves). Newton's equation (or the Lagrangian and Hamiltonian description) on one side and Maxwell equations on the other side where describing the two aspects separately.

The physical examples we have mentioned required that both attributes (i.e. wave and particle ones) should be possessed by radiation (with the introduction of quanta of light, photons) and by particles. For instance electrons should have also wave attributes. By using the Eikonal 1-form $\vec{k}\,d\vec{x} - \omega\,dt$, where on space time $\vec{k}$ is the wave

The paper is in final form and no version of it will be submitted elsewhere.

vector and ω the frequency, the correspondence between particle attributes and wave attributes was achieved by the Einstein-de Broglie identification

$$\vec{p}\,d\vec{x} - H\,dt = \hbar\,(\vec{k}\,d\vec{x} - \omega\,dt)\,.$$

Reading this relation from left to right allows to associate a wave vector and a frequency with the momentum and the energy respectively; reading it from right to left it is possible to associate "particle concepts" with wave attributes. These "new objects" have also been called "waviscles". It took of the order of two decades before a sound formulation of the "new mechanics" was proposed. It appeared in the form of Schroedinger equations and Heisenberg equations, rather different proposals which where proved to be equivalent very soon.

They represented a radical departure from the old mechanics and provided a unification of concepts like particle and wave. The carrier space for Schroedinger's description was a space of square integrable functions on space (or space-time with time playing the role of a parameter), the carrier space for Heisenberg's description was the space of the infinite dimensional matrices where an associative product was used (inspired by the Ritz combination principle coming from atomic spectra) [1]. The unification of the two views was achieved by realizing that these infinite dimensional matrices were just operators on the Hilbert space of square integrable functions used by Schroedinger. From there onward the theory developed into a quantum theory for fields and particles and by now it is believed to be our basic theory for the description of all physical phenomena. In particular it is believed that the classical description should be derivable from the quantum one when an appropriate limit is considered. This is very much like the limit we consider to get geometric optics from wave optics (it would give the classical limit in the Schroedinger description). Here we would like to consider the limit in the Heisenberg picture, because we expect that the quantum commutator of matrices would give rise to the classical Poisson bracket among functions on phase space. In particular this would allow us to tackle the main question of this paper. We know that many classical physical systems admit alternative Hamiltonian descriptions (this phenomenon is quite usual for completely integrable systems): how to obtain these alternative classical Hamiltonian descriptions as limit of quantum description?

From what we have said, it is quite natural to look for alternative commutation relations (alternative Lie algebra structures) on the space of infinite matrices (operators). Here the requirement that this Lie product defines derivations on the associative algebra of matrices (Dirac called quantum Poisson brackets those Lie products that would satisfy this condition) puts very strong conditions, indeed Dirac [2] proved that all quantum Poisson brackets have necessarily the form

$$[A, B] = \lambda(A \cdot B - B \cdot A)$$

i.e. they are multiples of the skew symmetrization of the associative product.

Therefore in looking for alternative quantum Poisson brackets one should look for alternative associative products on the space of operators ("infinite matrices").

For a better understanding of the classical limit and the comparison with alternative Hamiltonian descriptions for classical systems, it is convenient to consider the Weyl-Wigner picture of Quantum Mechanics. In this picture the "ambient space" for both, classical and quantum descriptions, is the phase-space.

In this description, quantum mechanics appears to be associated with a deformation of the commutative point-wise product into a non-local and non-commutative product. When the deformation parameter (here identified with the Planck constant) tends to zero, the commutator of the deformed product reproduces the Poisson bracket. Therefore in this description the quantum-classical comparison is more transparent.

The approach we are going to take is the following one: we first review the Schroedinger approach and consider the problem of alternative Hamiltonian descriptions for quantum evolution on Hilbert spaces, then we consider the equation of motion in the Heisenberg form and mention alternative description in this setting, afterwards we consider the connection of the two via the von Neumann approach [3].

A simple concrete example is considered. It will be used as a paradigmatic example to illustrate alternative descriptions in the various situations we have considered.

2. Schroedinger Picture

This formulation was arrived at by following the analogy of the correspondence between wave optics and geometric optics. It was accepted by the community of classical physicists more easily than the analogous formulation by Heisenberg in terms of infinite matrices. It goes as follows:

On square integrable functions on space, time is considered to be a parameter, one writes the evolution equation $(i\hbar\partial_t - H)\psi = 0$. The wave function is interpreted physically to represent a probability amplitude. This interpretation allows to write mean values, dispersion-free states and most of what goes with the notion of probability.

In this framework the classical limit is recovered by using eikonal variables. After this change of variables has been performed one finds two equations, one giving Hamilton-Jacobi equation and the other one is a "transport equation" on configuration space.

If we consider Schroedinger equation on square-integrable functions $\Psi : \mathbb{R}^3 \to \mathbb{C}$, associated with the Schroedinger operator

$$\mathcal{H} = -\frac{\hbar}{2m}\Delta + V(r),$$

where Δ is the Laplacian operator and $V(r)$ is the potential, we have

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar}{2m}\Delta\Psi + V(r)\Psi .$$

It is possible to rewrite this equation in terms of a pair of real valued functions [4]

$$\frac{d}{dt}\begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{\hbar}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\delta H_1}{\delta p} \\ \frac{\delta H_1}{\delta q} \end{pmatrix}$$

where

$$p(r,t) = \Im m\Psi(r,t), \quad q(r,t) = \Re e\Psi(r,t)$$

and

$$H_1(p,q) = \frac{1}{2}\int d^3r \left\{\frac{\hbar^2}{m}[(\nabla p)^2 + (\nabla q)^2] + V(r)(p^2+q^2)\right\}$$

with $\frac{\delta H_1}{\delta p}$ and $\frac{\delta H_1}{\delta q}$ representing the functional gradient of H_1 with respect to the real L_2 scalar product. This system is then a Hamiltonian dynamical system with respect to the Poisson bracket defined by

$$\{F,G\}_1 = \frac{1}{\hbar}\int d^3r \left(\frac{\delta F}{\delta q}\frac{\delta G}{\delta p} - \frac{\delta F}{\delta p}\frac{\delta G}{\delta p}\right).$$

Previous equations of motion are also Hamiltonian with a different Hamiltonian and a different symplectic structure. We have

$$\frac{d}{dt}\begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{\hbar}\begin{pmatrix} 0 & -\mathcal{H} \\ \mathcal{H} & 0 \end{pmatrix}\begin{pmatrix} \frac{\delta H_0}{\delta p} \\ \frac{\delta H_0}{\delta q} \end{pmatrix}$$

where

$$H_0 = \frac{1}{2}\int d^3r \; (p^2 + q^2)$$

and $\mathcal{H}$ is the Schroedinger operator previously defined. The new Poisson bracket, for any two functionals $F(q,p)$ and $G(q,p)$, is given by

$$\{F,G\}_0 = \int d^3r \left(\frac{\delta F}{\delta q}\mathcal{H}\frac{\delta G}{\delta p} - \frac{\delta F}{\delta p}\mathcal{H}\frac{\delta G}{\delta p}\right).$$

In conclusion, the same vector field can be given two different Hamiltonian descriptions. As a matter of fact one can generate infinitely many conserved functionals defined by

$$H_n(p,q) = \frac{1}{2}\int d^3r(p\mathcal{H}^n q + q\mathcal{H}^n p) \equiv \int d^3r(\Psi^*\mathcal{H}^n\Psi).$$

They are all in involution with respect to either one of previous Poisson brackets:

$$\{H_n, H_m\}_0 = 0 = \{H_n, H_m\}_1 .$$

This situation generalizes the one for finite dimensional Hamiltonian systems.

3. Eikonal coordinates [4]

It is possible to introduce coordinates (ρ, π), defined by

$$p(r,t) = \sqrt{\rho}\sin\frac{S(r,t)}{\hbar}; \quad q(r,t) = \sqrt{\rho}\cos\frac{S(r,t)}{\hbar}$$

with

$$\pi = \frac{S(r,t)}{2\hbar}.$$

The Hamiltonian H_1 becomes

$$K_1[\rho,\pi] = H_1(p,q) = \int d^3r \frac{\hbar^2}{2m}\rho\left\{[\frac{(\nabla\rho)^2}{4\rho^2} + (\nabla\pi)^2] + V\right\}$$

with corresponding Hamilton equations

$$\frac{d\pi}{dt} = -\frac{1}{\hbar}\frac{\delta K_1}{\delta\rho}$$

$$\frac{d\rho}{dt} = \frac{1}{\hbar}\frac{\delta K_1}{\delta \pi}\,.$$

More explicitly

$$\frac{\partial \pi}{\partial t} = \frac{\hbar}{2m}\frac{\Delta(\sqrt{\rho})}{\sqrt{\rho}} - \frac{\hbar}{m}(\nabla\pi)^2 - \frac{V}{\hbar}$$

$$\frac{\partial \rho}{\partial t} = -\frac{2\hbar}{m}\mathrm{div}(\rho\nabla\pi)\,.$$

Quantities ρ and $\vec{J} = \hbar\rho\frac{\nabla S}{m}$ represent the probability density and the current density respectively. By making more explicit these equations on finds

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{m} + V = \frac{\hbar^2}{2m}\frac{\Delta(\sqrt{\rho})}{\sqrt{\rho}}$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\frac{\nabla S}{m}) = 0\,.$$

If terms of order $\hbar^2$ and higher are neglected, our system describes a classical flow of particles without mutual interactions, in the potential V, with action S, invariant density ρ and velocity field $v = \frac{\nabla S}{m}$, satisfying

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{m} + V = 0$$

which represent the Hamilton-Jacobi equation associated with $H = \frac{p^2}{2m} + V$, and $\frac{\partial \rho}{\partial t}+\mathrm{div}(\rho\frac{\nabla S}{m}) = 0$ which represent the continuity equation for the probability density $\rho = \Psi^*\Psi$ and its current $\rho\frac{\nabla S}{m} = \frac{\hbar^2}{2im}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*)$.

We learn from this example that the classical limit of the Schroedinger equation provides us with a classical description on the cotangent bundle of the configuration space.

We consider now Schroedinger equation on an abstract Hilbert space $\mathbb{H}$, without a specific realization in terms of square integrable functions on some appropriate space [5].

Equations of motion have the expression

$$i\hbar\frac{d}{dt}\Psi = H\Psi\,.$$

Here H is some Hermitian operator with respect to the inner Hermitian product on the Hilbert space. From previous example we have learned that the vector field is Hamiltonian with respect the symplectic structure defined by the imaginary part of the Hermitian structure defined on $\mathbb{H}$.

Indeed, $h = g+i\omega$ denotes a decomposition of the Hermitian structure into a positive definite symmetric part g and a skew-symmetric part ω.

If H is a self-adjoint operator with respect to the Hermitian structure h , by exponentiation it defines a unitary flow. The two structures g and ω determine a Kaehler structure. Looking for alternative Hamiltonian descriptions for our dynamical equations on $\mathbb{H}$ amounts to search for other Kaehler structures which are compatible with the topology in $\mathbb{H}$, are compatible with the complex structure in $\mathbb{H}$, are invariant under the dynamical flow. It is easy to construct a large family of such Kaehler structures [6]. The simplest one is a family of Kaehler structures which are constant on $\mathbb{H}$. They have the form

$$(\Psi, \Phi) = \langle \Psi, A\Phi \rangle$$

i.e.

$$h_A(\Psi, \Phi) \equiv h(\Psi, A\Phi) .$$

It is easy to show that the scalar product defined by h_A is invariant if and only if $[H, A] = 0$. In our previous example we took A to be H itself and conserved functionals were associated with powers of H.

It is clear that by selecting, in the centralizer of H, all operators A which have the property that h_A is a new Hermitian structure on $\mathbb{H}$ we find a large family of alternative Hamiltonian description of our dynamical evolution. In the Schroedinger picture h_A will be written as

$$(\Psi, \Phi)_A = \frac{1}{2} \int d^3r (\Psi^*, A\Phi) .$$

4. VON NEUMANN-HEISENBERG EQUATION AND ITS RELATION WITH SCHROEDINGER EQUATION [7]

Starting with

$$i\hbar \frac{d}{dt} \Psi = H\Psi$$

and using Dirac notation $\langle \mid \rangle$ for the standard Hermitian structure, we have

$$i\hbar \frac{d}{dt} |\Psi\rangle = H |\Psi\rangle \quad , \; i\hbar \frac{d}{dt} \langle \Psi | = - \langle \Psi | H$$

where H is supposed to act on the right or on the left as the case may be. Introducing the projector

$$\rho_\Psi = \frac{|\Psi\rangle \langle \Psi |}{\langle \Psi | \Psi \rangle}$$

we find

$$i\hbar \frac{d}{dt} \rho_\Psi = [H, \rho_\Psi]$$

which is called the von Neumann evolution equation for the density matrix ρ.

In more formal terms, we have defined a projection

$$\mathbb{C} - \{0\} \to \mathbb{H} - \{0\} \xrightarrow{\pi} \mathbb{P}\mathbb{H}$$

from the Hilbert space to the projective Hilbert space. If we denote by π the projection $\pi : |\Psi\rangle \longmapsto \rho_\Psi$ we find that von Neumann equation on $\mathbb{P}\mathbb{H}$ is π-related with the Schroedinger equation on $\mathbb{H}$.

The major interest for this equation has to do with the possibility of going beyond pure states (i.e. projectors). Indeed it is possible to extend this equation to any convex combination of projectors, i.e. density matrices or mixed states. As a matter of fact we may even consider arbitrary combinations with real coefficients and get trace-class operators and by completion we go to Hilbert-Schmidt operators. Disregarding mathematical subleties, we can write equation of motion in the form

$$i\hbar\frac{d}{dt}A = [H, A]$$

with A any Hermitian operator, as a matter of fact we shall also consider non-Hermitian operators. The evolution equation on the space of operators is usually associated with the name of Heisenberg who considered "infinite matrices".

It is now clear that alternative Hermitian structures like h_A will give rise to a new product on the space of operators of the form

$$M \underset{A}{\bullet} N = MAN$$

i.e. we define a new associative product on the space of operators which is compatible with the dynamical evolution if and only if $[A, H] = 0$. Compatibility here means that the equation of motion have the form

$$i\hbar\frac{d}{dt}B = H_A AB - BAH_A = [H_A, B]_A \ .$$

Equating the two expressions obtained for the two different products, we find

$$H_A AB - BAH_A = HB - BH$$

i.e.

$$(H_A A - H)B - B(H - AH_A) = 0 \, .$$

Using the fact that the dynamics is compatible with the two products, i.e.

$$H_A A = AH_A$$

we get that $H_A A - H$ must be a central element in the algebra of operators, i.e. $H_A A - H = \lambda \mathbf{1}$, because h_A is a Hermitian structure, we find

$$H_A = A^{-1}(H + \lambda \mathbf{1}) \, .$$

Thus we have found again that appropriate choice of operators in the centralizer of H will provide us with alternative Heisenberg descriptions for the dynamical evolution on the space of observables.

To consider the classical limit of this description it is convenient to introduce the Weyl-Wigner picture of Quantum Mechanics.

5. Weyl-Wigner Picture [8, 9]

Given a symplectic vector space (E, ω), a Weyl map is a strongly continuous map from E to unitary operators on some Hilbert space $\mathbb{H}$:

$$W : E \to \mathcal{U}(\mathbb{H})$$

satisfying the condition

$$W(e_1)W(e_2)W^\dagger(e_1 + e_2) = \mathbf{1}e^{\frac{i}{2}\omega(e_1, e_2)} \, .$$

It is a projective unitary representation of the Abelian vector group associated with E.

A theorem due to von Neumann says that there exists such a map for any finite dimensional symplectic vector group. Indeed the Hilbert space $\mathbb{H}$ can be realized as the space of square integrable functions on any Lagrangian subspace of E. By using a Lagrangian subspace L it is possible to decompose E into

$$E = L \oplus L^* = T^*L = L^* \oplus (L^*)^* = T^*(L^*) .$$

The Lebesgue measure is a translational invariant measure on L and we have a specific realization of W.

We define

$$U = W|_{L^*} , \quad V = W|_L$$

and their action on $\mathcal{L}^2(L, d^n x)$ is given by

$$\begin{aligned} (V(y)\Psi)(x) &= \Psi(x+y) \\ (U(\alpha)\Psi)(x) &= e^{i\alpha(x)}\Psi(x) \end{aligned}$$

for $x, y \in L$, $\alpha \in L^*$, $\Psi \in \mathcal{L}^2(L, d^n x)$.

The strong continuity requirement in the definition of W allows to use Stone's theorem and we get

$$W(v) = e^{iR(v)} , \quad v \in E$$

with $R(v)$, the infinitesimal generator of one parameter unitary group $W(tv)$, $t \in \mathbb{R}$, depending linearly on v.

When we select a complex structure J,

$$J : E \to E , \quad J^2 = -1 ,$$

it is possible to define "creation" and "annihilation" operators

$$a(v) = \frac{1}{\sqrt{2}} \{R(v) + iR(Jv)\}$$

$$a^{\dagger}(v) = \frac{1}{\sqrt{2}} \{R(v) - iR(Jv)\} .$$

With this complex structure we also associate an inner product on E by setting

$$< v_1, v_2 >= \omega(Jv_1, v_2) - i\omega(v_1, v_2) .$$

The Wigner map is the inverse of the Weyl map, i.e. with any operator A acting on $\mathbb{H}$ we associate a function f_A on the symplectic space E by setting

$$f_A(v) = \mathrm{Tr} A W(v) , \ v \in E .$$

Clearly this map will be defined only on suitable operators A.

Let us analyze a little more this map. With any vector $(x, \alpha) \in L \oplus L^*$ we associate the unitary operator

$$W(x, \alpha) = \exp[i \frac{1}{\hbar}(x\hat{p} + \alpha\hat{q})] .$$

Here $\hat{p}$ and $\hat{q}$ represent the infinitesimal generators, according to Stone's theorem, associated with vectors $(0,1)$ and $(1,0)$ in $L \oplus L^*$, respectively. With any function on $L \oplus L^*$, say $\tilde{g}$, we associate its Fourier transform

$$g(p,q) = \frac{1}{(2\pi)^3} \int d^3x d^3\alpha \tilde{g}(x,\alpha) e^{i(xp+\alpha q)} ;$$

with this function we associate an operator by replacing the exponential $e^{i(xp+\alpha q)}$ with the unitary operator $W(x,\alpha)$, i.e. $\exp[i\frac{i}{\hbar}(x\hat{p} + i\alpha\hat{q})]$. This correspondence between functions and operators is a unitary isomorphism (for Hilbert-Schmidt operators on L and square integrable functions on $L \oplus L^*$).

It allows to induce a product on functions on phase-space by setting

$$(g_1 * g_2)(v) = \mathrm{Tr} A_1 A_2 W(v) .$$

If we use different expressions for W, say $W'(x,\alpha) = (\exp i\alpha\hat{Q})(\exp ix\hat{P})$ or

$$W'' = \exp((x + i\alpha)a^\dagger - (x - i\alpha)a)$$

with

$$a = \frac{\hat{Q} + i\hat{P}}{2}, \quad a^\dagger = \frac{\hat{Q} - i\hat{P}}{2} .$$

We find a different associative and therefore different deformed products. Because they are associated with different orderings of the Weyl operators, we consider them to be equivalent [10]. The transformation taking from one product to another is usually called a gauge transformation [11].

If we introduce complex coordinate $z = x + i\alpha$, $z^* = x - i\alpha$ the operator W'' is usually denoted by D (for displacement) and we have

$$D(z) = \exp(za^\dagger - z^* a) .$$

This operator is used to introduce coherent states and Bargmann-Fock representation.

Going back to the Weyl map, it is easy to see that $W : E \to \mathcal{U}(\mathbb{H})$ is an equivariant map with respect to symplectic transformation on (E,ω). If T is any symplectic transformation we have $\nu_T(W(v)) \equiv W(Tv)$.

In particular it says that infinitesimal Hamiltonian generators of one-parameter groups with quadratic functions on E will give rise to unitary transformations. More specifically, they define automorphisms of the unitary group and therefore they will be associated with unitary transformations acting via conjugation [7] on $\mathcal{U}(\mathbb{H})$. It is instructive to consider the example of the one dimensional harmonic oscillator.

Here $E = \mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}^*$. For any two vectors (x_1,α_1), (x_2,α_2) we have a symplectic structure $\omega((x_1,\alpha_1),(x_2,\alpha_2)) = \alpha_1(x_2) - \alpha_2(x_1)$. Moreover $R(x,\alpha) = x\hat{p} + \alpha\hat{q}$. The Hamiltonian

$$H = \frac{1}{2}(\hat{p}^2 + \hat{q}^2)$$

determines the time evolution in the Heisenberg picture given by

$$\nu_t(A) = A_t = e^{iHt} A e^{-iHt} .$$

For the particular case of $\hat{p}$ and $\hat{q}$, the action of the automorphism group is given by

$$\begin{aligned}\hat{p}_t &= \hat{p}\cos t - \hat{q}\sin t\\ \hat{q}_t &= \hat{p}\sin t + \hat{q}\cos t\,.\end{aligned}$$

This gives the action on $R(x,\alpha)$, $\nu_t R(x,\alpha) = R(x_t,\alpha_t)$

$$\begin{aligned}\nu_t[R(x,\alpha)] &= x\hat{p}_t + \alpha\hat{q}_t = (x\cos t + \alpha\sin t)\hat{p} + (-x\sin t + \alpha\cos t)\hat{q}\\ &= R(x_t,\alpha_t)\end{aligned}$$

i.e.

$$x_t = x\cos t + \alpha\sin t\,, \quad \alpha_t = -x\sin t + \alpha\cos t\,.$$

Clearly this transformation is symplectic on $\mathbb{R}^2$ with respect to the symplectic structure defined by $\alpha_1(x_2) - \alpha_2(x_1)$.

In complex coordinates, by denoting with $|0>$ the vacuum state or the fundamental state of $H = \frac{1}{2}(\hat{p}^2 + \hat{q}^2)$, coherent states are defined by

$$|z> = D(z)|0> \ .$$

Where, we recall, $D(z) = \exp(za^\dagger - z^* a)$. According to our previous computations,

$$|z_t> = D(z_t)|0> = \nu_t(D(z))|0> \ .$$

This relation explains the meaning of the equivariance of the Weyl map and the fact that the classical evolution associated with the Harmonic oscillator corresponds exactly to the quantum evolution.

6. Wigner map and $\star$-products

To consider the classical limit within the Heisenberg picture, we shall give more details on the Wigner map.

We start with a symplectic vector space $\mathbb{R}^2 = \mathbb{R}\oplus\mathbb{R}$, $\omega = dx\wedge dy$.

The map $\hat{A}\mapsto f_A$ given by $f_A(x,y) =$Tr $\hat{A}W(x,y)$ induces a product

$$(f_A \star f_B)(x,y) = \mathrm{Tr}\hat{A}\hat{B}W(x,y)\,.$$

By using the specific form of $W(x,y)$ it is possible to provide either an explicit form in terms of bidifferential operators or an integral form. We have

$$(f\star g)(x,y) = f(x,y)\, e^{i\frac{\hbar}{2}(\overleftarrow{\frac{\partial}{\partial x}}\overrightarrow{\frac{\partial}{\partial y}} - \overleftarrow{\frac{\partial}{\partial y}}\overrightarrow{\frac{\partial}{\partial x}})}\, g(x,y)$$

where a standard notation for physicists has been used, i.e. $\overleftarrow{\frac{\partial}{\partial x}}$ and $\overrightarrow{\frac{\partial}{\partial y}}$ mean that the operators act on the left or on the right, respectively.

As $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ commute, we can rewrite the $\star$-product in the following form

$$\begin{aligned}(f\star g)(x,y) &= f(x'+i\frac{\hbar}{2}\frac{\overrightarrow{\partial}}{\partial y}, y'-i\frac{\hbar}{2}\frac{\overrightarrow{\partial}}{\partial x})\,g(x,y)\Bigg|_{x'=x,\,y'=y}\\ &= f(x,y)\,g(x'-i\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial y}, y'+i\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial x})\Bigg|_{x'=x,\,y'=y}\end{aligned}$$

The expression in terms of bidifferential operators is very convenient when either one of f or g is a polynomial.

In integral form, the product has the following expression

$$(f\star g)(x,y) =$$

$$\int \frac{dx'dy'dx''dy''}{\hbar^2\pi^2} f(x',y')g(x'',y'')\exp[-\frac{2i}{\hbar}(y(x'-x'')+y'(x''-x)+y''(x-x')]\,.$$

The cyclic expression in the exponential represents twice the area of the phase space triangle with vertices $(x,y),(x',y'),(x'',y'')$ [12].

To recover the Poisson structure on the phase space, we notice that

$$\lim_{\hbar\to 0}\frac{f\star g-g\star f}{\hbar} = i\,\{f,g\}\ .$$

Therefore, the expression in terms of bidifferential operators, provides to first order in $\hbar$ the Poisson Bracket.

Equation of motion are given by

$$i\hbar\frac{df}{dt} = f_H\star f - f\star f_H\,.$$

All operations we can perform at the operational level have a counterpart, modulo suitable restrictions on the operators involved, at the level of $\star$-product.

If we compute the $\star$-product of linear functions, for instance, we find

$$(ax+by)\star(mx+ny) = (ax+by)(mx+ny)+\frac{i\hbar}{2}(an-bm)\,,$$

i.e. the pointwise product of linear functions differs from the nonlocal product by a term proportional to the symplectic area of the two vectors $(a,b),(m,n)$.

It is equally easy to show that for quadratic functions the $\star$-commutator coincides with usual Poisson bracket. This result is the counter part of the equivariance we have discussed for the Weyl map. Indeed:

$$(ax^2+by^2+cxy)\star(\alpha x^2+\beta by^2+\gamma xy) = (ax^2+by^2+cxy)(\alpha x^2+\beta by^2+\gamma xy)+$$

$$i\hbar(x^2(a\gamma-c\alpha)+y^2(c\beta-b\gamma)+2xy(a\beta-b\alpha)-\tfrac{i\hbar^2}{2}(a\beta+b\alpha-\tfrac{1}{2}c\gamma)$$

and

$$[(ax^2 + by^2 + cxy), (\alpha x^2 + \beta by^2 + \gamma xy)] =$$

$$i2\hbar(x^2(a\gamma - c\alpha) + y^2(c\beta - b\gamma) + 2xy(a\beta - b\alpha).$$

We have so far defined an associative $\star$-product between functions on the phase space by inducing it from the usual associative product of operators, via the Wigner map. If we rewrite the $\star$-product, in its integral form, using complex variables z, z', z'' we get

$$(f \star g)(z) = \frac{1}{\hbar^2\pi^2} \int d^2z' d^2z'' f(z') g(z'')\, L(z' - z, z'' - z)$$

with

$$L(z' - z, z'' - z) = \exp \frac{1}{\hbar}[(z' - z)(z'' - z)^* - (z' - z)^*(z'' - z)].$$

It is natural now to look for the most general $\star$-product on functions on the phase space which may be given by an integral expression like

$$(f \star g)(z) = \frac{1}{\hbar^2\pi^2} \int d^2z' d^2z'' f(z') g(z'')\, K(z', z'', z)$$

requiring only associativity, and forgetting any correspondence between operators and functions like the one given by the Wigner map. The associativity condition imposes on K the following requirement:

$$\int K(x, y, z)\, K(z, r, s)\, d^2z = \int K(x, z, s)\, K(z, y, r)\, d^2z.$$

This problem has been solved [13] within the subclass of kernels satisfying the following invariance property

$$K(z', z'', z) = F(z' - z,\, z'' - z).$$

The result is:

$$(f \star g)(z) = \frac{1}{\hbar^2\pi^2} \int d^2z' d^2z'' f(z') g(z'')\, F(z' - z,\, z'' - z)$$

with

$$F(z' - z,\, z'' - z) =$$

$$\int d^2x d^2y \Omega(x)\Omega(y)\Omega^{-1}(x + y)\, e^{-(x(z'-z)^* - x^*(z'-z) + y(z''-z)^* - y^*(z''-z))} e^{k(xy^* - yx^*)},$$

where Ω is an arbitrary function and k an arbitrary complex number. For special values of Ω and k we get, as a particular cases, the pointwhise commutative product of functions for $\Omega = 1$ and $k = 0$, or the $\star$-product induced by the Wigner map for $\Omega = 1$ and k real.

7. Classical limit of the deformed associative product

Now we shall consider the classical limit of the special deformation of the associative product given by

$$A \underset{K}{\bullet} B = A e^{\lambda K} B \,.$$

Clearly, we are getting

$$f_A \underset{K}{\star} f_B = f_A \star f_\lambda \star f_B \,,$$

where $f_\lambda(x,y) = \mathrm{Tr}\, e^{\lambda K} W(x,y)$.

It is convenient to write this product in the form

$$f \star f_\lambda \star g = [f(x + i\frac{\hbar}{2}\frac{\overrightarrow{\partial}}{\partial y}, y - i\frac{\hbar}{2}\frac{\overrightarrow{\partial}}{\partial x})\, f_\lambda(x,y)]\, g(x - i\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial y}, y + i\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial x})$$

as explained in the previous section.

When both f and g are linear functions, we have

$$[(a(x + i\frac{\hbar}{2}\frac{\overrightarrow{\partial}}{\partial y}) + b(y - i\frac{\hbar}{2}\frac{\overrightarrow{\partial}}{\partial x}))f_\lambda][m\,(x - i\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial y}) + n\,(y + i\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial x})] =$$

$$(axf_\lambda + ai\frac{\hbar}{2}\frac{\partial f_\lambda}{\partial y} + byf_\lambda - ib\frac{\hbar}{2}\frac{\partial f_\lambda}{\partial x})\,[m\,(x - i\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial y}) + n\,(y + i\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial x})] \,.$$

Collecting the terms, and using again the symbols f, f_λ, g we have the general expression

$$f \star f_\lambda \star g = f f_\lambda\, g + i\frac{\hbar}{2} f_\lambda \{f,g\} + i\frac{\hbar}{2} f\,\{f_\lambda, g\} + i\frac{\hbar}{2}\,\{f, f_\lambda\}\, g + O(\hbar^2) \,.$$

It appears that the associative point-wise product (the term of zero order in $\hbar$) is changed with the multiplication by f_λ. The Poisson Bracket is actually a Jacobi Bracket[1]. What is more relevant, however, for the problem we have in mind, is the fact that all these brackets, for different f_λ, are compatible pairwise. In particular, this implies that we are not getting in complete generality alternative Poisson brackets for bi-Hamiltonian classical systems [14]. This negative result requires that we consider more general deformations of the associative products on the space of operators.

8. General deformations of the associative products [15, 16, 17, 18, 19]

Let $(\mathcal{A}, \star)$ be an associative algebra and V be an $\mathcal{A}$–bimodule. We may assume that $\Psi : \mathcal{A} \to \mathrm{End}(V)$ is a linear representation (say a left-action) and $\Psi' : \mathcal{A} \to \mathrm{End}(V)$ a linear antirepresentation (say a right action) which commute.

[1] We recall that a Jacobi bracket is defined by a pair (X, Λ), a vector field X and a bivector field Λ, satisfying the two following properties: $i) L_X \Lambda = 0$, $ii) [\Lambda, \Lambda] = -2X \wedge \Lambda$, where the commutator for Λ is the Schouten bracket. On functions, they define a Lie algebra structure given by

$$[f, g] = \Lambda(df, dg) + f L_X g - g L_X f \,.$$

In the present situation, X is the Hamiltonian vector field associated with f_λ and Λ is the bivector $\Lambda = f_\lambda \Lambda_0$, Λ_0 being the one we obtain for $f_\lambda = 1$.

A k−cochain is a k−linear mapping from $\mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A}$ (k times) into V. The space of k−cochains, $C^k(\mathcal{A}, V)$ can be considered an additive group. For each $k \in \mathbb{N}$ we introduce a coboundary operator

$$\delta : C^k(\mathcal{A}, V) \to C^{k+1}(\mathcal{A}, V)$$

by setting

$$\begin{aligned}(\delta\alpha)(a_1, a_2, \ldots, a_{k+1}) &= a_1\alpha(a_2, \ldots, a_{k+1}) + \sum_{j=1}^{k}(-1)^J \alpha(a_1, \ldots, a_j \star a_{j+1}, \ldots, a_{k+1}) + \\ & (-1)^{k+1}\alpha(a_1, \ldots, a_k)a_{k+1}\,.\end{aligned}$$

It follows that $\delta \circ \delta = 0$. As usual we define cocicles in $C^k(\mathcal{A}, V)$ to be kernel of δ, $\delta\alpha = 0$, we denote them $\mathcal{Z}^k(\mathcal{A}, V)$, and coboundaries in $C^k(\mathcal{A}, V)$ to be the image of δ, $\delta(C^{k-1}(\mathcal{A}, V)) = \mathcal{B}^k(\mathcal{A}, V)$.

The quotient group $\mathcal{H}^k(\mathcal{A}, V) = \mathcal{Z}^k(\mathcal{A}, V)/\mathcal{B}^k(\mathcal{A}, V)$ is called the cohomology group.

When $k = 1$, we find

$$(\delta\alpha)(a_1, a_2) = a_1\alpha(a_2) + \alpha(a_1)a_2 - \alpha(a_1 \star a_2)$$

i.e. it is defined to be the deviation from being a derivation of the associative algebra.

For $k = 2$

$$(\delta\alpha)(a_1, a_2, a_3) = a_1\alpha(a_2, a_3) - \alpha(a_1 \star a_2, a_3) + \alpha(a_1, a_2 \star a_3) - \alpha(a_1, a_2)a_3\,.$$

We shall use these objects to consider deformations of the associative product. If $(\mathcal{A}, \cdot)$ is an associative algebra with unity, a new product is provided by

$$A \underset{K}{\bullet} B = AKB\,.$$

If we consider $T_K : \mathcal{A} \to \mathcal{A}$, $A \mapsto KA$, we find

$$T_K(A \underset{K}{\bullet} B) = T_K(A)T_K(B)\,.$$

We may generalize (and must, if we want to have non compatible Poisson Brackets in the classical limit) in the following way.

We consider an associative product

$$\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \quad , \quad (A, B) \mapsto AB$$

and a linear map $N : \mathcal{A} \to \mathcal{A}$. The map

$$\mu_N : (A, B) \mapsto A \underset{N}{\bullet} B = N(A)B + AN(B) - N(AB)$$

is a bilinear and defines a new algebra structure $(\mathcal{A}, \mu_N)$.

In previous terminology, $A \underset{N}{\bullet} B = (\delta_\mu N)(A, B)$. The obstruction for the linear map N to be a homomorphism of associative products is given by μ−Nijenhuis torsion of N

$$T_N(A, B) = N(A \underset{N}{\bullet} B) - N(A)N(B)\,.$$

We say that the linear map $N : \mathcal{A} \to \mathcal{A}$ is a μ-Nijenhuis tensor if $T_N(A, B) = 0$, for any $A, B \in \mathcal{A}$. We may show:

Theorem. The product μ_N is associative iff the μ-Nijenhuis torsion T_N of N is a two-Hochshild cocycle of the algebra $\mathcal{A}$, i.e.

$$\delta_\mu T_N(A,B,C) = AT_N(B,C) - T_N(AB,C) + T_N(A,BC) - T_N(A,B)C = 0\,.$$

If this is the case, μ_N is an associative product compatible with μ, i.e. $\mu + \lambda\mu_N$ are associative product for all λ. In particular if N is a $\mu-$Nijenhuis tensor, then μ_N is an associative product on $\mathcal{A}$ which is compatible with μ. The proof of this theorem follows by direct computation.

An interesting example of the Nijenhuis tensor is provided by the following construction.

If $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ is a decomposition of an associative algebra $\mathcal{A}$ into two subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ with projection P_1 and P_2 respectively, we may set

$$N = \lambda_1 P_1 + \lambda_2 P_2$$

and show that N is a Nijenhuis tensor.

At the level of 2×2 matrices, say

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\,,\; B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

We may consider $\mathcal{A}_1$ to be the algebra of upper triangular matrices

$$P_1 A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}\,,\; P_1 B = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}$$

and $\mathcal{A}_2$ be the (commutative) algebra of strictly lower-triangular matrices

$$P_2 A = \begin{pmatrix} 0 & 0 \\ a_{21} & 0 \end{pmatrix}\,,\; P_2 B = \begin{pmatrix} 0 & 0 \\ b_{21} & 0 \end{pmatrix}.$$

The new associative matrix multiplication associated with P_1 will be

$$A \circ B = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$$

or, by using P_2

$$A \circ' B = \begin{pmatrix} a_{12}b_{21} & 0 \\ 0 & a_{21}b_{12} \end{pmatrix}.$$

We notice that the decomposition

$$A = P_1(A) + P_2(A)$$

allows to write the product as

$$\begin{aligned} A \underset{P_1}{\circ} B &= P_1(A)B + AP_1(B) - P_1(AB) \\ &= P_1(A)P_1(B) + P_2(P_1(A)P_2(B) + P_2(A)P_1(B))\,. \end{aligned}$$

As an example in the space of operators, we may consider a and $a^\dagger$ and attribute a grading to the space of polynomials in a and $a^\dagger$, by attributing degree 1 to $a^\dagger$ and degree -1 to a. By selecting an ordering procedure we may consider monomials of positive degree, negative degree or zero degree. They will define subalgebras and we may apply previous procedures to generate new associative products. These subalgebras are invariant under the dynamical evolution defined by the Harmonic Oscillator, indeed we have

$$\frac{d}{dt}a = -ia \ , \ \frac{d}{dt}a^\dagger = ia^\dagger \, .$$

The relation between Nijenhuis tensors in the associative and Lie algebra cases is described by the following theorem.

If N is a μ−Nijenhuis tensor for an associative algebra $(\mathcal{A}, \mu)$, then N is a Nijenhuis tensor for the Lie algebra $(\mathcal{A}, [,])$, where $[A, B] = AB - BA$, and

$$[A, B]_N = A \underset{N}{\circ} B - B \underset{N}{\circ} A$$

is the deformed Lie bracket.

In connection with alternative Hamiltonian descriptions for a given dynamics, if the dynamical evolution defines a derivation with respect to μ, then it will define also a derivation with respect to μ_N, whenever it commutes with N. Therefore Nijenhuis tensors which commute with a given dynamical evolution will provide alternative Hamiltonian descriptions.

At this point we should consider specific examples and show that in the classical limit we may get alternative Poisson Brackets and look for their compatibility. These aspects will appear elsewhere.

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