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In: Jan Slovák and Martin Čadek (eds.): Proceedings of the 18th Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1999. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 59. pp. 155--160.

Persistent URL: http://dml.cz/dmlcz/701634

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 59 (1999) pp. 155-160

HARTOGS PHENOMENA FOR HERMITIAN VECTOR BUNDLES

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ABSTRACT. This article is an expanded version of the conference paper "Extension techniques for holomorphic vector bundles", presented at the 18th Winter School on Geometry and Physics, Srni, Czech Republic, January 10th – 17th, 1998. The following is a summary of recent results of the author concerning removable singularities for Hermitian- holomorphic vector bundles, and applications of these techniques to complex gauge fields.

1. HERMITIAN AND HOLOMORPHIC CONNECTIONS

Let X be an n–dimensional complex manifold, with smooth tangent bundle TX of real rank $2n$. X comes equipped with an operator $J \in C^\infty(X, TX \otimes (TX)^*)$ such that $J^2 = -id$, which splits the complexification of TX into two sub–bundles of complex rank n, corresponding to the eigenvalues $\mathbf{i}$, $-\mathbf{i}$ of J. Formally

$$\mathbf{C} \otimes_{\mathbf{R}} TX \cong T^{1,0}X \oplus T^{0,1}X ,$$

where the $-\mathbf{i}$–eigenbundle $T^{1,0}X$ corresponds to the "holomorphic tangent bundle" of X. Now consider a unitary vector bundle $\mathcal{E} \to X$, i.e., a smooth complex vector bundle equipped with a Hermitian inner product allowing the appropriate reduction of structure group to $\mathbf{U}(r)$. Moreover $\mathcal{E}$ will be a "holomorphic" vector bundle if and only if there exists a partial connection

$$\overline{\partial}_{\mathcal{E}} : C^\infty(\mathcal{E}) \to C^\infty((T^{0,1}X)^* \otimes \mathcal{E})$$

such that $\overline{\partial}_{\mathcal{E}}^2 = 0$. A fundamental property of such bundles is the existence of a unique connection ∇ which is compatible with both the Hermitian and the holomorphic structure of $\mathcal{E}$. On any sufficiently small open neighbourhood $U \subset X$, ∇ may be represented by a matrix of smooth one–forms $\mathbf{A}$, which splits into a pair of matrices $\mathbf{A}^{1,0} + \mathbf{A}^{0,1}$ corresponding to one–forms in dz_ν and $d\bar{z}_\mu$ respectively. Compatibility with respect to the Hermitian structure then entails that $\mathbf{A} \in C^\infty(U, (TX)^* \otimes \mathbf{u}(r))$, where $\mathbf{u}(r)$ denotes the Lie algebra of $\mathbf{U}(r)$, hence the Hermitian adjoint $\mathbf{A}^* = -\mathbf{A}$. Conversely, compatibility of ∇ with the holomorphic structure of $\mathcal{E}$ entails that $\mathbf{A}^{0,1}$ is holomorphically gauge–equivalent to zero, i.e., there exists a matrix g of linearly independent sections of $\mathcal{E}$ such that the linear system $\overline{\partial} g + \mathbf{A}^{0,1} g = \mathbf{0}$ is satisfied on U. The holomorphic structure $\overline{\partial}_{\mathcal{E}}$ is then represented locally by $\overline{\partial} + \mathbf{A}^{0,1}$, while the curvature F_∇ is represented by a form $\Sigma_{\mu,\nu} F_{\mu,\nu} dz_\nu \wedge d\bar{z}_\mu$ of bi–type (1,1) (cf. Atiyah [1]).

Conversely, suppose $\mathcal{E} \to X$ is a Hermitian vector bundle with connection ∇ and curvature form F_∇ of type $(1,1)$. Now on any sufficiently small neighbourhood $U \subset X$ it follows that $\overline{\partial}\mathbf{A}^{0,1} + \mathbf{A}^{0,1} \wedge \mathbf{A}^{0,1} = \mathbf{0}$. This is precisely the integrability condition

The paper is in final form and no version of it will be submitted elsewhere.

specified by the Newlander–Nirenberg theorem for global existence of a uniquely determined holomorphic structure $\overline{\partial}_{\mathcal{E}}$ (cf., e.g., [5]), with which ∇ is clearly compatible. Hence there is a one–one correspondence between holomorphic vector bundles and Hermitian connections of curvature type $(1,1)$. "Hermitian- holomorphic" vector bundles are consequently those which enjoy both structures.

Henceforth let Ω^1_X denote the holomorphic cotangent sheaf associated with $(T^{1,0})^*$. Given a holomorphic vector bundle $\mathcal{E} \xrightarrow{\pi} X$, a "holomorphic connection" on $\mathcal{E}$ is a $\mathbb{C}$-linear map

$$\hat{\nabla} : \mathcal{O}_X(\mathcal{E}) \to \Omega^1_X(\mathcal{E})$$

which obeys the Leibniz rule:

$$\hat{\nabla}(f\sigma) = df \otimes \sigma + f\hat{\nabla}\sigma \quad \forall f \in \mathcal{O}_X, \sigma \in \mathcal{O}_X(\mathcal{E}),$$

and is realised by the splitting of the exact sequence

$$0 \to \mathcal{E}nd(\mathcal{E}) \to J(\mathcal{E}) \xrightarrow{\pi_*} \mathcal{T}X \to 0 \ (1).$$

$\mathcal{T}X$ here denotes the sheaf of sections associated with $T^{1,0}X$, while $J(\mathcal{E})$ denotes the direct image under π of the sheaf of germs of holomorphic vector fields tangent to the total space of $\mathcal{E}$ which vary linearly along the fibres. Note that the tangent bundle along each fibre of $\mathcal{E}$ is trivial, hence any vector field lying in the kernel of π_* determines a linear endomorphism of that fibre. Moreover, the obstruction to splitting, and hence global existence of $\hat{\nabla}$, determines a unique cohomology class $\omega_{\mathcal{E}} \in H^1(X, \Omega^1\mathcal{E}nd(\mathcal{E}))$ (cf. [2]). In order to step outside the strictly holomorphic viewpoint, one may give the following alternative definition. Suppose ∇ is a Hermitian connection on $\mathcal{E}$ such that F_∇ is of type $(2,0)$. On any sufficiently small neighbourhood of X, the $\mathbf{A}^{0,1}$-component of ∇ clearly satisfies the Newlander–Nirenberg condition, hence we may write $\nabla = \nabla^{1,0} + \nabla^{0,1}$, where $\nabla^{0,1} = \overline{\partial}_{\mathcal{E}}$, while $\nabla^{1,0} = \hat{\nabla}$. Note that holomorphic vector bundles do not in general admit holomorphic connections unless X is Stein, in which case the cohomology $H^1(X, \Omega^1\mathcal{E}nd(\mathcal{E}))$ will vanish as a result of Cartan's Theorem B. Now the curvature F_∇ of a Hermitian connection compatible with the complex structure of $\mathcal{E}$ also represents a Dolbeault class in $H^1(X, \Omega^1\mathcal{E}nd(\mathcal{E}))$. If F_∇ represents the trivial class, then there exists $\varphi \in C^\infty(X, \Omega^1\mathcal{E}nd(\mathcal{E}))$ such that $\overline{\partial}_{\mathcal{E}}\varphi = F_\nabla$. In the affine space of connections on $\mathcal{E}$, $\hat{\nabla} := \nabla - \varphi$ defines a new connection, such that

$$F_{\hat{\nabla}} = -\partial\varphi + \varphi \wedge \varphi - \mathbf{A}^{1,0} \wedge \varphi - \varphi \wedge \mathbf{A}^{1,0}$$

is clearly of type $(2,0)$, hence $\hat{\nabla}$ is holomorphic. The cohomology classes represented by F_∇ and $\omega_{\mathcal{E}}$ are consequently the same. The relationship between the uniquely determined Hermitian connection ∇ and existence of a holomorphic connection $\hat{\nabla}$ will result in a new approach to removable singularities for the Yang–Mills and Bogomolny equations, to be discussed in the next section.

2. Extension Problems

The term "Hartogs phenomenon" may loosely be applied to a large class of results in complex analysis, in which a complex analytic structure, defined initially on the complement of a closed subset of a domain in $\mathbb{C}^{n+1}$, is to be extended to the entire domain by means of a specially constructed figure. A typical Hartogs figure H consists of a union of two sets, the first of which corresponds to the cartesian product of a polydisc $D \subseteq \mathbb{C}^n$ with an annulus $\mathcal{A} \subset \mathbb{C}$, while the second corresponds to the product of an open ball strictly contained in D with a disc which "fills in" the annulus. Hartogs observed that for a holomorphic function f defined on this union, D provides a holomorphic parametrization of the Cauchy integral formula around $\mathcal{A}$, hence defining a holomorphic function $\hat{f}$ on the smallest polydisc Δ containing $D \times \mathcal{A}$. Moreover, it follows from the Cauchy–Morera theorem that $\hat{f} \equiv f$ when restricted to the second component of H, hence $\hat{f}$ becomes the unique holomorphic extension of f to Δ. The utility of Hartogs figures in a wide range of extension problems, from continuation of holomorphic and meromorphic functions to extension of analytic subvarieties and coherent analytic sheaves, has been surveyed by Siu [10], drawing together the work of several authors. There are numerous examples which illustrate the problem of extension of holomorphic vector bundles. If

$$\varpi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}_n$$

denotes the natural quotient map, then an elementary example of a bundle $\mathcal{E} \to \mathbb{C}^{n+1} \setminus \{0\}$ which cannot be extended is given by the pullback $\varpi^* \mathcal{T}\mathbb{CP}_n$ of the holomorphic tangent sheaf when $n \geq 2$. Note that if the pullback were trivial, the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{CP}_n} \to \mathcal{O}^{n+1}(1) \to \mathcal{T}\mathbb{CP}_n \to 0$$

would induce an exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}^{n+1} \to \mathcal{O}^n \to 0$$

over $\mathbb{C}^{n+1} \setminus \{0\}$, which necessarily splits since $H^1(\mathbb{C}^{n+1} \setminus \{0\}, \mathcal{O}) = 0$ when $n \geq 2$. The existence of linearly independent sections

$$F_i : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}^{n+1}, \quad 1 \leq i \leq n,$$

of $\mathcal{O}(\mathcal{E})$ would then imply $det(F_1(z), ..., F_n(z), z)$ is a holomorphic function on $\mathbb{C}^{n+1}$ which admits an isolated zero at the origin–a contradiction. Further examples are provided by the null–correlation and Tango bundles on $\mathbb{CP}_n$, $n \geq 3$, and the Horrocks–Mumford bundle on $\mathbb{C}^5 \setminus \{0\}$ (cf. [9]).

Consider the long–exact cohomology sequence derived from (1):

$$0 \to H^0(X, \mathcal{E}nd(\mathcal{E})) \to H^0(X, J(\mathcal{E})) \to$$
$$\to H^0(X, \mathcal{T}X) \underset{\delta}{\to} H^1(X, \mathcal{E}nd(\mathcal{E})) \to \ldots \quad (2).$$

For the case of X a domain of $\mathbb{C}^n$, note that any non–vanishing holomorphic vector field $\xi \in H^0(X, \mathcal{T}X)$ determines a class $\delta(\xi) \in H^1(X, \mathcal{E}nd(\mathcal{E}))$ which is the obstruction to a global holomorphic lift of ξ from X to the total space of $\mathcal{E}$. When $\delta(\xi)$ is trivial, the lift in fact determines a holomorphic Lie derivative (or "relative connection") L_ξ, acting on sections of $\mathcal{E}$, hence $\delta(\xi)$ may be identified with the contraction $F_\nabla \rfloor \xi$. In particular, when X is simply a Hartogs figure H, with ξ lying tangent to the annular fibres $\mathcal{A}$, then existence of an endomorphism $\psi \in C^\infty(X, \mathcal{E}nd(\mathcal{E}))$ such that $\overline{\partial}_{\mathcal{E}} \psi = F_\nabla \rfloor \xi$ determines a trivialisation of $\mathcal{E}$ over H. The idea is simply to obtain a trivialisation of $\mathcal{E}$ over the polydisc D, then apply parallel transport with respect to L_ξ. A covariantly constant frame will then exist on each annular fibre of H, provided holonomy is trivial. Now the second component of

H consists of simply connected discs parametrised by an open ball inside D, and holonomy is parametrised complex–analytically, hence uniqueness of analytic continuation implies that trivial holonomy over the ball extends to D. Unique extension of $\mathcal{E}$ to Δ then follows directly. A global version of this idea is the following theorem of N. P. Buchdahl and the author (cf. [4]).

Theorem 1. *Let $Z \subset X$ be an analytic subset of complex codimension at least two, and $\mathcal{E} \to X \setminus Z$ a holomorphic vector bundle. If $\mathcal{E}$ admits a holomorphic connection then there exists a unique holomorphic bundle $\overline{\mathcal{E}} \to X$ such that $\overline{\mathcal{E}}\,|_{X \setminus Z} \cong \mathcal{E}$. If X is a Stein manifold, then $\overline{\mathcal{E}}$ exists if and only if $\mathcal{E}$ admits a holomorphic connection.*

A local corollary of this theorem [6] may be related to removable singularities for the anti self–dual Yang–Mills equation, since F_∇ on any subset of Euclidean $\mathbb{R}^4$ is anti self–dual precisely when it has type $(1,1)$ with respect to any complex structure which is compatible with the chosen orientation.

Corollary 1. *Consider a ball $B \subseteq \mathbb{C}^2$, a Hermitian-holomorphic vector bundle $\mathcal{E} \to B \setminus \{0\}$, and a non-vanishing vector field ξ on B. If there exists $\psi \in C^\infty(B \setminus \{0\}, \mathcal{E}nd(\mathcal{E}))$ such that*

$$\overline{\partial}_{\mathcal{E}}\psi = F_\nabla \rfloor \xi$$

then $\mathcal{E}$ is trivial.

In particular, if ∇ is anti self–dual, then there exists a unique anti self–dual extension to B. The relative exactness condition above may be compared with the "finite energy" condition

$$\|F_\nabla\|^2 := \int_{B\setminus\{0\}} -tr(F_\nabla \wedge *F_\nabla) < \infty$$

(where $*F_\nabla$ denotes the Hodge dual) used for removable singularities of general Yang–Mills fields by Uhlenbeck [11]. The finite energy condition has also been applied by Bando [3] to extend hermitian–holomorphic vector bundles across the origin in $\mathbb{C}^2$. While the L^2–curvature hypothesis is certainly the most natural one, the hypothesis of existence of relative holomorphic connections allows us to consider extension problems across more general singularity sets. Nevertheless, the precise relationship between the two criteria is a question of great interest. Before proceeding to further extension results for Yang–Mills fields, however, the following global result addresses extension across totally real submanifolds [7] in a similar manner to those above.

Theorem 2. *Let M be a real-analytic manifold of dimension at least three, and X a complex manifold corresponding to the complexification of M. If $\mathcal{E} \to X \setminus M$ is a holomorphic vector bundle which admits a holomorphic connection, then there exists a unique holomorphic bundle $\overline{\mathcal{E}} \to X$ such that $\overline{\mathcal{E}}\,|_{X \setminus M} \cong \mathcal{E}$. If X is a Stein manifold, then $\overline{\mathcal{E}}$ exists if and only if $\mathcal{E}$ admits a holomorphic connection.*

A further specialisation of the argument [7] leads to the following

Corollary 2. *Let L be a real line in $\mathbb{R}^4$, B a ball in $\mathbb{C}^2$, and $\mathcal{E} \to B \setminus L$ a Hermitian-holomorphic vector bundle. Moreover, let ξ be a non-vanishing holomorphic vector field on B for which the flow intersects L transversely. If there exists $\psi \in C^\infty(B \setminus L, \mathcal{E}nd(\mathcal{E}))$ such that*

$$\overline{\partial}_{\mathcal{E}}\psi = F_\nabla \rfloor \xi$$

then $\mathcal{E}$ is trivial.

As a final application, consider B a ball in $\mathbb{R}^3$, and $\mathcal{E} \to B \setminus \{0\}$ a unitary vector bundle, with hermitian connection ∇ represented locally by a potential $\mathbf{A} = A_1 dx_1 + A_2 dx_2 + A_3 dx_3$. The *static Bogomolny equation* for monopoles is a field equation on $\mathbb{R}^3$ which may be realised as a time–independent reduction of the anti self–dual Yang–Mills equation on $\mathbb{R}^4$. In particular, if $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ simply deletes the t–coordinate, then

$$\pi^* F_\nabla + \pi^*(*F_\nabla) \wedge dt \quad (3)$$

naturally defines an anti self–dual two–form on $\mathbb{R}^4$ (note that the Hodge star in (3) refers to forms on $\mathbb{R}^3$). A Hermitian connection ∇' on $\pi^*\mathcal{E}$ for which (3) corresponds to $F_{\nabla'}$ can be constructed in each local frame by defining $\mathbf{A}' := \mathbf{A} + \varphi dt$ such that the (Bogomolny) equation

$$F_\nabla = 2 * (d\varphi - [\mathbf{A}, \varphi])$$

is satisfied. Here φ represents the "electrostatic potential" of the monopole field (cf. [8]). Recall that ∇' is compatible with the holomorphic structure of $\pi^*\mathcal{E}$ over $\pi^* B \subseteq \mathbb{C}^2$, hence $\overline{\partial}_{\pi^*\mathcal{E}}$ is represented locally by matrices of the form

$$\mathbf{A}^{0,1} := \frac{1}{2}(A_1 + \mathrm{i}A_2)d\bar{z}_1 + \frac{1}{2}(A_3 + \mathrm{i}\varphi)d\bar{z}_2,$$

with respect to complex coordinates $z_1 = x_1 + \mathrm{i}x_2,\ z_2 = x_3 + \mathrm{i}t$. Define an operator

$$D : C^\infty(\mathcal{E}) \to C^\infty((\mathbb{C} \otimes T\mathbb{R}^3)^* \otimes \mathcal{E})$$

such that for any section σ of $\mathcal{E}$,

$$D(\sigma) := \overline{\partial}_{\pi^*\mathcal{E}} \pi^*(\sigma) \mid_{\mathbb{C}\otimes T\mathbb{R}^3} .$$

As a slight abuse of notation we shall use "D" also to denote the induced operator on $\mathcal{E}nd(\mathcal{E})$.

Corollary 3. *(cf. [7]) Let $F_\nabla = \Sigma_{\mu,\nu} F_{\mu,\nu} dx_\mu \wedge dx_\nu$ represent a static monopole field on $B \setminus \{0\} \subseteq \mathbb{R}^3$. If there exists $\psi \in C^\infty(B \setminus \{0\}, \mathcal{E}nd(\mathcal{E}))$ such that*

$$D\psi = F_\nabla \rfloor \frac{\partial}{\partial z_1},$$

then there exists a unique connection over B which is gauge–equivalent to ∇ over $B \setminus \{0\}$.

Proof. Note

$$F_{\nabla'} \rfloor \frac{\partial}{\partial z_1} = \pi^*(F_\nabla \rfloor \frac{\partial}{\partial z_1}) - \pi^*(*F_\nabla \rfloor \frac{\partial}{\partial z_1})dt \ (4),$$

while it is easily checked that

$$F_\nabla \rfloor \frac{\partial}{\partial z_1} = -\mathrm{i}(F_{1,2} d\bar{z}_1 + (*F_\nabla \rfloor \frac{\partial}{\partial z_1})dx_3) \ (5).$$

Moreover, $\psi \in C^\infty(B \setminus \{0\}, \mathcal{E}nd(\mathcal{E}))$ implies that in any local frame

$$\overline{\partial}_{\pi^*\mathcal{E}} \pi^*(\psi) = \pi^*(D\psi) - \mathrm{i}(\frac{\partial}{\partial x_3} + \frac{1}{2}(A_3 + \mathrm{i}\varphi))\pi^*(\psi)dt \ (6).$$

But $D\psi = F_\nabla \rfloor \frac{\partial}{\partial z_1}$, together with (5), implies

$$\frac{\partial \psi}{\partial x_3} + \frac{1}{2}(A_3 + \mathrm{i}\varphi)\psi = -\mathrm{i}(*F_\nabla \rfloor \frac{\partial}{\partial z_1}),$$

hence from (4) and (6) it follows that $\overline{\partial}_{\pi^*\mathcal{E}}\pi^*(\psi) = F_{\nabla'} \rfloor \frac{\partial}{\partial z_1}$. Now consider any ball $B' \subseteq \mathbb{R}^4$ such that $B' \cap \mathbb{R}^3 = B$, let L correspond to the t–axis, and apply the previous corollary. The unique anti self–dual connection obtained on B' is gauge–equivalent to ∇', hence time–independent, and satisfies the Bogomolny equation on B. □

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