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DEFORMATIONS OF MINIMAL SURFACES OF $\mathbb{R}^3$ CONTAINING PLANAR GEODESICS

HUBERT GOLLEK

Abstract

We study an operator deforming an arbitrary minimal curve, i. e., a meromorphic curve of infinitesimal arc length 0 in 3-dimensional complex space $\mathbb{C}^3$, into meromorphic curves of prescribed infinitesimal arc length d. A meromorphic function h is the deformation parameter. For $d = 0$ this deformation preserves the class of minimal curves and yields a deformation of minimal surfaces. We show that the class of surfaces containing a planar geodesic is preserved under deformations with functions of the form $\sqrt{\pm i h}$ where h is real.

1 Introduction

There is a well known bijective correspondence between minimal surfaces in $\mathbb{R}^3$ in conformal parametrization and minimal curves in $\mathbb{C}^3$, i. e., parametrized curves $\Phi : \mathcal{U} \subset \mathbb{C} \longrightarrow \mathbb{C}^3$ such that $\langle \Phi', \Phi' \rangle = 0$, where $\langle .,. \rangle$ is the bilinear complex extension of the Euclidean scalar product of $\mathbb{R}^3$.

We will assume that the components of the curve Φ are meromorphic functions on an open subset $\mathcal{U}$ of $\mathbb{C}$. This bijection is established by assigning to Φ the parametric surface $\mathbf{x}(u, v) = \mathrm{Re}\,(\Phi(u + i\,v))$. The mapping $\mathbf{x}(u, v)$ so constructed is a minimal surface of $\mathbb{R}^3$ in conformal parametrization, defined and regular outside the points of $\mathcal{U}$ where Φ has a pole or where Φ' vanishes. The poles of Φ correspond to topological ends of $\mathbf{x}$ and the zeroes of Φ' to branch points of $\mathbf{x}$ (see [1], [2] or [9]).

Consequently, solving the minimal surface equation in $\mathbb{R}^3$ is equivalent to solving the ordinary differential equation ${\Phi_1'}^2 + {\Phi_2'}^2 + {\Phi_3'}^2 = 0$ for the components Φ_1, Φ_2, Φ_3 of Φ. One tool to achieve this by integrations is the *Weierstraß formula*. Putting $\Phi_1' = f/2\left(1 - g^2\right)$, $\Phi_2' = i\,f/2\,(1 + g^2)$, $\Phi_3' = f\,g$ where f and g are arbitrary meromorphic functions on $\mathcal{U}$, and integrating yields the minimal curve

(1.1) $$\mathbf{Wei}_{f,g}(z) = \int_{z_0}^{z} f(\zeta)/2\left(1 - g^2(\zeta), i\left(1 + g(\zeta)^2\right), 2\,g(\zeta)\right) \mathrm{d}\zeta.$$

*The paper is in final form and no version of it will be submitted elsewhere.

Conversely, any minimal curve Φ with $\Phi_2' - i\,\Phi_3' \neq 0$ can be represented as $\Phi = \mathbf{Wei}_{f,g}$ with

(1.2) $f = \Phi_1' - i\,\Phi_2'$ and $g = \dfrac{\Phi_3'}{\Phi_1' - i\,\Phi_2'}$.

One practical advantage of the Weierstraß formula is the direct access to basic geometric quantities such as *Gauß map* $\mathbf{U} : \mathbb{C} \longrightarrow S^2$, *first* and *second fundamental form* $\mathbf{I}$ and $\mathbf{II}$ and the *gaussian curvature* k of the surface $\mathbf{x}$ associated to $\mathbf{Wei}_{f,g}$. Putting $\lambda^2 = |f|^2 (1+|g|^2)^2/4$ one gets

$$(1.3)\quad \begin{cases} \mathbf{I} = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix}, & \mathbf{II} = \begin{pmatrix} -\mathrm{Re}\,(f\,g') & \mathrm{Im}\,(f\,g') \\ \mathrm{Im}\,(f\,g') & \mathrm{Re}\,(f\,g') \end{pmatrix}, \\ \mathbf{U} = \dfrac{1}{|g|^2+1} \begin{pmatrix} 2\,\mathrm{Re}(g) \\ 2\,\mathrm{Im}(g) \\ |g|^2 - 1 \end{pmatrix}, & k = -\left(\dfrac{4|g'|}{|f|\,(1+|g^2|)^2}\right)^2. \end{cases}$$

These and other simple relations are useful for the construction of embedded minimal surfaces, invariant with respect to certain groups of congruence transformations of $\mathbb{R}^3$ (see for instance in [6] or [7]).

An other solution of the equation $\langle \Phi', \Phi' \rangle = 0$ is obtained by putting

(1.4) $\Phi'(z) = \alpha'(z) - i\,\alpha'(z) \times \gamma(z), \quad z \in \mathcal{U} \subset \mathbb{C},$

where $\times$ denotes the cross product of $\mathbb{C}^3$ associated to the scalar product $\langle , \rangle$ and (α, γ) is an *analytic strip* of $\mathbb{R}^3$, i. e., a pair $(\alpha(t), \gamma(t))$ of regular analytically parametrized real curves such that $\langle \alpha', \gamma \rangle = 0$ and $|\gamma'| = 1$. The curves α and γ in (1.4) are local holomorphic extensions of α and γ to an open subset $\mathcal{U}$ of $\mathbb{C}$. We denoted them by the same symbols.

Integration of (1.4) leads to the *Björling-representation formula*, denoted here as $\mathbf{Bj}_{\alpha,\gamma}(z) = \alpha(z) - i \int \alpha'(\zeta) \times \gamma(\zeta) d\zeta$ (see also [2] or [3]). We have $\alpha(t) = \mathrm{Re}\left(\mathbf{Bj}_{\alpha,\gamma}(t)\right)$ for all real t in the domain of definition of α. Therefore the minimal surface $\mathbf{x}$ associated to the minimal curve $\Phi(z) = \mathbf{Bj}_{\alpha,\gamma}(z)$ contains the curve α. Moreover, $\gamma(t)$ is the unit surface normal of $\mathbf{x}$ along $\alpha(t)$ and the minimal surface $\mathbf{x}$ is uniquely determined by these properties. The operator $(\alpha, \gamma) \longrightarrow \mathbf{Bj}_{\alpha,\gamma}$ commutes with the actions of $\mathrm{SO}(3,\mathbb{R}) \subset \mathrm{SO}(3,\mathbb{C})$ on real curves and minimal curves respectively, while the real part of $\mathbf{Bj}$ commutes with the actions of $\mathrm{SO}(3,\mathbb{R})$ on real analytic curves and minimal surfaces.

Specializing to the case of a planar curve $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^2 \subset \mathbb{R}^3$ and choosing for γ the normal vector of α one obtaines the following representation formula

$$(1.5)\quad \mathbf{Bj}^*_\alpha(z) = \left(a(\zeta), b(\zeta), i \int_{z_0}^{z} \sqrt{a'^2(\zeta) + b'^2(\zeta)}\, d\zeta\right), \quad (\alpha(t) = (a(t), b(t), 0)).$$

This formula describes all minimal surfaces intersecting the xy-plane $\mathbb{R}^2 \subset \mathbb{R}^3$ perpendicularly in the curve α. These surfaces are invariant under the orthogonal reflection of $\mathbb{R}^3$ with respect to the xy-plane and α is a geodesic of these surfaces. If, for instance, α is a circle in $\mathbb{R}^2$ then $\mathbf{Bj}_\alpha(z)$ is the minimal curve of a catenoid. It is easily observed that the representation formula $\mathbf{Bj}^*$ commutes with real analytic transformations $t : \mathbb{R} \longrightarrow \mathbb{R}$ of the parameter, i. e., $\mathbf{Bj}^*_{\alpha\circ t} = \mathbf{Bj}^*_\alpha \circ t$.

A minimal curve is of type $\Phi = \mathbf{Bj}^*_\alpha$ if and only if its first and second component are real functions. An other criterion can be formulated in terms of the Weierstraß-functions $f = a' - i\,b'$ and $g = -i\,s'/f$ of Φ, where $(a(t), b(t), 0) = \alpha(t)$ and $s' = \sqrt{a'^2 + b'^2}$.

Since a and b are real functions we have $\sqrt{a'^2 + b'^2} = ||f||$ for real arguments. Therefore the condition

(1.6) $i\,f(t)\,g(t) = ||f(t)||$ for real t

for Φ to be of the form $\Phi = \mathbf{Bj}^*_\alpha$ is necessary and sufficient.

In section 2 we discuss properties of the parameter of E. Study of minimal curves and its relations to the Weierstraß and the Björling formula. Some facts are provided that are necessary for considerations of the subsequent sections.

Section 3 is devoted to the construction of a rational operator $\Delta_{\Phi,h,d}$ deforming a minimal curve Φ with $\Phi_1' - i\,\Phi_2' \neq 0$ into a meromorphic curve of prescribed infinitesimal arc length d. Δ depends on an arbitrary meromorphic function h. For $\Phi = \mathbf{Wei}^*{}_f$ we show that the operator $\mathbf{Def} : (f, h, d) \longrightarrow \mathbf{Def}_{f,h,d} = \Delta_{\Phi,h,d}$ is invariant under reparametrizations (see proposition 3.1 for the precise meaning of this invariance). This result leads to an global analogue of the operator $\mathbf{Def}$ acting from the space of triples (f, h, ω), consisting of a function f, a meromorphic vector field h, and a meromorphic 1-form ω on a Riemann surface $\mathcal{R}$ into the space of meromorphic mappings Ω of $\mathcal{R}$ into $\mathbf{C}^3$ such that $i\,\omega$ is the differential of the arc length of Ω. Δ and $\mathbf{Def}$ reduce for $d = 0$ to a deformation operator of minimal curves and a rational representation formula of minimal curves respectively. We show that basic geometric quantities of $\Delta_{\Phi,h,d}$ can be expressed algebraically in terms of the natural parameter of $\Delta_{\Phi,h,0}$ and the minimal curvature of Φ.

Finally, in section 4 we show that $\Delta_{\Phi,h,0}$ is consistent with the special Bjoerling formula $\mathbf{Bj}^*$ for functions h of the form $h(z) = \sqrt{i}\,h_1(z)$, where h_1 is a real function. We display some examples: deformation of the catenoid with the functions $h(t) = a\left(\cos^3 t + \sin^3 t\right)$.

2 E. Study's Parameter

Next we discuss some basic properties of two invariants of minimal curves, the natural parameter p and the minimal curvature κ. The latter is sometimes named after E. Study, (see [11] and also [1] and [10]). They resemble the classical invariants *arc length* and *curvature* of real curves in Euclidean 3-space. We note that also an orthogonal moving frame for such curves can be defined in such the way that Frenét's equations hold in almost the same form as in the 3-dimensional real case, the role of torsion in the minimal case being performed by the logarithmic derivative κ'/κ (see for instance [5]).

The invariants p and κ will be needed for the definition of the deformation $\mathbf{Def}_{\Phi,h}$ of a minimal curve of section 3 below. Let $\Phi : \mathcal{U} \longrightarrow \mathbf{C}^3$ be a minimal curve in general position, i. e., Φ', Φ'', Φ''' are assumed to be linearly independent. Choosing a branch of the function $z^{1/4}$ define $\omega = \langle\Phi''(z), \Phi''(z)\rangle^{1/4}\,dz$. One can show that ω

is a nonvanishing meromorphic differential and that any function p_Φ with $\mathrm{d}p_\Phi = \omega$ is locally invertible. An other expression for $\omega = p'_\Phi \mathrm{d}z$ is given by

$$(2.1)\ p'_\Phi(z) = \sqrt{-\langle \Phi'(z) \times \Phi''(z), \mathrm{v}(z)\rangle \,/\, \langle \Phi'(z), \mathrm{v}(z)\rangle},$$

where $\mathrm{v}(z)$ is an arbitrary complex vector function (see [1]). Write $p = p_\Phi$ for short, assume that Φ is locally reparametrized to p, and denote by $\Phi_{,p}, \Phi_{,pp}, \ldots$ the derivatives with respect to p. We call this p the *natural parameter* of Φ and have $\langle \Phi_{,pp}, \Phi_{,pp}\rangle = -1$.

The *minimal curvature* κ_Φ of Φ is defined as $\kappa_\Phi = \sqrt{\langle \Phi_{,ppp}(z), \Phi_{,ppp}(z)\rangle}$. Again, the definition depends on the choice of a branch of the square root function but in applications below only the function κ_Φ^2 and its derivatives will appear. The invariant κ_Φ transforms as a function, i. e., for a holomorphic parameter transform $t : \mathcal{V} \longrightarrow \mathcal{U}$ one has $\kappa_{\Phi \circ t} = \kappa_\Phi \circ t$. Equation (2.5) below gives κ_Φ^2 in arbitrary parametrization.

The derivative p' of $\mathrm{Wei}_{f,g}$ is $p'(z) = \sqrt{-i\, f(z)\, g'(z)}$. Therefore, by $\mathrm{Wei}^*{}_g = \mathrm{Wei}_{i/g',g}$ we obtain a representation formula for minimal curves in natural parametrization. The square of the minimal curvature κ_f of $\mathrm{Wei}^*{}_f$ is the *Schwarz derivative* of f:

$$(2.2)\ \kappa_f^2 = f'^{-2}(z)\left(3 f''^2(z) - 2 f'(z)\, f^{(3)}(z)\right).$$

If κ_f is an arbitrary meromorphic function, a minimal curve with minimal curvature κ_f can be constructed by solving the differential equation (2.2) for f and putting $\Phi = \mathrm{Wei}^*{}_f$. The result is not unique. The Schwarz derivative is an invariant of the group $\mathrm{Sl}(2,\mathbb{C})$, acting on functions by fractional linear transformations $f \longrightarrow (a\, f + b)/(c\, f + d)$ and f is determined uniquely up to such transformations. One can show that fractional linear transformations of $\mathrm{Sl}(2,\mathbb{C})$ on functions correspond under Wei^* to actions of elements of $\mathrm{SO}(3,\mathbb{C})$ given by the group homomorphism μ assigning to a matrix $\mathrm{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gl}(2,\mathbb{C})$ the complex orthogonal matrix

$$(2.3)\ \mu(\mathrm{g}) = \frac{1}{2\det(\mathrm{g})}\begin{pmatrix} a^2 - b^2 - c^2 + d^2 & i\,(a^2 + b^2 - c^2 - d^2) & 2\,(cd - ab) \\ -i\,(a^2 - b^2 + c^2 - d^2) & a^2 + b^2 + c^2 + d^2 & 2i\,(ab + cd) \\ 2\,(bd - ac) & -2i\,(ac + bd) & 2\,(bc + ad) \end{pmatrix}.$$

More precisely, if $\tilde{f} = (a\, f + b)/(c\, f + d)$ then $\mathrm{Wei}^*{}_{\tilde{f}} = \mu(g)\, \mathrm{Wei}^*{}_f + \mathrm{c}$, where c is a constant vector. In other words:

Proposition 2.1 *The derivative of* Wei* *considered as an operator acting from meromorphic functions to derivatives of minimal curves, is equivariant with respect to the homomorphism (2.3) of* $Gl(2,\mathbb{C})$ *to* $SO(3,\mathbb{C})$.

A complete proof of the following proposition can be found in [5] or [10].

Proposition 2.2 *A minimal curve is uniquely determined up to translations and transformations with elements of the group* $SO(3,\mathbb{C})$ *by its natural parameter and its minimal curvature.*

If the derivatives $\Phi_{,p}$, $\Phi_{,pp}$, $\Phi_{,ppp}$ of a minimal curve are linear independent all other derivatives of Φ can be expressed as linear combinations of them with coefficients that must be expressions in terms of these invariants. The following relation gives the coefficients in the case of the derivative of order 4:

(2.4) $\Phi_{,pppp} = \kappa\,\kappa_{,p}\,\Phi_{,p} + \kappa^2\Phi_{,pp}.$

Since it is substantial for the constructions of the next section let us give a short proof: We infer from $\langle\Phi_{,p},\Phi_{,p}\rangle = 0$, $\langle\Phi_{,pp},\Phi_{,pp}\rangle = -1$ and $\langle\Phi_{,ppp},\Phi_{,ppp}\rangle = \kappa^2$ by successive differentiation $\langle\Phi_{,p},\Phi_{,pp}\rangle = 0$, $\langle\Phi_{,p},\Phi_{,ppp}\rangle = 1$, $\langle\Phi_{,pp},\Phi_{,ppp}\rangle = 0$. Differentiating once more we obtain $\langle\Phi_{,pppp},\Phi_{,p}\rangle = 0$, $\langle\Phi_{,pppp},\Phi_{,pp}\rangle = -\kappa^2$, $\langle\Phi_{,pppp},\Phi_{,ppp}\rangle = \kappa\,\kappa_{,p}$. We use the last three equations to show, that the coefficients a_1, a_2, a_3 in a tentative 'ansatz' $\Phi_{,pppp} = a_1\,\Phi_{,p} + a_2\,\Phi_{,pp} + a_3\,\Phi_{,ppp}$ come out as $a_3 = 0$, $a_2 = \kappa^2$ and $a_1 = \kappa\,\kappa_{,p}$.

Similar considerations give an formula for the minimal curvature of a minimal curve in arbitrary parametrization: The chain rule of order 3 gives $\Phi_{,zzz} = \Phi_{,ppp}\,p_{,z}^3 + 3\,\Phi_{,pp}\,p_{,z}\,p_{,zz} + \Phi_{,p}\,p_{,zzz}$ and from the above list of scalar products $\langle d^i\Phi/dp^i, d^j\Phi/dp^j\rangle$ of derivatives of various order we obtain the following formula.

(2.5) $\kappa_\Phi^2 = p_{,z}^{-6}\left(\langle\Phi_{,zzz},\Phi_{,zzz}\rangle + 9\,p_{,z}^2\,p_{,zz}^2 - 2\,p_{,z}^3\,p_{,zzz}\right).$

Let us return to the formula $\mathbf{Bj}^*$. If a minimal surface intersects the (x,y)-plane perpendicularly along a curve α then the natural parameter and the minimal curvature of the corresponding minimal curve depend only on the curvature of α. In fact, by the invariance of the representation formula $\mathbf{Bj}^*$ under real parameter transformations we can assume that α is given by its natural equation in the form

(2.6) $\alpha(t) = \int_{t_0}^{t} (\cos(w(\zeta)), \sin(w(\zeta)))\,d\zeta,$

where $w'(t)$ is the curvature of α. Put $\Phi = \mathbf{Bj}_\alpha^*$. A direct computation with (2.1) yields p'_Φ, while κ_Φ is computed with (2.5). Altogether we have the following.

Proposition 2.3 *Let $\alpha(t)$ be a parametrized analytic planar curve with $\langle\alpha',\alpha'\rangle = 1$, $w'(t)$ its signed curvature and $\Phi = \mathbf{Bj}_\alpha^*$.*

Then the derivative p' of the natural parameter and the minimal curvature κ of Φ are given by

(2.7) $p'(z) = \sqrt{-i\,w'(z)}$ *and* $\kappa^2(z) = \dfrac{4\,w'^4(z) - 7\,w''^2(z) + 4\,w'(z)\,w^{(3)}(z)}{4\,i\,w'^3(z)}.$

Consequently, these invariants satisfy the equation

(2.8) $i\,\kappa^2(z)\,p'^4(z) = p'^6(z) - 5\,p''^2(z) + 2\,p'(z)\,p^{(3)}(z)$

Conversely, if the minimal curvature and the natural parameter of a minimal curve Φ satisfy (2.8) then there exists a real analytic planar curve α with $\langle\alpha',\alpha'\rangle = 1$ such that Φ agrees with $\mathbf{Bj}_\alpha^*$ up to a translation and a transformation of $SO(3,\mathbb{C})$.

Proof. Equation (2.8) is obtained by eliminating w' from (2.7). For the proof of the converse consider a minimal curve Φ such that p'_Φ and κ_Φ satisfy (2.8), define the curvature w of the prospective curve α by $w'(z) = i\,p'^2(z)$ and α by (2.6). Expressing κ_Φ in terms of w by (2.8) yields an expression that agrees with the second term of (2.7).

Therefore the natural parameter and the minimal curvature of Φ and $\mathbf{Bj}^*_\alpha$ are the same. By proposition 2.2 Φ and $\mathbf{Bj}^*_\alpha$ must be equal up to a translation and a certain complex congruence transformation of $SO(3,\mathbb{C})$. ■

Corollary 1 *Locally, any minimal curve Φ can be reparametrized to a minimal curve of the form $\mathbf{t}\,\mathbf{Bj}^*_\alpha$ for some real analytic planar curve α and a complex orthogonal transformation $\mathbf{t} \in SO(3,\mathbb{C})$.*

Proof. Given Φ, determine the derivative p'_Φ of its natural parameter and its minimal curvature κ_Φ. A parameter transformation t yields. $p'_{\Phi\circ t} = t'\ (p'_\Phi \circ t)$ and $\kappa_{\Phi\circ t} = \kappa_\Phi \circ t$. Inserting $p'_{\Phi\circ t}$ and $\kappa_{\Phi\circ t}$ in (2.8) one obtains an ordinary differential equation of order 2 for t. Transforming Φ to new parameters with a solution of this eqation gives a minimal curve $\Phi \circ t$ whose natural parameter and minimal curvature satisfy (2.8). Now refer to proposition 2.3 to see that $\Phi \circ t$ is the transform of a curve of type $\mathbf{Bj}^*_\alpha$ by an element of $SO(3,\mathbb{C})$. ■

3 Deformations

At first we turn for a moment to arbitrary meromorphic curves Ω and show that they can be represented in a unique way as a linear combination of the derivatives of an appropriate minimal curve Φ in natural parametrization. The coefficients of this linear combination involve the infinitesimal arc length d of Ω, the minimal curvature κ of Φ and a free function h. This yields a rational representation formula not only of minimal curves but also of meromorphic curves with prescribed arc length function, especially a general solution of the ordinary differential equation $\Omega'^2_1+\Omega'^2_2+\Omega'^2_3 = -d^2$ by algebraic expressions (formula (3.5) below).

Theorem 3.1 *Let Φ be a nonplanar minimal curve with minimal curvature κ and h and d two arbitrary meromorphic functions. Let p be the natural parameter of Φ and define two other curves Δ and Ψ by*

(3.1) $\Delta_{\Phi,h,d} = \left(d + h_{,pp} - h\,\kappa^2\right)\Phi_{,p} - h_{,p}\Phi_{,pp} + h\Phi_{,ppp}.$ *and* $\Psi_{\Phi,h,d} = \Phi + \Delta_{\Phi,h,d}.$

Then both $\Delta_{\Phi,h,d}$ and $\Psi_{\Phi,h,d}$ have infinitesimal arc length $i\,d$. Moreover, $(\Phi,h,d) \longrightarrow \Delta_{\Phi,h,d}$ is an injective mapping onto the space of meromorphic curves Ω with $\Omega'_2 - i\,\Omega'_3 \neq 0$.

Proof. Let us abbreviate $\Delta_{\Phi,h,d}$ and $\Psi_{\Phi,h,d}$ to Δ (resp. Ψ) and write $\Phi', \Phi'', h', \ldots$ instead of $\Phi_{,p}, \Phi_{,pp}, h_{,p}, \ldots$. We infer from (2.4) that

(3.2) $\Delta' = \left(d' + h''' - h'\,\kappa^2 - h\,\kappa\,\kappa'\right)\ Phi' + d\,\Phi''.$

Since $\langle\Phi',\Phi'\rangle = \langle\Phi',\Phi''\rangle = 0$ and $\langle\Phi'',\Phi''\rangle = -1$ we obtain immediately $\langle\Delta',\Delta'\rangle = -d^2$ and $||\Psi||^2 = \langle\Phi'+\Delta',\Phi'+\Delta'\rangle = 2\ \langle\Phi',\Delta'\rangle - d^2 = -d^2$.

For a given curve Ω with $\Omega'_2 - i\,\Omega'_3 \neq 0$ a tripel (Φ,h,d) with $\Omega = \Delta_{\Phi,h,d}$ is determined as follows. At frist we obtain immediately d from $\langle\Omega',\Omega'\rangle = -d^2$. Next we can assume that $\Phi = \mathbf{Wei}^*{}_f = \mathbf{Wei}_{i/f',f}$ for some function f. We use the Weierstraß formula and (3.2) to compute

$$(3.3)\ \Omega' = l\,\Phi' + d\,\Phi'' = \frac{i}{2f'^2}\begin{pmatrix} (l - l\,f^2)\,f' - 2\,d\,f\,f'^2 + d\,(f^2-1)\,f'' \\ i\,\left(l\,(f^2+1)\,f' + 2\,d\,f\,f'^2 - d\,(f^2+1)\,f''\right) \\ 2\,\left(d\,f'^2 + f\,(l\,f' - d\,f'')\right) \end{pmatrix}$$

with $l = d' + h''' - h'\,\kappa^2 - h\,\kappa\,\kappa'$. But the special form of the factor l is unimportant. We obtain from (3.3) the following generalization of (1.2)

$$(3.4)\ f = \frac{\Omega_3' - i\,d}{\Omega_1' - i\,\Omega_2'} = \frac{\Omega_3' - \sqrt{\langle\Omega',\Omega'\rangle}}{\Omega_1' - i\,\Omega_2'}.$$

Thus $\Phi = \mathbf{Wei}^*{}_f$ has been determined. Finally from (3.1) and $\langle\Phi',\Phi'''\rangle = 1$ we get $h = \langle\Phi',\Omega\rangle$. ∎

Putting $\Phi = \mathbf{Wei}^*{}_f$ in $\Delta_{\Phi,h,d}$ we obtain the following rational representation formula, denoted by $(f,h,d) \longrightarrow \Delta_{f,h,d}$, for meromorphic curves with prescribed infinitesimal arc length function d.

$$(3.5)\ \frac{i}{2f'^3}\begin{pmatrix} f'\left(2ff'^2h' + h'f'' - f^2h'f'' + f'h'' - f^2f'h''\right) - d\,(f^2-1)\,f'^2 \\ \quad + h\left(-2f'^4 + 2ff'^2f'' - f''^2 + f^2f''^2 + f'f''' - f^2f'f'''\right), \\ i\left(f'\left(-2ff'^2h' + h'f'' + f^2h'f'' + f'h'' + f^2f'h''\right) + d\,(f^2+1)\,f'^2\right. \\ \quad \left. + h\left(2f'^4 - 2ff'^2f'' - f''^2 - f^2f''^2 + f'f''' + f^2f'f'''\right)\right), \\ 2\left(dff'^2 - f'^3h' - fhf''^2 + f'^2\,(fh'' - hf'') + ff'\,(h'f'' + hf''')\right) \end{pmatrix}$$

$\Delta_{f,h,d}$ is a bijection onto the set of curves Ω with $\Omega_1' - i\,\Omega_2' \neq 0$ whose inverse is given by elementary operations.

Moreover, proposition 3.1 below shows an interesting invariance of $\Delta_{f,h,d}$ under parameter transformations. It shows that for any Riemann surface $\mathcal{R}$ there exists a natural bijective nonlinear differential operator $\tilde{\Delta}$ acting from the space of tripels (f, h, ω), where f is a meromorphic function on $\mathcal{R}$, h a meromorphic vector field and ω a meromorphic differential 1- form into the space of meromorphic mappings Ω of $\mathcal{R}$ into $\mathbb{C}$ such that $\langle \mathrm{d}\Omega, \mathrm{d}\Omega\rangle = -\omega \otimes \omega$. Namely, if in a local coordinate system z on an open subset $\mathcal{W} \subset \mathcal{R}$ one has $\mathrm{h} = h\,\partial/\partial z$ and $\omega = d\,\mathrm{d}z$ then by (3.6), putting $\tilde{\Delta}(f,\mathrm{h},\omega)|_{\mathcal{W}} = \Delta(f,h,d)$, this local expression does not depend on the choice of the local coordinate.

Let $\mathcal{U} \subset \mathbb{C}$ be open and denote by $\mathbf{A}_{\mathcal{U}}$ the algebra of all meromorphic functions on $\mathcal{U}$ and by $\mathrm{M}_{\mathcal{U}} = \mathbf{A}_{\mathcal{U}} \times \mathbf{A}_{\mathcal{U}} \times \mathbf{A}_{\mathcal{U}}$ the space of all meromorphic mappings of $\mathcal{U}$ into $\mathbb{C}^3$.

Proposition 3.1 *Denote by $\Delta_{\mathcal{U}}$ the operator mapping $(f,h,d) \in \mathbf{A}_{\mathcal{U}} \times \mathbf{A}_{\mathcal{U}} \times \mathbf{A}_{\mathcal{U}}$ to the curve $\Delta_{f,h,d} \in \mathrm{M}_{\mathcal{U}}$ defined by (3.5). $\Delta_{\mathcal{U}}$ is a bijective map onto the set of meromorphic curves Ω with $\Omega_1' - i\,\Omega_2' \neq 0$. If $\mathcal{V} \subset \mathbb{C}$ ia an other open subset and $t : \mathcal{V} \longrightarrow \mathcal{U}$ a holomorphic parameter transformation then*

$$(3.6)\ \Delta_{\mathcal{U}}(f,h,d) \circ t = \Delta_{\mathcal{V}}\left(f \circ t, \frac{h \circ t}{t'}, t'\,(d \circ t)\right).$$

Moreover, if $\mathbf{r} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl(2, \mathbb{C})$, $l_{\mathbf{r}} f = \dfrac{a f + b}{c f + d}$ *then*

(3.7) $\mu(\mathbf{r})\ (\det(\mathbf{r})\, \Delta(f, h, d)) = \Delta(l_{\mathbf{r}} f, \det(\mathbf{r})\, h, \det(\mathbf{r})\, d)$.

The proof is elementary but involved. It can be achieved with the help of a computer algebra system such as *Mathematica* (see [4]). A non-electronic proof can be found in [5].

Minimal curves come in as the special case $d = 0$. The curves $\Delta_{\Phi,h,0}$ and $\Psi_{\Phi,h,0}$ are new minimal curves. We denote them by $\mathbf{Var}_{\Phi,h} = \Delta_{\Phi,h,0}$ and $\mathbf{Def}_{\Phi,h} = \Phi + \Delta_{\Phi,h,0}$ and call them *variation* and *deformation* of Φ by h respectively.

Proposition 3.2 *If* Φ *is a minimal curve and* p *its natural parameter then, the derivative* π' *of the natural parameter of* $\mathrm{Var}_{\Phi,h}$ *is*

(3.8) $\pi'^2 = h''' - h' \kappa^2 - h \kappa \kappa'$.

In the case $\Phi = \mathbf{Wei}^*{}_f$ *the expression (2.2) for the minimal curvature gives*

(3.9) $\pi'^2 = f'^{-3} \left(3 h f''^3 - f' f'' .(3 h' f'' + 4 h f''') + f'^3 h''' + f'^2 \left(2 h' f''' + h f^{(4)}\right)\right)$.

Moreover, if $\mathcal{D}(f, h)$ *denotes the differential operator defined by the right hand side of (3.9), then under a holomorphic coordinate change* t*,* $\mathcal{D}(f, h)$ *transforms according to the rule* $\mathcal{D}(f \circ t, (h \circ t)/t') = t'^2\ (\mathcal{D}(t, h) \circ t)$.

Write again $\Delta = \Delta_{f,h,d}$ for short. Then $\langle \Delta', \Delta' \rangle = -d^2$. On can ask for the complex length $\langle \Delta'', \Delta'' \rangle$ of the second derivative. This highly involved expression in f, h and d can be expressed by π and the Schwarz derivative κ_f in a simple way. Solving (3.9) for $f^{(4)}$ and substituting $f^{(4)}$ and $f^{(5)}$ in $\langle \Delta'', \Delta'' \rangle$ leads to a significant simplification:

$$\langle \Delta'', \Delta'' \rangle = d \left(2 d'' + 3 d f''^2 / f'^2 - 2 d f^{(3)} / f' - \pi^4 + 4 \pi d \pi' - 4 \pi^2 d' - 4 d'^2\right)$$

Replacing here $f^{(3)}$ according to (2.1) gives an expression depending only on π, κ_f and d:

(3.10) $\langle \Delta'', \Delta'' \rangle = -\pi^4 + d^2 {\kappa_f}^2 + 4 \pi d \pi' - 4 \pi^2 d' - 4 d'^2 + 2 d d''$

In a similar way an expression for $\langle \Delta''', \Delta''' \rangle$ is obtained:

$$(3.11) \begin{cases} \langle \Delta''', \Delta''' \rangle = & \pi^4 \kappa_f^2 - d^2 \kappa_f^4 + 4 \pi^3 \pi'' + 6 d \kappa_f \left(d' \kappa_f' - \kappa_f d''\right) \\ & -4 \pi \left(2 d \kappa_f^2 \pi' - 3 d' \pi'' + 6 \pi' d''\right) \\ & +2 \pi^2 \left(-6 \pi'^2 + 3 \kappa_f^2 d' + d \kappa_f \kappa_f' + d^{(3)}\right) \\ & +3 \left(4 \pi'^2 d' + 3 \kappa_f^2 d'^2 - 3 d''^2 + 2 d' d^{(3)}\right) \end{cases}$$

For $d(z) = i$ the operator $\Delta_{f,h,i}$ describes meromorphic curves of constant speed 1 in $\mathbb{C}^3$. Curvature $\tilde{\kappa}$ and torsion $\tilde{\tau}$ of $\Delta_{f,h,i}$ are given by $\tilde{\kappa} = \sqrt{-\pi^4 - \kappa_f^2 + 4 i \pi \pi'}$ and $\tilde{\tau} = -\tilde{\kappa}^{-2} \left(\pi^6 + \pi^2 \kappa_f^2 - 6 i \pi^3 \pi' - 2 \pi'^2 - i \kappa_f \kappa_f' - 2 \pi \pi''\right)$.

4 Deformations and the Björling Formula

We are going to show that the Björling formula behaves well under deformations. Assume that a planar unit speed curve α is defined in terms of its curvature w' by (2.6). We will refer to α as the *base curve* of Bj^*_α. The variation of Bj^*_α must be a differential expression in w' and h to be computed from definition (3.1) and equations (2.7). The explicit result is the following

Proposition 4.1 *If α is a real planar curve given by its curvature w' as in (2.6) then the variation of $\mathrm{Bj}^*_\alpha(z)$ with a function $\sqrt{i}\,h(z)$ is the minimal curve*

$$(4.1)\ \ \mathrm{Var}_{\mathrm{Bj}^*_\alpha,\sqrt{i}h} = \frac{1}{4w'^{\frac{7}{2}}}\begin{pmatrix} -4\sin(w)h'w'^3+3\cos(w)hw''^2-2\cos(w)hw'w'''- \\ 2w'^2\left(2\cos(w)h''+h\sin(w)w''\right), \\ 4\cos(w)h'w'^3+3h\sin(w)w''^2-2h\sin(w)w'w'''+ \\ w'^2\left(-4\sin(w)h''+2\cos(w)hw''\right), \\ i\left(4w'^2h''+h\left(4w'^4-3w''^2+2w'w'''\right)\right) \end{pmatrix}.$$

Again the proof is elementary but involved. We conclude from this that for real functions h the curve $\mathrm{Var}_{\mathrm{Bj}^*_\alpha,\sqrt{i}h}$ is again of type Bj^*. The base curve β of $\mathrm{Var}_{\mathrm{Bj}^*_\alpha,\sqrt{i}h}$ is given by the first and second component of the vector (4.1). If one tries to express β directly in terms of the components a and b of the curve $\alpha = (a,b)$ one arrives at very involved terms. There are relatively simple expressions for the infinitesimal arc length ds^2_β and the signed curvature κ_β of β:

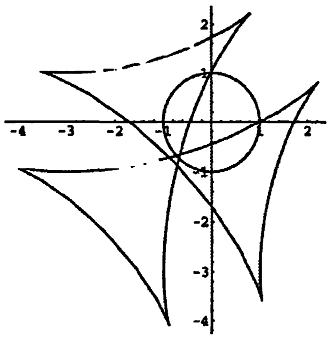
$$(4.2)\ \left\{\begin{array}{rcl} ds^2_\beta &=& \left\{4\,w'^2\left(2\,w'\,h^{(3)}-3\,h''\,w''\right)+2\,h'\,w'\left(4\,w'^4-3\,w''^2+2\,w'\,w^{(3)}\right)\right. \\ &+& \left.h\left(4\,w'^4\,w''+21\,w''^3-22\,w'\,w''\,w^{(3)}+4\,w'^2\,w^{(4)}\right)\right\}\left(8\,w'^{\frac{9}{2}}\right)^{-1} \\ \kappa_\beta &=& 8\,w'^{\frac{11}{2}}\left\{4\,w'^2\left(-3\,h''\,w''+2\,w'\,h^{(3)}\right)\right. \\ &+& 2\,h'\,w'\left(4\,w'^4-3\,w''^2+2\,w'\,w^{(3)}\right) \\ &+& \left.h\left(4\,w'^4\,w''+21\,w''^3-22\,w'\,w''\,w^{(3)}+4\,w'^2\,w^{(4)}\right)\right\}^{-1}. \end{array}\right.$$

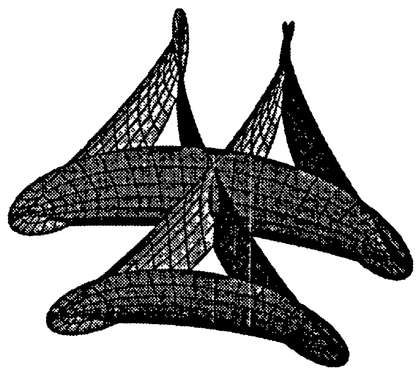
Let us have a look at an example. The catenoid is the surface of revolution of the catenary. The associated minimal curve of the catenoid and of its deformation with a function $\sqrt{i}\,h$ are

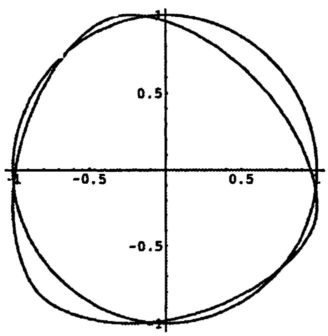
$$\Phi_{\mathrm{cat}}(z) = \begin{pmatrix} \cos(z) \\ \sin(z) \\ -i\,z \end{pmatrix} \text{ and } \mathrm{Var}_{\Phi_{\mathrm{cat}},\sqrt{i}\,h}(z) = \begin{pmatrix} -\cos(z)\,h'(z)+\sin(z)\,h''(z) \\ -\sin(z)\,h'(z)-\cos(z)\,h''(z) \\ i\,(h(z)+h''(z)) \end{pmatrix}$$

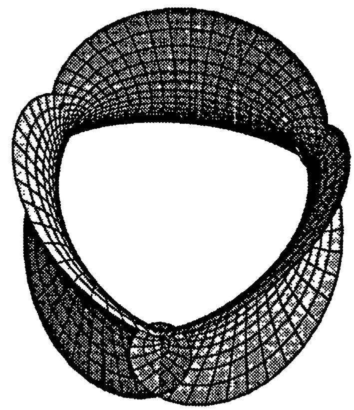
The derivative of the natural parameter and minimal curvature of Φ_{cat} are constant: $p'_{\mathrm{cat}} = \kappa_{\mathrm{cat}} = \sqrt{i}$. For $\mathrm{Var}_{\Phi_{\mathrm{cat}},\sqrt{i}h}$ these invariants are $p' = \sqrt{h'+h'''}$ and $\kappa^2 = i\left(4\,h'^2+5\,h''^2+4\,h'\,h'''+10\,h''\,h''''+5\,h''''^2-4\,h'\,h^{(5)}-4\,h'''\,h^{(5)}\right)\left(2\,(h'+h''')\right)^{-3}$ respectively.

Two simple examples give a visual idea of the Björling formula and its deformation.









Although the catenoid is a regular embedded minimal surface, it seems that almost all deformations loose these properties. The deformed surfaces have branchpoints and selfintersections. We choose for the first picture $h(z) = a\left(\sin^3(z) + \cos^3(z)\right)$ with a real parameter a and draw for $a = 1$ the base curve of the catenoid, i.e. the unit circle, together with that of the deformed curve. For a smaller deformation the curve of intersection with the xy-plane has no singularities but the associated minimal surface has still branch points and selfintersections. We choose $a = 1/13$ to obtain the configuration of the second picture.

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