Jiří M. Tomáš

Natural operators on vector fields on the cotangent bundles of the bundles of (k, r)-velocities

In: Jan Slovák and Martin Čadek (eds.): Proceedings of the 17th Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1998. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 54. pp. [113]--124.

Persistent URL: http://dml.cz/dmlcz/701621

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 54 (1998), pp. 113-124

NATURAL OPERATORS ON VECTOR FIELDS ON THE COTANGENT BUNDLES OF THE BUNDLES OF (k,r)-VELOCITIES

JIŘÍ TOMÁŠ

ABSTRACT. We classify all natural operators $TM \to TT^*T^r_kM$ for $\dim M \geq k+2$ and give their geometrical description. KEYWORDS. Natural bundle, natural operator, vector field, Weil bundle, B-admissible A-velocity.

1. PRELIMINARIES

We give another contribution to the theory of Weil bundles. Our investigations come out from the general result of Kolář, who classified all natural operators $T \to TT^A$, transforming vector fields on manifolds to vector fields on Weil bundles. Our result presents another step to the solution of the general problem of the classification of all natural operators $T \to TT^*T^A$ for arbitrary Weil algebra A. Some partial results were found by Kolář, ([5]) for $A = \mathbb{R}$, Kobak for $A = \mathbb{D}$, ([1]) and for $A = \mathbb{D}^2_1$ in [8].

All natural operators are considered on the category $\mathcal{M}f_m$ of smooth manifolds and local diffeomorphisms. We follow the basic terminology used in [5]. Our approach is based on the covariant definition of Weil bundles and we essentially use the concept of B-admissible A-velocity, [2]. $\mathbb{D}^r_k$ denotes the Weil algebra $J^r_0(\mathbb{R}^k, \mathbb{R})$ of jets and $\mathbb{D}$ denotes the algebra of dual numbers.

We essentially need the following result of Kolář, [3]. Let F be a natural bundle and $Y : FM \to TFM$ be a vector field. $\widetilde{Y}$ denotes the function $T^*FM \to \mathbb{R}$ defined as follows: $\widetilde{Y}(w) = \langle Y(p(w)), w\rangle$, where p is the cotangent bundle projection $p : T^*FM \to FM$. Let N_F denote the vector space of natural operators $T \to TF$ and suppose it to be finite dimensional. Fixing any basis $A_1, \ldots, A_n$ of N_F, the dual vector space N^*_F can be identified with $\mathbb{R}^n$. If there is a function $j : N^*_F \to (T^*F)_0\mathbb{R}^m$ satisfying

$$\langle A, u\rangle = \tilde{A}(\frac{\partial}{\partial x^1})(ju)$$

1991 *Mathematics Subject Classification.* 58A20, 53A55.
This paper is in final form and no version of it will be submitted for publication elsewhere.
The author was supported by the grant No. 201/96/0079 of the GA ČR

for every $A \in N_F$, $u \in N_F^*$ and the orbit of $j(N_F^*)$ with respect to the stability group of the origin and the vector field $\frac{\partial}{\partial x^1}$ is dense in $(T^*F)_0\mathbb{R}^m$, we have the bijection $S : C^\infty(N_F^*, \mathbb{R}) \to \mathrm{Nop}(T, T^*F \times \mathbb{R})$ defined as follows

$$(\mathrm{Dh})_M X = h(\widetilde{A_{1,M}X}, \dots \widetilde{A_{n,M}X}) : T^*FM \to \mathbb{R}$$

provided Nop denotes the set of all natural operators. This implies, that every natural operator $T \to C^\infty(T^*F, \mathbb{R})$ is of the form Dh.

2. Absolute Natural Operators $T \to TTT^A$

In this section, we follow the general result of Kolář, giving the full classification of all natural operators $T \to TT^B$ for any Weil algebra B. We investigate in more details the case $B = A \otimes \mathbb{D}$ for any Weil algebra A and the algebra of dual numbers $\mathbb{D}$. We give the geometrical description of those operators and for the case $A = \mathbb{D}_k^r$ express the base of absolute operators by means of A-admissible A-velocities. Moreover, we obtain the coordinate expression of those operators.

The Weil algebra $A \otimes \mathbb{D}$ is identified with $A \times A$ with the multiplication defined as follows: $(a, b)(c, d) = (ac, ad + bc)$. Let $\mathrm{Aut}(B)$ denote the group of all algebra automorphisms on B. It is a closed subgroup of $\mathrm{GL}(B)$, so it is a Lie subgroup. Every element of its Lie algebra $D \in \mathcal{A}ut(B)$ is tangent to a one-parameter subgroup $d(t)$ and determines a vector field $D_M = \frac{\partial}{dt}|_0(d(t))_M$ on every bundle T^BM. The constant map $X \mapsto D_M$ forms the natural operator $\mathrm{op}(D)_M : TM \to TT^BM$. Furthermore, we remind that a derivation of B is a linear map $D : B \to B$ satisfying $D(ab) = D(a)b + aD(b)$ for all $a, b \in B$. Let $\mathrm{Der}\, B$ denote the set of all derivations of B. The classical result ([5]) yields the identification between $\mathcal{A}ut(B)$ and $\mathrm{Der}\, B$. Furthermore, for every natural bundle F we have the flow operator $\mathcal{F}$, defined by $\mathcal{F}(X) = \frac{\partial}{\partial t}|_0 F(Fl_t^X)$. According to [4] ,[5] we have the following action of B on tangent vectors of T^BM. If $m : \mathbb{R} \times TM \to TM$ is the multiplication of the tangent vectors on M by reals, applying the functor T^B we obtain $T^Bm : T^B\mathbb{R} \times T^BTM \to T^BTM$. Since $T^BTM = T^{B\otimes\mathbb{D}}M$ and $T^B\mathbb{R} = B$, where $\mathbb{D}$ is the algebra of dual numbers, we have constructed a map $B \times TT^BM \to TT^BM$. The coordinate expression of the action of $c \in B$ is $c(a_1, \dots, a_m, b_1, \dots, b_m) = (a_1, \dots, a_m, cb_1, \dots, cb_m)$ for all $a_1, \dots, a_m, b_1, \dots, b_m \in B$. This is a natural affinor [5] and we denote it by $af_M(c) : TT^BM \to TT^BM$.

Proposition 1 (*Kolář* [4],[5]). *All natural operators $T \to TT^B$ are of the form $af(c) \circ \mathcal{T}^B + \mathrm{op}(D)$ for any $c \in B$.*

Now, we are going to discuss the case $B = A \otimes \mathbb{D}$. We prove the following lemma.

Lemma 2. *Let A be a Weil algebra, $\mathbb{D}$ be the algebra of dual numbers. A linear map $D : A \times A \to A \times A$ is a derivation of $A \otimes \mathbb{D}$ if and only if D is of the form*

$$D(a, b) = (D_1(a), D_2(a) + D_1(b) + kb) \tag{1}$$

where $D_1, D_2 \in \operatorname{Der} A, \quad k \in A \quad a, b \in A.$

Proof. From the definition of a derivation and the multiplication in $A \otimes \mathbb{D}$ one can immediately verify, that the formula (1) defines a derivation.

Conversely, let $f(a,b) = (f_1(a) + f_2(b), f_3(a) + f_4(b))$ be a derivation of $A \otimes \mathbb{D}$. Obviously, f_1, f_2, f_3, f_4 are linear maps $A \to A$. The assumption of a derivation on f can be written in the form $(f_1(ac) + f_2(ad + bc), f_3(ac) + f_4(ad + bc)) = (f_1(a)c + f_2(b)c + af_1(c) + af_2(d), f_1(a)d + f_2(b)d + f_3(a)c + f_4(b)c + af_3(c) + af_4(d) + bf_1(c) + bf_2(d))$ Let us compare the first components of the last equation. If we put $b = d = 0$, we obtain $f_1(ac) = f_1(a)c + af_1(c)$, which is the derivation condition for f_1. Let l denote $f_2(1)$. Substituting $d = 1, \quad b, c = 0$ we deduce $f_2(a) = la$.

Let us consider the second components of the recent equation. Setting $b = d = 0$ yields $f_3 \in \operatorname{Der} A$. Let $k = f_4(1)$. If we put $b = c = 0$ and $d = 1$ we obtain $f_4(a) = f_1(a) + ka$. Finally, we put $a = c = 0$, which follows $0 = f_2(b)d + bf_2(d) = 2lbd$. We obtain $l = 0$, which completes the proof. □

Lemma 2 enables us to consider following three basic systems of derivations of $A \otimes \mathbb{D}$.

$$
\begin{aligned}
&D(a,b) = (D_1(a), D_1(b)), && \text{where } D_1 \in \operatorname{Der} A \\
(2) \qquad &D(a,b) = (0, D_2(a)), && \text{where } D_2 \in \operatorname{Der} A \\
&D(a,b) = (0, kb) && \text{for any } k \in A
\end{aligned}
$$

The exponential mapping $exp : \mathcal{A}ut(A \otimes \mathbb{D}) \to \operatorname{Aut}(A \otimes \mathbb{D})$ defines a bijection between $\mathcal{A}ut(A \otimes \mathbb{D})$ and the connected component of the unit in $\operatorname{Aut}(A \otimes \mathbb{D})$, which yields the following three systems of automorphisms

$$
\begin{aligned}
&f(a,b) = (f_1(a), f_1(b)), \quad \text{where } f_1 = expD_1 \\
(3) \qquad &f(a,b) = (a, b + D_2(a)) \\
&f(a,b) = (a, kb)
\end{aligned}
$$

For any Weil algebra B, every element $D \in \operatorname{Der} B$ determines an absolute natural operator $\operatorname{op}(D)$. The following lemma gives the geometrical desription of such natural operators for $B = A \otimes \mathbb{D}$, where A is any Weil algebra.

Lemma 3. *Let* $D : A \otimes \mathbb{D} \to A \otimes \mathbb{D}$ *be a derivation. Then the natural operator* $\operatorname{op}(D) : T \to TTT^A$ *is of the form*

$$
(4) \qquad \mathcal{T} \circ \operatorname{op}(D_1) + \mathcal{V} \circ \operatorname{op}(D_2) + Taf(k) \circ L_{T^A}
$$

where $\mathcal{T}$ *denotes the flow prolongation of the tangent bundle functor,* $\mathcal{V}$ *denotes the vertical lift* $TT^A \to TTT^A$, L_{T^A} *denotes the Liouville vector field on* TT^A *and* $D_1, D_2 \in \operatorname{Der} A$, $k \in A$.

Proof. Let us consider A as a factor of polynomials $\mathbb{R}[\tau_1, \ldots, \tau_k]/I$, where I is an ideal of finite codimension. Let us investigate the first formula from (2). We prove, that $\operatorname{op}(D) = \mathcal{T} \circ \operatorname{op}(D_1)$. Every element of $TT^A\mathbb{R}^m$ is of the form $(\frac{y^i_\alpha}{\alpha!}\tau^\alpha, \frac{z^i_\alpha}{\alpha!}\tau^\alpha)$, where τ^α are the generators of A as a vector space. Let e denote the unit in $\operatorname{Aut}(A)$. It holds $\mathcal{T}(\operatorname{op}(D_1))(\frac{y^i_\alpha}{\alpha!}\tau^\alpha, \frac{z^i_\alpha}{\alpha!}\tau^\alpha) = \frac{d}{dt}|_0 TFl^{\operatorname{op}(D_1)}(t,e)(\frac{y^i_\alpha}{\alpha!}\tau^\alpha, \frac{z^i_\alpha}{\alpha!}\tau^\alpha) = (\frac{d}{dt}|_0 TFl^{D_1}(t,e)(\frac{y^i_\alpha}{\alpha!}\tau^\alpha, \frac{z^i_\alpha}{\alpha!}\tau^\alpha)_{i=1,\dots,m} = (\frac{d}{dt}|_0 T exp(tD_1)(t,e)(\frac{y^i_\alpha}{\alpha!}\tau^\alpha, \frac{z^i_\alpha}{\alpha!}\tau^\alpha))_{i=1,\dots,m} = \frac{d}{dt}|_0(\frac{y^i_\alpha}{\alpha!}\sum_{n=0}^\infty \frac{t^n D_1^n(\tau^\alpha)}{n!\alpha!}, \frac{\partial(exp(tD_1))^i_\alpha}{\partial y^j_\beta} z^j_\beta) = (\frac{y^i_\alpha}{\alpha!}D_1(\tau^\alpha), \frac{z^i_\alpha}{\alpha!}D_1(\tau^\alpha)) = (\operatorname{op}(D_1)(\frac{y^i_\alpha}{\alpha!}\tau^\alpha), \operatorname{op}(D_1)(\frac{z^i_\alpha}{\alpha!}\tau^\alpha)) = \operatorname{op}(D)(\frac{y^i_\alpha}{\alpha!}\tau^\alpha, \frac{z^i_\alpha}{\alpha!}\tau^\alpha)$. The fact, that $\operatorname{op}(D) = \mathcal{V} \circ \operatorname{op}(D_2)$ for $D(a,b) = (0, D_2(a))$ is obvious.

Finally, the Liouville vector field L_{T^A} as a vector field generated by the one-parameter group of homotheties of the vector bundle $TT^A \to T^A$ has the integral curve in the neighbourhood of (a,b) given by $\gamma(t) = (a, tb)$. It holds $\frac{d}{dt}|_1 af(k) \circ \gamma(t) = \frac{d}{dt}|_1(a, tkb) = \operatorname{op}(D)(a,b)$ for $D(a,b) = (0, kb)$, which proves our claim. □

Absolute natural operators can be searched by means of A-admissible A-velocities ([2]). It follows from the existence of the bijection between B-admissible A-velocities and natural transformations $i : T^B \to T^A$ given by $i^{j^A f}(j^B g) = j^A(g \circ f)$. Moreover, there is a bijection between the natural transformations of this kind and $\operatorname{Hom}(B, A)$, which follows that the absolute natural operators can be searched by reparametrizations.

Let $A = \mathbb{D}^r_k \otimes \mathbb{D}$. The algebra $\mathbb{D}^r_k$ can be considered as an algebra op polynomials $\mathbb{R}[\tau_1, \dots, \tau_k]$ factorized by the ideal of polynomials of degree at least $r+1$. The algebra $\mathbb{D}$ is considered as the algebra of polynomials of t factorized by the ideal $\langle t^2 \rangle$. Every A-admissible A-velocity is of the form

$$\begin{aligned} & a^1_\alpha \tau^\alpha + b^1_\gamma \tau^\gamma t \\ & \vdots \\ & \vdots \\ & a^k_\alpha \tau^\alpha + b^k_\gamma \tau^\gamma t \\ & a_\alpha \tau^\alpha + b_\gamma \tau^\gamma t \end{aligned} \tag{5}$$

where α and γ are multiindices satisfying $1 \le |\alpha| \le r$ and $0 \le |\gamma| \le r$.

The conditions of A-admissibility together with our limiting to the connected component of the unit in $\operatorname{Aut}(A)$ yield $a_\alpha = 0$ for $1 \le |\alpha| \le r$ and $b^j_0 = 0$ for $1 \le j \le k$. Every element of $T^A\mathbb{R}^m$ can be considered in the form

$$\frac{y^i_\alpha}{\alpha!}\tau^\alpha + \frac{z^i_\alpha}{\alpha!}\tau^\alpha t; \qquad 0 \le |\alpha| \le r \tag{6}$$

which defines the canonical coordinates on $T^A\mathbb{R}^m$. The reparametrizatiom $\tau_i \mapsto \tau_i + \delta^j_i a \tau^\beta$; $|\beta| \ge 1$ yields the natural operator

$$A^j_\beta = \sum_{|\alpha+\beta| \le r+1} \frac{\alpha_j}{\alpha_j + \beta_j} \frac{(\alpha+\beta)!}{\alpha!} (y^i_\alpha \frac{\partial}{\partial y^i_{\alpha+\beta-\{j\}}} + z^i_\alpha \frac{\partial}{\partial z^i_{\alpha+\beta-\{j\}}}) \tag{7}$$

where the bottom multiindex $\alpha+\beta-\{j\}$ denote the sum of multiindices α and β by components decreased by one at the j-th component. The reparametrization $\tau_i \mapsto \tau_i + \delta_i^j a\tau^\beta t$; $|\beta| \geq 1$ yields the natural operator

$$\bar{A}_\beta^j = \sum_{|\alpha+\beta|\leq r+1} \frac{\alpha_j}{\alpha_j+\beta_j} \frac{(\alpha+\beta)!}{\alpha!} y_\alpha^i \frac{\partial}{\partial z^i_{\alpha+\beta-\{j\}}} \tag{8}$$

and the reparametrization $t \mapsto t + \delta_i^j a\tau^\beta t$; $|\beta| \geq 0$ yields the natural operator

$$A^\beta = \sum_{|\alpha+\beta|\leq r} \frac{(\alpha+\beta)!}{\alpha!} z_\alpha^i \frac{\partial}{\partial z^i_{\alpha+\beta}} \tag{9}$$

The natural operator $A_\beta^j = \mathcal{T}\circ\mathrm{op}(D_\beta^j)$, where D_β^j denotes the derivation $D : \mathbb{D}_k^r \to \mathbb{D}_k^r$ given by $D(\tau_i) = \delta_i^j \tau^\beta$, which follows from Lemma 3. Similarly, $\bar{A}_\beta^j = \mathcal{V}\circ\mathrm{op}(D_\beta^j)$ and $A^\beta = Taf(\tau^\beta)\circ L_{TA}$.

3. Natural Operators $T \to TT^*T_k^r$

In this Section, we determine all natural operators $T \to TT^*T_k^r$ by means of $\mathbb{D}_k^r \otimes \mathbb{D}$-admissible $\mathbb{D}_k^r \otimes \mathbb{D}$-velocities and give the geometrical description of those operators.

We remind the natural equivalence $s : TT^* \to T^*T$ by Modugno, Stefani, [7] and the natural equivalence $t : TT^* \to T^*T^*$ by Kolář, Radziszewski, [6] . Let x^i be the standard coordinates on $\mathbb{R}^m$ and $p_i dx^i$ define the additional coordinates p_i on $T^*\mathbb{R}^m$. Let x^i, p_i induce the coordinates $X_1^i = dx^i, P_i = dp_i$ on $TT^*\mathbb{R}^m$ and $\xi_i dx^i + \eta^i dp_i$ define the additional coordinates ξ_i, η^i on $T^*T^*\mathbb{R}^m$. Furthermore, let $Y^i = dx^i$ be the coordinates on $T\mathbb{R}^m$ and $\alpha_i dx^i + \beta_i dY^i$ define the additional coordinates α_i, β_i on $T^*T\mathbb{R}^m$. Then

$$\begin{aligned} s(x^i, p_i, X_1^i, P_i) &= (x^i, Y^i, \alpha_i, \beta_i) \qquad \text{where } Y^i = X_1^i, \alpha_i = P_i, \beta_i = p_i \\ t(x^i, p_i, X_1^i, P_i) &= (x^i, p_i, \xi_i, \eta^i) \qquad \text{where } \xi_i = P_i, \eta^i = -X_1^i \end{aligned} \tag{11}$$

Let $A : T \to TTT_k^r$ be a natural operator and $\widetilde{A} : T \to C^\infty(T^*TT_k^r, \mathbb{R})$ be its associated natural operator . If we consider the natural operator $\widetilde{A}\circ s \circ t^{-1} : T \to C^\infty(T^*T^*T_k^r, \mathbb{R})$ satisfying the assumption of the linearity on fibers of the vector bundle $T^*T^*T_k^r \to T^*T_k^r$, we can construct the natural operator $\widetilde{\widetilde{A}} : T \to TT^*T_k^r$, since the functions linear on fibers of the natural bundle $T^*T^*T_k^r \to T^*T_k^r$ are in the canonical bijection with vector fields on $T^*T_k^r$.

Let y_α^i, z_α^i be the coordinates on TT_k^r defined in (6). We define the additional coordinates on $T^*T_k^r\mathbb{R}^m$ by $p_i^\alpha dy_\alpha^i + q_i^\alpha dz_\alpha^i$. Then we can obtain the following natural operators $T \to TT^*T_k^r$

$$\widetilde{\widetilde{A}}{}_\beta^{j\,\prime} = \sum_{|\alpha+\beta|\leq r+1} \frac{\alpha_j}{\alpha_j+\beta_j} \frac{(\alpha+\beta)!}{\alpha!} \left(y_\alpha^i \frac{\partial}{\partial y^i_{\alpha+\beta-\{j\}}} - q_i^{\alpha+\beta-\{j\}} \frac{\partial}{\partial q_i^\alpha}\right) \tag{12}$$

$$\widetilde{\widetilde{A^\beta}} = \sum_{|\alpha+\beta|\le r} \frac{(\alpha+\beta)!}{\alpha!} q_i^{\alpha+\beta} \frac{\partial}{\partial q_i^\alpha} \tag{13}$$

where q_i^α are the additional coordinates on $T^*T_k^r$ defined by $q_i^\alpha dy_\alpha^i$. Furthermore, let

$$N_\alpha = af(\tau^\beta) \circ \mathcal{T}T_k^r \tag{14}$$

Clearly, $\widetilde{\widetilde{N_\alpha}}$ are the non-absolute natural operators $T \to TT^*T_k^r$; $0 \le |\alpha| \le r$, where $\mathcal{T}T_k^r$ denotes the flow prolongation of the natural bundle TT_k^r.

The recent construction will be used essentially for searching for the natural operators $T \to VT^*T_k^r$, where $VT^*T_k^r$ denotes the vertical bundle of the vector bundle $T^*T_k^r \to T_k^r$. Since we do not classify all natural operators $T \to C^\infty(T^*TT_k^r, \mathbb{R})$, other natural operators $T \to TT^*T_k^r$ are searched directly. The following lemmas enable the reduction of our problem to the problem of the classification of natural operators $T \to VT^*T_k^r$. First we need the following lemma from [5] .

Lemma 4 ([5]). *Let* $V_{p,q} = \underbrace{V \times \ldots \times V}_{p-\text{times}} \times \overbrace{V^* \times \ldots \times V^*}^{q-times}$, *where V denotes the vector space $\mathbb{R}^m$ with the standard action of G_m^1. Then it holds*

(a) All smooth G_m^1-equivariant maps $V_{p,q} \to V$ are of the form

$$\sum_{j=1}^{p} g_j(\langle x_k, y_l\rangle)x_j,$$

where $g_j : \mathbb{R}^{pq} \to \mathbb{R}$ are any smooth functions, $j, k = 1, \ldots, p$, $l = 1, \ldots, q$.

(b) All smooth G_m^1-equivariant maps $V_{p,q} \to V^*$ are of the form

$$\sum_{l=1}^{q} h_l(\langle x_k, y_h\rangle)y_l,$$

where $h_l : \mathbb{R}^{pq} \to \mathbb{R}$ are any smooth functions, $k = 1, \ldots, p$, $h, l = 1, \ldots, q$.

(c) All smooth G_m^1-invariant functions $V_{p,q} \to \mathbb{R}$ are of the form $g(\langle x_k, y_h\rangle)$ for any smooth function $g : \mathbb{R}^{pq} \to \mathbb{R}$ and $k = 1, \ldots, p$, $h = 1, \ldots, q$

Since $T^*T_k^r$ is the natural bundle of order $r+1$, we are searching for G_m^{r+2}-equivariant maps $(J^{r+1}T)_0\mathbb{R}^m \times (T^*T_k^r)_0\mathbb{R}^m \to (TT^*T_k^r)_0\mathbb{R}^m$ over the identity on $(T^*T_k^r)_0\mathbb{R}^m$, which are in the canonical bijection with natural operators $T \to TT^*T_k^r$ according to the general theory. Let us denote

$$N_{1,\alpha} = af(\tau^\alpha t) \circ \mathcal{T}T_k^r \tag{15}$$

We prove the following lemma.

Lemma 5. *Let* $h : (J^{r+1}T)_0\mathbb{R}^m \times (T^*T_k^r)_0\mathbb{R}^m \to \mathbb{R}^m$ *be a* G_m^{r+2}*- equivariant mapping,* $m \geq k+1$*,* X *be a vector field on* $\mathbb{R}^m$*. Then it holds*

$$W^i(j_0^{r+1}X, y_\alpha^i, q_i^\alpha) = h^0(\widetilde{N_{1,\lambda}(X)}(y_\alpha^i, q_i^\alpha), \widetilde{\bar{A}_\beta^j}(y_\alpha^i, q_i^\alpha))X^i + h^p(\widetilde{N_{1,\lambda}(X)}(y_\alpha^i, q_i^\alpha), \widetilde{\bar{A}_\beta^j}(y_\alpha^i, q_i^\alpha))y_p^i \tag{16}$$

where $1 \leq p \leq k$, $1 \leq |\alpha| \leq r$, $0 \leq |\lambda| \leq r$, *and* h^0, $h^p : \mathbb{R}^N \to \mathbb{R}$ *are any smooth functions for* $N = (k+1)\sum_{l=1}^r C(l+k-1, l-1) + 1$.

Proof. We are searching for equivariant maps $(J^rT)_0\mathbb{R}^m \times (T^*T_k^r)_0\mathbb{R}^m \to T$, since the independence of W^i on $X^{j_1,\dots j_{r+1}}$ is given by the formula for the action of B_m^{r+2}, which is of the form

$$\bar{X}^i_{j_1,\dots j_{r+1}} = X^i_{j_1\dots j_{r+1}} + a^i_{j_1\dots j_{r+1}l}X^l \tag{17}$$

where $X^i_{j_1\dots j_p}$ denote the canonical coordinates of $j_0^{r+1}X$, $a^i_{j_1\dots j_p}$ denote the canonical coordinates of G_m^{r+1} and B_m^s denote the set $\{j_0^s\varphi \in G_m^s; j_0^{s-1}\varphi = j_0^{s-1}\mathrm{id}_{\mathbb{R}^m}\}$. Fixing any element $(j_0^rX, y_\alpha^i, q_i^\mu) \in (J^rT)_0\mathbb{R}^m \times (T^*T_k^r)_0\mathbb{R}^m$ for $0 \leq |\alpha| \leq r$, $0 \leq |\mu| \leq r$, we can achieve $j_0^rX = j_0^r(\frac{\partial}{\partial x^1})$ by means of G_m^{r+1} on a dense subset of $(J^rT)_0\mathbb{R}^m \times (T^*T_k^r)_0\mathbb{R}^m$. Let C_0 denote the set of all r-jets of constant vector fields on $\mathbb{R}^m$, which is a G_m^1-equivariant subset. If we put $S_0 = C_0 \times (T^*T_k^r)_0\mathbb{R}^m$, it holds according to Lemma 4

$$W^i = g^0(X^iq_i^\lambda, y_\alpha^iq_i^\beta)X^i + g^\gamma(X^iq_i^\lambda, y_\alpha^iq_i^\beta)y_\gamma^i \tag{18}$$

for $1 \leq |\gamma|, |\alpha| \leq r$, $0 \leq |\beta|, |\lambda| \leq r$. From the cooincidence of $\widetilde{N_{1,\lambda}}$ with $X^iq_i^\lambda$ together with the coordinate expression of the absolute operators $\widetilde{\bar{A}_\beta^j}$ we can deduce

$$W^i = g^0(\widetilde{N_{1,\lambda}}, \widetilde{\bar{A}_\beta^j}, y_p^iq_i^0, y_\mu^iq_i^\nu)X^i + g^\gamma(\widetilde{N_{1,\lambda}}, \widetilde{\bar{A}_\beta^j}, y_p^iq_i^0, y_\mu^iq_i^\nu)y_\gamma^i \tag{19}$$

where $0 \leq |\lambda|, |\nu| \leq r$, $1 \leq |\beta|, |\gamma| \leq r$, $2 \leq |\mu| \leq r$, $j, p \in \{1, \dots, k\}$. We gradually annihilate all excessive arguments of g^0, g^γ by G_m^{r+1} preserving S_0 and the value of W^i. By the action of G_m^1 on S_0 we can manage on a dense subset $S_1 \subseteq S_0$ $X^i = \delta_1^i$, $y_p^i = \delta_{p+1}^i$. The formula for the action of B_m^s on $y^i_{l_1\dots l_s}$, $2 \leq s \leq r$ is of the form $\bar{y}^i_{l_1\dots l_s} = y^i_{l_1\dots l_s} + a^i_{l_1+1\dots l_s+1}$. It follows, that we can annihilate all y_α^i, $|\alpha| \geq 2$ and

$$W^i = g^0(\widetilde{N_{1,\lambda}}, \widetilde{\bar{A}_\beta^j}, y_p^i\hat{q}_i^0, 0, \dots, 0)X^i + g^p(\widetilde{N_{1,\lambda}}, \widetilde{\bar{A}_\beta^j}, y_p^i\hat{q}_i^0, 0, \dots, 0)y_p^i \tag{20}$$

where $\hat{q}_i^0$ denotes the new value of q_i^0 obtained by the composition of the actions of B_m^l. Since $y_p^i\hat{q}_i^0$ can be annihilated by the action of $G_m^{r+1} \cap \mathit{Diff}_0^1\,\mathbb{R}^m$, the functions g^0, g^p depend only on $\widetilde{N_{1,\lambda}}$ and $\widetilde{\bar{A}_\beta^j}$. Renaming these functions to h^0, h^p we prove our claim, since $W^i = h^0(\widetilde{N_{1,\lambda}}, \widetilde{\bar{A}_\beta^j})X^i + h^p(\widetilde{N_{1,\lambda}}, \widetilde{\bar{A}_\beta^j})y_p^i$ can be extended to $(J^{r+1}T)_0\mathbb{R}^m \times (T^*T_k^r)_0\mathbb{R}^m$. □

The following lemma reduces our problem to the classification of natural operators $T \to VT^*T_k^r$, where $VT^*T_k^r$ denotes the vector bundle $T^*T_k^r \to T_k^r$.

Lemma 6. *Let $A_M : TM \to TT^*T^r_kM$ be a natural operator, $m \geq k+1$. There are smooth functions $h^\beta_j : \mathbb{R}^N \to \mathbb{R}$ and $h^\alpha_0 : \mathbb{R}^N \to \mathbb{R}$ such that*

$$A_M - h^\alpha_0((\widetilde{N_{1,\lambda}})_M, (\widetilde{\bar{A}^p_\mu})_M)(\widetilde{\widetilde{N_\alpha}})_M - h^\beta_j((\widetilde{N_{1,\lambda}})_M, (\widetilde{\bar{A}^p_\mu})_M)(\widetilde{\widetilde{A^j_\beta}})_M$$

is a natural operator $TM \to VT^*T^r_kM$, where $1 \leq |\beta|, |\mu| \leq r, j, p \leq r$, $0 \leq |\alpha|, |\lambda| \leq r$ and $N = (k+1)\sum_{l=1}^r C(l+k-1, l-1) + 1$.

Proof. Let $Y^i_\alpha = dy^i_\alpha$, $Q^\alpha_i = dq^\alpha_i$ define the additional coordinates on $TT^*T^r_kM$. The action of G^{r+2}_m on Y^i_α is of the form $\bar{Y}^i_\alpha = Y^i_\alpha + a^i_l Y^l_\alpha$ whenever $Y^i_\gamma = 0$ for every multiindex γ satisfying $|\gamma| < |\alpha|$. Applying Lemma 5 to Y^i_0 we obtain $Y^i_0 = h^0_0(\widetilde{N_{1,\lambda}}, \widetilde{\bar{A}^p_\mu})X^i + h^j_0(\widetilde{N_{1,\lambda}}, \widetilde{\bar{A}^p_\mu})y^i_j$, which follows that the natural operator $A - h^0_0(\widetilde{N_{1,\lambda}}, \widetilde{\bar{A}^p_\mu})\widetilde{\widetilde{TT^r_k}}$ satisfies $Y^i_0 = h^j_0(\widetilde{N_{1,\lambda}}, \widetilde{\bar{A}^p_\mu})y^i_j$. We prove, that $h^j_0(\widetilde{N_{1,\lambda}}, \widetilde{\bar{A}^p_\mu}) = 0$. If $y^i_j = \delta^i_{j+1}$, $j^{r+1}_0X = j^{r+1}_0(\frac{\partial}{\partial x^1})$, the transformation law of the action B^{r+2}_m on Q^0_i is of the form $\bar{Q}^0_i = Q^0_i - a^p_{il_1+1\dots l_r+1l}Y^l_0 q^{l_1\dots l_r}_p$. If we put $a^p_{il_1+1\dots l_r+1l} = 0$ except of $a^1_{il+1\dots l+1}$, we obtain $Y^i_l = 0$ whenever $q^{l\dots l}_1 \neq 0$ since such an element of G^{r+2}_m does not affect the value of any element from $(T^*T^r_k)_0\mathbb{R}^m$.

The rest of the proof is made by the induction in respect to $|\beta|$. If the natural operator A satisfies $Y^i_\gamma = 0$ for every multiindex γ satisfying $|\gamma| < |\beta|$, Lemma 5 and the coordinate form of $\widetilde{\widetilde{\bar{A}^p_\mu}}$ yield that $A - h^\beta_0(\widetilde{N_{1,\lambda}}, \widetilde{\bar{A}^p_\mu})\widetilde{\widetilde{N_\beta}} - h^\beta_j(\widetilde{N_{1,\lambda}}, \widetilde{\bar{A}^p_\mu})\widetilde{\widetilde{A^j_\beta}}$ satisfies $Y^i_\beta = 0$ for some functions $h^\beta_0, h^\beta_j : \mathbb{R}^N \to N$. □

Now we are going to investigate natural operators $T \to VT^*T^r_k$. Every natural operator $A_M : TM \to TT^*T^r_kM$ can be expressed by

$$A_MX(y^i_\alpha, q^\alpha_i) = Y^i_\alpha(j^{r+1}_0X, y^i_\alpha, q^\alpha_i)\frac{\partial}{\partial y^i_\alpha} + Q^\alpha_i(j^{r+1}_0X, y^i_\alpha, q^\alpha_i)\frac{\partial}{dq^\alpha_i} \tag{21}$$

Let $\pi^\alpha_i dy^i_\alpha + \rho^i_\alpha dq^\alpha_i$ define the additional coordinates on $T^*T^*T^r_kM$. Every natural operator of this kind is identified with $Y^i_\alpha\pi^\alpha_i + Q^\alpha_i\rho^i_\alpha$, which is a natural operator $T \to C^\infty(T^*T^*T^r_k, \mathbb{R})$ satisfying the linearity on fibers of $T^*T^*T^r_k \to T^*T^r_k$.

Natural operators $f_M : TM \to C^\infty(T^*T^*T^r_kM, \mathbb{R})$ are in the bijective correspondence with natural operators $g_M : TM \to C^\infty(T^*TT^r_kM, \mathbb{R})$ given by $g_M = f_M \circ t_{T^r_kM} \circ s^{-1}_{T^r_kM}$. Let $z^i_\alpha = dy^i_\alpha$ define the additional coordinates on TT^r_kM and $r^\alpha_i dy^i_\alpha + s^\alpha_i dz^i_\alpha$ define the additional coordinates on $T^*TT^r_kM$. The natural equivalence yields

$$z^i_\alpha = -\rho^i_\alpha, \qquad r^\alpha_i = \pi^\alpha_i, \qquad s^\alpha_i = q^\alpha_i \tag{22}$$

Since we are searching only for natural operators $T \to VT^*T^r_k$, it holds $Y^i_\alpha = 0$. Thus we are searching for natural operators $g_M : TM \to C^\infty(T^*TT^r_kM, \mathbb{R})$ which are independent on r^α_i and linear in z^i_α. The formula (22) enables us to write q^α_i instead of s^α_i. The following lemma describes all natural operators of the recent kind independent on r^α_i.

Lemma 7. *For* $\dim M \geq k+2$ *every natural operator* $g_M : TM \to C^\infty(T^*T^r_k M, \mathbb{R})$ *independent on* r^α_i *is of the form*

$$h(\widetilde{N_{1,\lambda}}, \widetilde{\bar{A}^j_\beta}, \widetilde{A^\gamma}) \tag{23}$$

where h *is any smooth function.*

Proof. Let $\dim M = k+2$. On a dense subset of $J^{r+1}TM \times_M TT^r_k M$ we can achieve by G^{r+2}_{k+2} the immersion element i, which is of the form $j^{r+1}_0 X = j^{r+1}_0(\frac{\partial}{\partial x^1})$, $y^i_p = \delta^i_{p+1}$, $z^i_0 = \delta^i_{k+2}$ while the other y^i_α and z^i_α are zeros. Lemma 4 (c) implies, that every natural operator in question is identified with some function, the arguments of which evaluate themselves over the element i as q^α_j, r^α_j, $0 \leq |\alpha| \leq r$, $1 \leq j \leq k+2$. Over the immersion element i, q^α_j coincide with the natural operators $\widetilde{N_{1,\alpha}}$, $\widetilde{\bar{A}^j_\beta}$, $\widetilde{A^\gamma}$ except of $q^0_2, \dots q^0_{k+1}$, which are annihilated by B^{r+2}_{k+2} stabilizing the immersion element i in the following way.

The change of the value of the element i is given by the following formulas given by the transformation laws of B^s_{k+2} on $(J^{r+1}T)_0\mathbb{R}^{k+2} \times (TT^r_k)_0\mathbb{R}^{k+2}$

$$\bar{y}^i_{l_1\dots l_s} = y^i_{l_1\dots l_s} + a^i_{l_1+1\dots l_s+1}, \qquad \bar{z}^i_{l_1\dots l_s} = z^i_{l_1\dots l_s} + a^i_{l_1+1\dots l_s+1\,k+2} \tag{24}$$

which follows, that $a^i_j = \delta^i_j$, $a^i_\alpha = 0$ for $2 \leq |\alpha| \leq r+1$ or if 1 is contained in the multiindex α. The transformation law for q^0_j over the element i is of the form $\bar{q}^0_j = q^0_j - a^h_{jl_1+1\dots l_s+1} q^{l_1\dots l_r}_h$. We can annihilate q^0_j for $2 \leq j \leq k+1$ by $a^1_{j\dots j}$ (the order of the bottom index being $r+1$) whenever $q^{j-1\dots j-1} \neq 0$.

The proof is almost the same for $\dim M > k+2$. □

Proposition 8. *Let* $A_M : TM \to TT^*T^r_k M$ *be a natural operator,* $\dim M \geq k+2$. *Then it holds*

$$A_M = h^\alpha(\widetilde{\widetilde{N_\alpha}})_M + h^\beta_j(\widetilde{\widetilde{A^j_\beta}})_M + h_\gamma(\widetilde{\widetilde{A^\gamma}})_M$$

where h^α, h^β_j, h_γ *are any smooth functions of* $(\widetilde{N_{1,\lambda}})_M$, $(\widetilde{\bar{A}^p_\mu})_M$ *for* $1 \leq |\beta|, |\mu| \leq r$, $1 \leq j, p \leq r$, $0 \leq |\alpha|, |\lambda|, |\gamma| \leq r$.

Proof. By Lemma 7, the natural operators $T \to VT^*T^r_k$ are searched among the functions $h(\widetilde{N_{1,\lambda}}, \widetilde{\bar{A}^j_\beta}, \widetilde{A^\gamma})$, which are linear in z^i_α. It holds:

$$Q^\alpha_i = \frac{\partial h(\widetilde{N_{1,\lambda}}, \widetilde{\bar{A}^j_\beta}, \widetilde{A^\delta})}{\partial \widetilde{A^\gamma}} \frac{(\alpha+\gamma)!}{\alpha!} q^{\alpha+\gamma}_i \tag{25}$$

which follows from the cordinate expreesion of $\widetilde{N_{1,\lambda}}, \widetilde{\bar{A}^j_\beta}, \widetilde{A^\gamma}$.

Since $\frac{\partial h(\widetilde{N_{1,\lambda}},\widetilde{\bar{A}^j_\beta},\widetilde{A^\delta})}{\partial \widetilde{A^\gamma}}$ is again a smooth combination of $\widetilde{N_{1,\lambda}}$, $\widetilde{\bar{A}^j_\beta}$, $\widetilde{A^\delta}$ and Q^α_i does not depend on any z^i_α, the formula (25) reduces to

$$Q^\alpha_i = \frac{\partial h(\widetilde{N_{1,\lambda}},\widetilde{\bar{A}^j_\beta},0)}{\partial \widetilde{A^\gamma}} \frac{(\alpha+\gamma)!}{\alpha!} q^{\alpha+\gamma}_i. \tag{26}$$

If we put $h_\gamma(\widetilde{N_{1,\lambda}},\widetilde{\bar{A}^j_\beta}) = \frac{\partial h(\widetilde{N_{1,\lambda}},\widetilde{\bar{A}^j_\beta},0)}{\partial \widetilde{A^\gamma}}$, we obtain, that every natural operator $T \to VT^*T^r_k$ is of the form $h_\gamma(\widetilde{N_{1,\lambda}},\widetilde{\bar{A}^j_\beta})\widetilde{\widetilde{A^\gamma}}$, which follows from the coordinate expression of $\widetilde{\widetilde{A^\gamma}}$. Applying Lemma 6 proves our claim. □

We notice some properties of the operation $\widetilde{\widetilde{\ }}$. Let $Y : TM \to TTM$ be a linear vector field, [5]. Let $\xi^i = dx^i$ define the additional coordinates on TM. Then the coordinate expression of Y is of the form

$$Y = X^i(x)\frac{\partial}{\partial x^i} + \eta^i_j(x)\xi^j\frac{\partial}{\partial \xi^i}. \tag{27}$$

Furthermore, let $\rho_i dx^i + \sigma_i d\xi^i$ define the additional coordinates on T^*TM. Then $\widetilde{Y}$ is of the form

$$\widetilde{Y} = X^i(x)\rho_i + \eta^i_j(x)\xi^j\sigma_i \tag{28}$$

If $w_i dx^i$ define the coordinates on T^*M and $\chi_i dx^i + \mu^i dw_i$ define the additional coordinates on T^*T^*M, then the natural equivalence $t \circ s^{-1} : T^*TM \to T^*T^*M$ yields

$$w_i = \sigma_i, \qquad \chi_i = \rho_i, \qquad \mu^i = -\xi^i \tag{29}$$

Under this transformation we obtain that $\widetilde{Y} = X^i(x)\chi_i - \eta^i_j(x)\mu^j w_i$. Since $\widetilde{Y}$ satisfies the linearity discussed in the beginning of this section, we obtain

$$\widetilde{\widetilde{Y}} = X^i(x)\frac{\partial}{\partial x^i} - \eta^i_j(x)w_i\frac{\partial}{\partial w_j} \tag{30}$$

which is the dual vector field to Y ([5]). If we put $Y = \mathcal{T}X$ for a vector field $X : M \to TM$, one can easily see, that $\widetilde{\widetilde{\mathcal{T}X}} = \mathcal{T}^*X$.

Substituting T^r_kM for M and $af(\tau^\alpha) \circ \mathcal{T}^r_k$ or $\mathrm{op}(D^j_\beta)$ for X, we obtain $\widetilde{\widetilde{N_\alpha}} = \mathcal{T}^* \circ af(\tau^\alpha) \circ \mathcal{T}^r_k$ or $\widetilde{\widetilde{A^j_\beta}} = \mathcal{T}^* \circ \mathrm{op}(D^j_\beta)$ respectively.

Now we are going to investigate the natural operators $\widetilde{\widetilde{A^\gamma}}$. From the coordinate expression of $\widetilde{\widetilde{A^0}}$, one can immediately deduce, that $\widetilde{\widetilde{A^0}} = \mathcal{L}_{T^*(T^r_k)}$, which is the Liouville vector field on the natural bundle $T^*(T^r_k) \to T^r_k$.

A vector bundle $EF \to F$ is identified with $EF \times_F EF$. The identification is given by $(x^i, y^p, 0, \xi^p) \simeq ((x^i, y^p), (x^i, \xi^p))$, where x^i are the coordinates on F, y^p are the fiber coordinates on EF and $\xi^p = dy^p$ are the additional coordinates on VEF. For a local diffeomorphism f, the coordinate expression of $VEFf$ is of the form $\bar{\xi}^p = \frac{\partial f^p}{\partial y^q}\xi^q$. If we put $EF = T^*T^r_k$, the natural operator $\mathcal{L}_{T^*(T^r_k)}$ is expressed by $q^\alpha_i \frac{\partial}{\partial q^\alpha_i}$ in our coordinates on $T^*T^r_k$. If we evaluate the coordinate form of the map $af(\tau^\gamma)^*$, we obtain $\bar{q}^\alpha_i = \frac{(\alpha+\gamma)!}{\alpha!} q^{\alpha+\gamma}_i$, which follows that $\widetilde{\widetilde{A^\gamma}} = Vaf(\tau^\gamma)^* \circ \mathcal{L}_{T^*(T^r_k)}$.

It remains to desribe the natural operators $\widetilde{N_{1,\lambda}}$ and $\widetilde{\bar{A}^j_\beta}$. It is obvious that $N_{1,\lambda} = \mathcal{V} \circ af(\tau^\lambda) \circ \mathcal{T}^r_k$. By Lemma 3 we have $\widetilde{\bar{A}^j_\beta} = \mathcal{V} \circ \mathrm{op}(D^j_\beta)$. Let us define the natural transformation $q_M : T^*TT^r_kM \to T^*T^r_kM$ by $q_M = p_{T^*T^r_kM} \circ s^{-1}_{T^r_kM}$, where $p_{T^*T^r_kM} : TT^*T^r_kM \to T^*T^r_kM$ is the tangent bundle projection and $s_{T^r_kM} : TT^*T^r_kM \to T^*TT^r_kM$ is the natural equivalence by Modugno, Stefani. If we consider the coordinates defined before Lemma 7, the formulas (21) and (22) imply that the natural transformation q_M is of the form $(y^i_\alpha, z^i_\alpha, r^\alpha_i, s^\alpha_i) \mapsto (y^i_\alpha, q^\alpha_i)$, where $q^\alpha_i = s^\alpha_i$. If $A : T \to TT^r_k$ is a natural operator, $A = Y^i_\alpha \frac{\partial}{\partial y^i_\alpha}$, then $\mathcal{V} \circ \widetilde{A_M} = Y^i_\alpha s^\alpha_i = Y^i_\alpha q^\alpha_i = \mathcal{V} \circ \widetilde{A_M} \circ q_M = \widetilde{A_M}$. It follows, that $\widetilde{N_{1,\lambda}}$ is identified with $\widetilde{af(\tau^\lambda) \circ \mathcal{T}^r_k}$ and $\widetilde{\bar{A}^j_\beta}$ is identified with $\widetilde{\mathrm{op}(D^j_\beta)}$, which follows, that Proposition 8 can be presented in the following form

Theorem 9. *Let $A_M : TM \to TT^*T^r_kM$ be a natural operator, $\dim M \geq k+2$. Then A is of the form*

$$h^\alpha(\widetilde{N_{1,\lambda}}, \widetilde{\mathrm{op}(D^p_\mu)})\mathcal{T}^* \circ af(\tau^\alpha) \circ \mathcal{T}^r_k + h^\beta_j(\widetilde{N_{1,\lambda}}, \widetilde{\mathrm{op}(D^p_\mu)})\mathcal{T}^* \circ \mathrm{op}(D^j_\beta) + h_\gamma(\widetilde{N_{1,\lambda}}, \widetilde{\mathrm{op}(D^p_\mu)})Vaf(\tau^\gamma)^* \circ \mathcal{L}_{T^*(T^r_k)}$$

where $\mathcal{T}^*$ is the flow prolongation of the cotangent bundle functor, $\mathcal{L}_{T^*(T^r_k)}$ is the Liouville vector field on the natural bundle $T^*(T^r_k) \to T^r_k$, V is the vertical bundle functor, $af(\tau^\gamma)^*$ is the dual map to $af(\tau^\gamma)$, h^α, h^β_j, h_γ are any smooth functions of $\widetilde{af(\tau^\lambda) \circ \mathcal{T}^r_k}$ and $\widetilde{\mathrm{op}(D^p_\mu)}$ for the same values of multiindices as in Proposition 8 and $\tau_1, \ldots, \tau_k$ are variables of polynomials forming the Weil algebra $\mathbb{D}^r_k$.

I thank prof. I. Kolář for his useful help and suggestions.

References

[1] Kobak P., *Natural liftings of vector fields to tangent bundles of 1-forms*, Mathematica Bohemica **116** (1991), 319-326.

[2] Kolář I., *Covariant Approach to Natural Transformations of Weil Bundles*, Comment. Math. Univ. Carolinae (1986).

[3] Kolář I., *On Cotagent Bundles of Some Natural Bundles*, Rendiconti del Circolo Matematico di Palermo Serie II-Numero **37** (1994), 115-120.

[4] Kolář I., *On the Natural Operators on Vector Fields*, Ann. Global Anal. Geometry **6** (1988), 109-117.

[5] Kolář I., Michor P. W., Slovák J., *Natural Operations in Differential Geometry*, Springer – Verlag, 1993.

[6] Kolář I., Radziszewski Z., *Natural Transformations of Second Tangent and Cotangent Bundles*, Czechoslovak Math. (1988), 274-279.

[7] Modugno M., Stefani G., *Some results on second tangent and cotangent spaces*, Quaderni dell'Università di Lecce (1978).

[8] Tomáš J., *Some natural operators on vector fields*, Archivum Mathematicum **31** (1995), 23 239-249.

Department of Mathematics
Faculty of Civil Engineering
Technical University Brno
Žižkova 17
602 00 Brno
Czech Republic