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DIFFERENTIAL GEOMETRY OVER THE STRUCTURE SHEAF: A WAY TO QUANTUM PHYSICS *

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Abstract

The formulation of differential geometry over the structure sheaf stresses the algebraic side of the theory. So it represents a proper starting point on the way to quantum physics carried by geometric observables.

1 Introduction

The motivation for this work is the aim to describe quantum physics in terms of noncommutative differential geometry. This unifies two aspects of a mathematical description of physics. The first is the quantum theoretic aspect. Quantum theory is an abstract theory of measurement which assigns to every possible measurement an observable and lays down the relations between them by an algebra structure on the set of all observables. The fundamental characteristics of quantum physics the uncertainty relations are encoded in the noncommutativity of this algebra. The uncertainty relations are signs for the existence of an interaction so we define

Definition 1 *Let $E_{\mathcal{O}}$ be a generating set for an observable algebra $\mathcal{O}$ and $\mathcal{T}(E_{\mathcal{O}})$ the free algebra generated by $E_{\mathcal{O}}$. We call a bilinear map*

$$W : E_{\mathcal{O}} \times E_{\mathcal{O}} \longrightarrow \mathcal{T}(E_{\mathcal{O}}) \tag{1}$$

an interaction structure for $\mathcal{O}$ if it provides the relations for the algebra $\mathcal{O}$ on the generating set i.e.

$$\mathcal{O} = \mathcal{T}(E_{\mathcal{O}})/J_W \tag{2}$$

with the ideal $J_W \subset \mathcal{T}(E_{\mathcal{O}})$ generated by the elements
$a \otimes b - b \otimes a - W(a,b)$ with $a, b \in E_{\mathcal{O}}$.

A nonvanishing interaction structure is equivalent to a noncommutative observable algebra. In this way noncommutativity enters the region of our interest.
The second aspect is the field theoretic one; this means a description of physics in terms of fields over some physical parameter spaces. The requirement of local coordinates on

*This paper is in final form and no version of it will be submitted for publication elsewhere.

these parameter spaces combined with an invariance of physics under special choices of such coordinates introduces the principle of relativity. The mathematical counterpart to this is the concept of manifolds in geometry, or after specializing to differentiable fields in differential geometry.
A first step to a unification of these two aspects is a formulation of differential geometry more accessible for quantum theory, or to say for deformation. For that we make an approach to differential geometry by the help of sheaves.

2 Differential geometry over the structure sheaf

Differential geometry is the theory of geometric objects over smooth manifolds [Ni]. We start by defining the smooth manifolds and then come to the objects.

2.1 Smooth manifolds

We call a space of the form $\mathbb{K}^n$ for a field $\mathbb{K}$ with a suitable order structure a coordinate space. Here for $\mathbb{K}$ we take $\mathbb{R}$. A manifold is a local coordinate space i.e. it looks locally like a coordinate space. The term "local" brings us into the category $\mathcal{Top}$ of topological spaces and continous maps; it reflects the restriction process ρ in the topology $\tau(X)$ of a topological space X. Dual to the restriction ρ is the inclusion which in contrast to ρ is a map $\in Mor(\mathcal{Top})$. It provides a topological space X with the structure of a category [Ha] by $Obj(X) = \tau(X)$ and

$$Mor(V,U) = \begin{cases} inclusion & : V \subset U \\ \emptyset & : V \not\subset U \end{cases}, \; U,V \in Obj(X). \tag{3}$$

Therefore local structures on topological spaces which always belong to a lifting of the restriction process to some fields over X can be described by contravariant functors from the category X to some suitable categories (mostly the category of abelian groups) [Ha]. These functors are the presheaves on X under whom the sheaves form a special subset fulfilling further conditions of local to global and gluing behaviour, the sheaf axioms [Ha]. The coordinate structures on manifolds in this approach are given by sheaves O_M of germs of functions. The special types of functions joining the required properties of the manifolds (e.g. differentiability) serve as local coordinate functions i.e. as the components of the chart maps. In connection with the restriction maps of the sheaf this is equivalent to the atlas approach to manifolds. The algebra structure of the field $\mathbb{R}$ provides the germs of the functions with a ring structure so that O_M becomes a sheaf of rings and the manifolds are defined by so called ringed spaces (M, O_M) [Ha]. A ringed space generally is a pair of a topological space M and a sheaf of rings O_M on M which fixes an additional structure on M, that's why it is called the structure sheaf of the ringed space. The ringed spaces form a category $\mathcal{R}$ where the morphisms $\phi \in Mor(X,Y)$ for $X = (M, O_M), Y = (N, O_N) \in Obj(\mathcal{R})$ are given by pairs

$$\phi \; : \; M \longrightarrow N \tag{4}$$
$$\phi^\sharp \; : \; \phi^{-1}O_N \longrightarrow O_M \tag{5}$$

of a continuous map ϕ and a sheaf morphism $\phi^\sharp$ ($\phi^{-1}O_N$ denotes the inverse image sheaf [Ha]). The local coordinate structure of a manifold is given by isomorphisms of special subspaces (neighbourhoods) to subspaces of a coordinate space. This forces to introduce the notion of local isomorphisms in the category of ringed spaces.

Definition 2 *A morphism $\phi \in Mor(X,Y)$ for $X,Y \in Obj(\mathcal{R})$ is called a local isomorphism if for every point $x \in M$ there exists an open set $U \in \tau(M)$ with $x \in U$ such that the restriction*
$\rho(\phi) : (U, \rho_U^M O_M) \to im(\rho(\phi))$ is an isomorphism in $\mathcal{R}$.

A local isomorphism in $\mathcal{R}$ respects the local coordinate structure of a manifold so these morphisms describe the coordinate transformations of manifolds. In the smooth category $\mathcal{C}^\infty$ which is the subcategory in $\mathcal{R}$ with $Obj(\mathcal{C}^\infty) = \{\text{smooth manifolds}\}$ the structure sheaf of an $X \in Obj(\mathcal{C}^\infty)$ is the sheaf of germs of smooth functions $O_M = C_M^\infty$. The local isomorphisms are the local diffeomorphisms $\mathcal{D}_0 \subset Mor(\mathcal{C}^\infty)$ and the coordinate transformations of a differentiable manifold X are elements in $Mor(X,X) \cap \mathcal{D}_0$.
Before we turn to the geometric objects we again stress the algebraic flavour of the sheaf approach to differential geometry. $\mathcal{A}_\mathbb{R}$ denotes the category of $\mathbb{R}$-algebras. In the case of compact topological spaces we can use the global section functor [Ha]

$$\begin{aligned} \Gamma \ : \ & \mathcal{R} \longrightarrow \mathcal{A}_\mathbb{R} \\ & X \mapsto \Gamma(M, C_M^\infty) \end{aligned} \tag{6}$$

to describe a differentiable manifold given by a ringed space (M, C_M^∞) in purely algebraic terms by the algebra $\Gamma(M, C_M^\infty) = C^\infty(M, \mathbb{R})$. The information about the manifold X in the form (M, C_M^∞) can be regained from the algebra $C^\infty(M,\mathbb{R})$ by use of the adjoint functor *spec′*. This comes out of the functor *spec* used in algebraic geometry [EH] by a restriction of the space $|spec|$ of prime ideals to the set of maximal ideals $|maxspec|$. The topology on $|maxspec|$ is induced from the Zariski-topology on $|spec|$. Applied to $C^\infty(M,\mathbb{R})$ this gives a homeomorphism between the topological spaces $maxspec C^\infty(M,\mathbb{R})$ and M. The sheaf C_M^∞ is constructed out of $C^\infty(M,\mathbb{R})$ by the usual localization procedures [EH]. So in the language of algebraic geometry a differentiable manifold is an affine scheme [EH]. From the quantum theoretic point of view this dual equivalence of categories raises Γ as quantitzation functor which assigns to the classical system (M, C_M^∞) its observable algebra $C^\infty(M,\mathbb{R})$.

2.2 Geometric objects

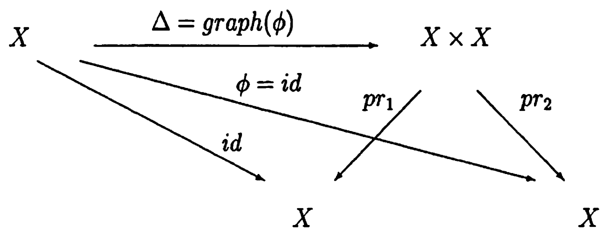
After stating the manifolds with the help of structure sheaves we turn now to the geometric objects on these spaces. Geometric objects [Ni] on a smooth manifold X are such objects which don't change their character under coordinate transformations (e.g. scalars remain scalars, vectors remain vectors). This invariance property we compose in a slightly more abstract language. We are looking for functors

$$g : \mathcal{C}^\infty \longrightarrow \mathcal{S}h \tag{7}$$

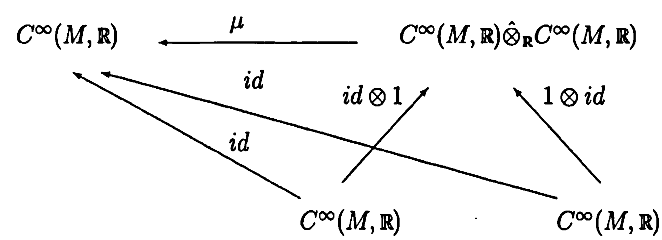
from the category of smooth manifolds into the category of sheaves which respect (lift) the local isomorphisms. As geometric objects we take the elements in the stalks of the sheaves assigned by such functors. The global sections deliver the fields of geometric objects, the geometric fields. Those functors can be derived from the structure sheaf in a canonical way. Therefor we use the required invariance properties of the geometric objects; the transformation behaviour of geometric objects does not depend on a special choice of coordinate transformation ϕ, so we take the simplest one, the identity $\phi = id \in \mathcal{D}_0$, and study its graph

$$graph(id) = \Delta : X \longrightarrow X \times X, \tag{8}$$

the famous diagonal, and perturbations of it. The diagonal morphism in a category with products is defined by the special form of a product diagram which belongs to the *id* isomorphism:



The dual equivalence given by the functors Γ and *spec'* determines the diagonal $\Delta = (\Delta, \Delta^\sharp)$ in terms of a coproduct diagram:



Here $\hat{\otimes}_{\mathbb{R}}$ means the completion of the algebraic tensor product such that

$$C^\infty(M, \mathbb{R})\hat{\otimes}_{\mathbb{R}}C^\infty(M, \mathbb{R}) = \Gamma(M \times M, C^\infty_{M\times M}).$$

In this way the multiplication map

$$\mu : C^\infty(M, \mathbb{R})\hat{\otimes}_{\mathbb{R}}C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R}) \tag{9}$$

fixes the sheaf morphism

$$\Delta^\sharp : \Delta^{-1}(C^\infty_M \hat{\otimes}_{\mathbb{R}} C^\infty_M) \longrightarrow C^\infty_M \tag{10}$$

over the diagonal in $\mathcal{T}op$. $\Delta^\sharp$ is a surjective homomorphism between C^∞_M-modules, therefore its kernel $I_\Delta := ker\Delta^\sharp$ is an ideal in
$\Delta^{-1}(C^\infty_M \hat{\otimes}_{\mathbb{R}} C^\infty_M)$ and defines a filtration

$$\Delta^{-1}(C^\infty_M \hat{\otimes}_{\mathbb{R}} C^\infty_M) \supset I_\Delta \supset I^2_\Delta \supset \dots . \tag{11}$$

This filtration yields the functors we are looking for. By the properties of the tensor product of homomorphisms a first one is immediately given by

$$\begin{aligned} p^\infty \;:\; \mathcal{C}^\infty \quad &\longrightarrow \quad \mathcal{S}h \\ (M, C^\infty_M) &\longrightarrow \Delta^{-1}(C^\infty_M \hat{\otimes}_{\mathbb{R}} C^\infty_M). \end{aligned} \tag{12}$$

This functor is universal in the sense that all other sheaves for geometric objects can be derived from it in a canonical way. For that we relax the identity on $\Delta^{-1}(C^\infty_M \hat{\otimes}_{\mathbb{R}} C^\infty_M)$ by a perturbation in the k-th order

$$id \longrightarrow j^k id = id + Mor(I^{k+1}_\Delta), \tag{13}$$

with $Mor(I^{k+1}_\Delta) \subset Mor(\Delta^{-1}(C^\infty_M \hat{\otimes}_{\mathbb{R}} C^\infty_M), \Delta^{-1}(C^\infty_M \hat{\otimes}_{\mathbb{R}} C^\infty_M))$. This class of morphisms acts as the identity on the k-jet sheaves of X which we define as the quotients

$$J^k C^\infty_M := \Delta^{-1}(C^\infty_M \hat{\otimes}_{\mathbb{R}} C^\infty_M)/I^{k+1}_\Delta. \tag{14}$$

The resulting functors

$$\begin{aligned} p^k \;:\; \mathcal{C}^\infty \quad &\longrightarrow \quad \mathcal{S}h \\ (M, C^\infty_M) &\longrightarrow J^k C^\infty_M \end{aligned} \tag{15}$$

respect local isomorphisms and the sections of the sheaves which assign these functors to a differentiable manifold are fields of geometric objects of order k. The summands of the associated graded module of the I_Δ-filtration

$$\Delta^{-1}(C^\infty_M \hat{\otimes}_{\mathbb{R}} C^\infty_M)/I_\Delta \oplus I_\Delta/I^2_\Delta \oplus I^2_\Delta/I^3_\Delta \oplus \dots \tag{16}$$

provide the homogeneous geometric fields. The first summand e.g. is isomorphic to the structure sheaf C^∞_M itself, its global sections the scalar functions $C^\infty(M, \mathbb{R})$ are the fields of order zero. The second summand I_Δ/I^2_Δ is the conormal module of Δ; its sections which are first order fields are isomorphic to the covectorfields on M
$\Gamma(M, I_\Delta/I^2_\Delta) \cong \Omega^1(M)$ [F1].

3 Outlook (Deformation)

As mentioned before the algebra $C^\infty(M,\mathbb{R})$ is the observable algebra for the system $X = (M, C^\infty_M)$. Because it is commutative in quantum theoretic terms it has vanishing interaction structure. In order to get interesting noncommutative observable algebras of geometric fields (geometric observables) one has to deform the algebra $C^\infty(M,\mathbb{R})$ by suitable interaction structures. The deep connection of interaction and measurement stated in quantum physics paves the way to such structures. In the commutative case of $C^\infty(M,\mathbb{R})$ the algebra structure is defined by pointwise addition and multiplication; this corresponds to a measurement of geometric objects of order zero at a point $x \in M$ by evaluation

$$ev_x : C^\infty_x \longrightarrow \mathbb{R}. \tag{17}$$

Nontrivial interaction structures arise by measurement of geometric objects of higher order. Such geometric measurements belong to geometric structures on smooth manifolds.

Definition 3 *A geometric structure of order k on a smooth manifold X is an isomorphism of dual sheaves*

$$q^k : I^k_\Delta / I^{k+1}_\Delta \longrightarrow \mathcal{H}om_{C^\infty_M}(I^k_\Delta / I^{k+1}_\Delta, C^\infty_M). \tag{18}$$

The most prominent geometric structures are the symmetric and antisymmetric first order ones which deliver Riemannian and symplectic geometry. A symplectic structure $q^1 = \omega$ e.g. defines as a nontrivial interaction structure the Poisson-structure $\{\,,\}$ on $C^\infty(M,\mathbb{R})$

$$\begin{aligned} W^1 \;:\; & C^\infty(M,\mathbb{R}) \times C^\infty(M,\mathbb{R}) \longrightarrow C^\infty(M,\mathbb{R}) \\ & (f,g) \mapsto W^1(f,g) = \{f,g\} := \omega(df,dg). \end{aligned} \tag{19}$$

The noncommutative observable algebra associated to this structure is the universal enveloping algebra of the Lie-algebra $(C^\infty(M,\mathbb{R}), \{,\})$. Equivalent to that we can deform the commutative algebra $C^\infty(M,\mathbb{R})$ in terms of star-products (with $\hbar = 1$) [We]

$$f \star g = fg + \sum_{k=1}^{\infty} \hbar^k B^k(f,g) \tag{20}$$

with $B^1(f,g) = \frac{1}{2}\omega(df,dg)$ and $B^k = q^k = 0$ for $k > 1$ [F2]. In the more general case of nonvanishing higher order geometric structures differentials of higher order

$$d_k : C^\infty(M,\mathbb{R}) \longrightarrow \Gamma(M, I^k_\Delta / I^{k+1}_\Delta) \tag{21}$$

are used and the influence of the higher order fields to the deformation is given by $B^k(f,g) = q^k(d_k f, d_k g)$ in the $\star_{\{q^k\}}$-product. The global section functor Γ combined with the deformation $\star_{\{q^k\}}$ so represents the quantization functor which assigns to an "interacting" space $(M, C^\infty_M, \{q^k\})$ its observable algebra.

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