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LIFTINGS OF 1–FORMS TO SOME NON PRODUCT PRESERVING BUNDLES

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ABSTRACT. The work is devoted to the question how to construct geometrically a 1–form on some non product preserving bundles by means of a 1–form on an original manifold M. First we will deal with liftings of 1–forms to higher order cotangent bundles. Then we will be concerned with liftings of 1–forms to the bundles which arise as a composition of the cotangent bundle with the tangent or cotangent bundle.
KEYWORDS. Higher order cotangent bundle, natural operator, lifting

1. INTRODUCTION

The aim of this paper is to study geometrical constructions of 1–forms on some non product preserving bundles by means of 1–forms on an original manifold M. Roughly speaking, the word geometrical means that the constructions in question can be defined independently of coordinate changes. Using a more general point of view, geometrical constructions are in fact natural differential operators of certain type, cf. [4]. If F is an arbitrary natural bundle, then the natural operators transforming 1–forms on a manifolf M into 1–forms on FM will be denoted by $T^* \rightsquigarrow T^*F$. In other words, natural operators of such a type are sometimes called liftings.

By the general theory, every product preserving bundle F can be expressed as a Weil bundle T^A corresponding to certain Weil algebra A, [4]. Mikulski has in [8] determined all natural operators $T^* \rightsquigarrow T^*T^A$ for every Weil bundle T^A. If F does not preserve products, then we have no general description of all natural operators $T^* \rightsquigarrow T^*F$. The simplest example of a non product preserving bundle is the classical cotangent bundle T^*. In [2] we have studied liftings of various kinds of tensor fields to the cotangent bundle and we have classified here all natural operators $T^* \rightsquigarrow T^*T^*$ transforming 1–forms to the cotangent bundle. In particular, we have proved that the pull-back and the classical Liouville 1–form are the only 1–forms on T^*M which can be geometrically constructed from a 1–form on M. Further, Mikulski has in [7] studied linear natural operators transforming 1–forms to the higher order tangent bundle $T^{(r)}$ which is defined by $T^{(r)}M = (T^{r*}M)^*$, $T^{r*}M = J^r(M,\mathbb{R})_0$. This paper is devoted to liftings of 1–forms to the higher order cotangent bundle T^{r*} and also to the bundles T^*T, TT^* and T^*T^*, where T is the tangent bundle. Except classification

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theorems we will also study some related geometrical questions. Using such a point of view, this paper is a continuation of [2].

All manifolds and maps are assumed to be infinitely differentiable.

2. Liftings of 1–forms to higher order cotangent bundles

The r-th order cotangent bundle is defined as the space $T^{r*}M = J^r(M, \mathbb{R})_0$ of all r-jets of smooth functions $\varphi : M \to \mathbb{R}$ with the target $0 \in \mathbb{R}$. Every local diffeomorphism $f : M \to N$ is then extended into a vector bundle morphism $T^{r*}f : T^{r*}M \to T^{r*}N$ defined by $j^r_x\varphi \mapsto j^r_{f(x)}(\varphi \circ f^{-1})$, where f^{-1} is constructed locally. Obviously, T^{r*} does not preserve products and for $r = 1$ we obtain the classical cotangent bundle T^*. The aim of this section is to study how an arbitrary 1–form on M can induce a 1–form on $T^{r*}M$, i.e. to study natural operators $T^* \rightsquigarrow T^*T^{r*}$. Denote by $(x^i, u_i, \dots, u_{i_1\dots i_r})$ the canonical coordinates on $T^{r*}M$ and by G^r_m the group of all invertible r-jets from $\mathbb{R}^m$ into $\mathbb{R}^m$ with the source and the target zero. Then the coordinates on G^r_m will be denoted by $(a^i_j, a^i_{jk}, \dots, a^i_{j_1\dots j_r})$, while the coordinates of an inverse element will be denoted by a tilde.

Let $\omega = \omega_i dx^i$ be an arbitrary 1–form on M and denote by $\pi^*\omega$ its pull-back to $T^{r*}M$ with respect to the vector bundle projection $\pi : T^{r*}M \to M$. Moreover, we have a canonical projection $\pi_r : T^{r*}M \to T^*M$, so that the classical Liouville 1–form $\lambda_M = u_i dx^i$ on T^*M induces the 1–form $\pi_r^*\lambda_M$ on $T^{r*}M$. Now we prove

Proposition 1. *All natural operators $T^* \rightsquigarrow T^*T^{r*}$ transforming 1–forms on M into 1–forms on $T^{r*}M$ are of the form*

$$\omega \mapsto c_1\pi^*\omega + c_2\pi_r^*\lambda_M \tag{1}$$

with any $c_1, c_2 \in \mathbb{R}$.

Proof. The proof is based on the canonical equivalence between natural operators in question and equivariant maps between corresponding standard fibres, [4]. Using such a point of view, r-th order natural operators $T^* \rightsquigarrow T^*T^{r*}$ are in a bijection with the G^{r+1}_m–equivariant maps

$$(J^rT^*)_0\mathbb{R}^m \oplus (T^{r*})_0\mathbb{R}^m \to (T^*T^{r*})_0\mathbb{R}^m. \tag{2}$$

The canonical coordinates on the standard fibre $(J^rT^*)_0\mathbb{R}^m$ will be denoted by $(\omega_i, \omega_{i,j}, \dots, \omega_{i,j_1\dots j_r})$ and the coordinates on $(T^{r*})_0\mathbb{R}^m$ are $(u_i, u_{ij}, \dots, u_{i_1\dots i_r})$. Moreover, the coordinate expression

$$\Omega = \alpha_i dx^i + \beta^i du_i + \beta^{ij} du_{ij} + \dots + \beta^{i_1\dots i_r} du_{i_1\dots i_r}$$

of a 1–form on $T^{r*}M$ defines the coordinates $(\alpha_i, \beta^i, \beta^{ij}, \dots, \beta^{i_1\dots i_r})$ on the standard fibre $(T^*T^{r*})_0\mathbb{R}^m$. In this way the coordinate form of (2) is

$$\begin{aligned}
\alpha_i &= \alpha_i(\omega_i, \omega_{ij}, \dots, \omega_{i,j_1\dots j_r}, u_i, u_{ij}, \dots, u_{i_1\dots i_r}),\\
\beta^i &= \beta^i(\omega_i, \omega_{ij}, \dots, \omega_{i,j_1\dots j_r}, u_i, u_{ij}, \dots, u_{i_1\dots i_r}),\\
&\dots\\
\beta^{i_1\dots i_r} &= \beta^{i_1\dots i_r}(\omega_i, \omega_{ij}, \dots, \omega_{i,j_1\dots j_r}, u_i, u_{ij}, \dots, u_{i_1\dots i_r}).
\end{aligned}$$

By the homotheties $\widetilde{a}^i_j = k\delta^i_j$ we have

$$k\alpha_i = \alpha_i(k\omega_i, k^2\omega_{ij}, \dots, k^{r+1}\omega_{i,j_1\dots j_r}, ku_i, k^2u_{ij}, \dots, k^r u_{i_1\dots i_r}),$$
$$\frac{1}{k}\beta^i = \beta^i(k\omega_i, k^2\omega_{ij}, \dots, k^{r+1}\omega_{i,j_1\dots j_r}, ku_i, k^2u_{ij}, \dots, k^r u_{i_1\dots i_r}),$$
$$\dots$$
$$\frac{1}{k^r}\beta^{i_1\dots i_r} = \beta^{i_1\dots i_r}(k\omega_i, k^2\omega_{ij}, \dots, k^{r+1}\omega_{i,j_1\dots j_r}, ku_i, k^2u_{ij}, \dots, k^r u_{i_1\dots i_r}).$$

Multiplying both sides of $\beta^{i_1\dots i_s}$ by k^s and then setting $k \to 0$ we obtain $\beta^{i_1\dots i_s} = 0$ for all $s = 1, \dots, r$. Moreover, by the theorem on homogeneous functions from [4] we have that α_i are linear in ω_i and in u_i and independent of all remaining coordinates. Up till now, we have deduced

$$\alpha_i = c_1\omega_i + c_2u_i, \quad \beta^i = 0, \dots, \beta^{i_1\dots i_r} = 0$$

which is the coordinate form of (1). Next, one evaluates easily the following transformation laws: $\overline{u}_i = \widetilde{a}^j_i u_j$ and $\overline{\omega}_i = \widetilde{a}^j_i \omega_j$. Moreover, since all $\beta's = 0$, then the transformation law of α_i can be expressed in the simple form $\overline{\alpha}_i = \widetilde{a}^j_i \alpha_j$. Then the full equivariance of α_i reads that c_1 and c_2 are arbitrary real numbers. We have also proved that all r-th order natural operators are reduced to the zero order ones. Finally, by the consequences of the Peetre theorem, [4], every natural operator in question has a finite order. □

Remark 1. We remark that W. Mikulski has in [6] classified all natural operators transforming vector fields to the r-th order cotangent bundle. Moreover, we have in [3] also determined all natural operators transforming $(0,2)$–tensor fields to this bundle and we have discussed here some related geometrical questions.

3. Liftings of 1–forms to the cotangent bundle of a tangent bundle

First we will be concerned with the question how a 1–form on a smooth manifold M can induce a 1–form on T^*TM. Denote by $(x^i, y^i = dx^i, p_idx^i + q_idy^i)$ the canonical coordinates on T^*TM. If $q_M : T^*M \to M$ is the bundle projection, then $q_{TM} : T^*TM \to TM$. Moreover, denoting by $p_M : TM \to M$ the tangent bundle projection and $s_M : TT^*M \to T^*TM$ the canonical isomorphism of Tulczyjev and Modugno and Stefani, [4], then the composition $v_{T^*M} := p_{T^*M} \circ s_M^{-1} : T^*TM \to T^*M$ is given by $(x^i, y^i, p_i, q_i) \mapsto (x^i, q_i)$. If $A \in T^*TM$, then the contraction $I_1 := \langle q_{TM}(A), v_{T^*M}(A)\rangle$ defines an invariant function on T^*TM, in coordinates

(3) $$I_1 = q_iy^i.$$

Quite analogously, the contraction with a 1–form $\omega = \omega_idx^i$ defines another function

(4) $$I_2 = \omega_iy^i.$$

Then the exterior differentials

(5) $$\Omega_1 := dI_1 = q_idy^i + y^idq_i$$

and

$$\Omega_2 := dI_2 = \omega_i dy^i + \omega_{i,j} y^i dx^j \tag{6}$$

are 1–forms on T^*TM. The well-known canonical involution $\kappa_M : TTM \to TTM$ of the iterated tangent bundle defines an analogous mapping

$$\iota_{TT^*TM} := Ts_M \circ \kappa_{T^*M} \circ T(s_M^{-1}) : TT^*TM \to TT^*TM,$$

$\iota_{TT^*TM}(x^i, y^i, p_i, q_i, dx^i, dy^i, dp_i, dq_i) = (x^i, dx^i, p_i, q_i, y^i, dy^i, dp_i, dq_i)$. Considering a 1–form on T^*TM as a linear mapping $TT^*TM \to \mathbb{R}$, we can define another 1–form on T^*TM by

$$\Omega_3 := \Omega_2 \circ \iota_{TT^*TM} = \omega_i dy^i + \omega_{i,j} y^j dx^i. \tag{7}$$

Moreover, let $\lambda_M = q_i dx^i$ be the canonical Liouville 1–form on T^*M and let $\lambda_{TM} = p_i dx^i + q_i dy^i$ be Liouville 1–form on T^*TM. Then we define

$$\Omega_4 := \lambda_{TM} - dI_1 = p_i dx^i - y^i dq_i. \tag{8}$$

Finally, we will denote by $\Omega_5 = \omega_i dx^i$ and $\Omega_6 = q_i dx^i$ the 1–forms on T^*TM which are defined by the pull-back of ω and λ_M, respectively. Now we prove

Proposition 2. *Let $dim M \geq 2$. Then all natural operators $T^* \rightsquigarrow T^*(T^*T)$ transforming 1–forms on M into 1–forms on T^*TM are of the form*

$$\omega \mapsto c_1(I_1, I_2)\Omega_1 + \cdots + c_6(I_1, I_2)\Omega_6 \tag{9}$$

where $c_1, \ldots, c_6 : \mathbb{R}^2 \to \mathbb{R}$ are arbitrary smooth functions.

Proof. Consider first the first order natural operators. Then it suffices to find all G_m^3–equivariant smooth maps

$$(J^1T^*)_0\mathbb{R}^m \oplus (T^*T)_0\mathbb{R}^m \to (T^*T^*T)_0\mathbb{R}^m.$$

The action of G_m^3 on $(J^1T^*)_0\mathbb{R}^m$ is

$$\begin{aligned} \overline{\omega}_i &= \widetilde{a}_i^j \omega_j, \\ \overline{\omega}_{i,j} &= \widetilde{a}_i^k \widetilde{a}_j^\ell \omega_{k,\ell} + \widetilde{a}_{ij}^k \omega_k \end{aligned} \tag{10}$$

and the action of the same group on $(T^*T)_0\mathbb{R}^m$ is

$$\begin{aligned} \overline{y}^i &= a_j^i y^j, \\ \overline{q}_i &= \widetilde{a}_i^j q_j, \\ \overline{p}_i &= \widetilde{a}_i^j p_j + \widetilde{a}_{\ell i}^j a_k^\ell q_j y^k. \end{aligned} \tag{11}$$

Next, the coordinate expression of a 1–form $\Omega = \alpha_i dx^i + \beta_i dy^i + \gamma^i dp_i + \delta^i dq_i$ on T^*TM defines the coordinates $(\alpha_i, \beta_i, \gamma^i, \delta^i)$ on the standard fibre $(T^*T^*T)_0\mathbb{R}^m$ with the following action of the group G^3_m

$$
\begin{aligned}
\overline{\alpha}_i &= \widetilde{a}^j_i \alpha_j + \widetilde{a}^k_{ji}\overline{y}^j \beta_k + a^j_{km}\widetilde{a}^m_i \gamma^k \overline{p}_j + a^j_{k\ell m}\widetilde{a}^m_i \widetilde{a}^\ell_n \gamma^k q_j \overline{y}^n + \\
&\quad + a^j_{k\ell}\widetilde{a}^\ell_{ni}\overline{y}^n \gamma^k \overline{q}_j + a^j_{km}\widetilde{a}^m_i \delta^k \overline{q}_j, \\
\overline{\beta}_i &= \widetilde{a}^j_i \beta_j + a^j_{k\ell}\widetilde{a}^\ell_i \gamma^k \overline{q}_j, \\
\overline{\gamma}^i &= a^i_j \gamma^j, \\
\overline{\delta}^i &= a^i_j \delta^j + a^i_{k\ell}\widetilde{a}^\ell_m \gamma^k \overline{y}^m.
\end{aligned}
\tag{12}
$$

Since $\omega_{i,j}$ are neither symmetric nor antisymmetric in i and j, it will be useful to introduce a new couple of coordinates by $S_{ij} = \frac{1}{2}(\omega_{i,j} + \omega_{j,i})$ and $R_{ij} = \frac{1}{2}(\omega_{i,j} - \omega_{j,i})$. Then $\omega_{i,j} = S_{ij} + R_{ij}$ and S_{ij} are symmetric and R_{ij} are antisymmetric in i and j. Using (10) we directly compute the transformation law of S_{ij} and R_{ij} on the kernel of the jet projection $G^2_m \to G^1_m$

$$
\begin{aligned}
\overline{S}_{ij} &= S_{ij} + \widetilde{a}^k_{ij}\omega_k, \\
\overline{R}_{ij} &= R_{ij}.
\end{aligned}
\tag{13}
$$

In what follows we will use the following auxiliary assertions.

Lemma 1. *Let $f : (J^1T^*)_0\mathbb{R}^m \oplus (T^*T)_0\mathbb{R}^m \to \mathbb{R}^m$ be an G^2_m–equivariant smooth mapping and let $dimM \geq 2$. Then it holds*

$$f^i = \varphi(q_i y^i, \omega_i y^i) y^i$$

where $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is an arbitrary smooth function.

Proof of Lemma 1. We have to determine all G^2_m–equivariant maps of the form $f^i = f^i(y^i, p_i, q_i, \omega_i, \omega_{i,j})$. Replace $\omega_{i,j}$ with S_{ij} and R_{ij}, so that $f^i = f^i(y^i, p_i, q_i, \omega_i, S_{ij}, R_{ij})$. By the tensor evaluation theorem from [4] and using the fact that $R_{ij}y^i y^j = 0$ we obtain $f^i = \varphi(p_i y^i, q_i y^i, \omega_i y^i, S_{ij} y^i y^j) y^i$. Then the equivariance on the kernel $G^2_m \to G^1_m$ yields

$$\varphi(p_i y^i, q_i y^i, \omega_i y^i, S_{ij}y^i y^j) = \varphi((p_i + \widetilde{a}^k_{i\ell} q_k y^\ell) y^i, q_i y^i, \omega_i y^i, (S_{ij} + \widetilde{a}^k_{ij}\omega_k) y^i y^j).$$

Put $\omega = (1, 0, \ldots, 0)$ and $q = (0, 1, 0, \ldots, 0)$. Then

$$\varphi(p_i y^i, y^2, y^1, S_{ij} y^i y^j) = \varphi((p_i + \widetilde{a}^2_{i\ell} y^\ell) y^i, y^2, y^1, (S_{ij} + \widetilde{a}^1_{ij}) y^i y^j).$$

If $\widetilde{a}^2_{ij} \neq 0$ and $\widetilde{a}^1_{ij} = 0$, then we see that φ is independent of $p_i y^i$. Analogously, $\widetilde{a}^1_{ij} \neq 0$ proves the independence of φ on $S_{ij} y^i y^j$ and the proof of Lemma 1 is finished.

Quite analogously we can prove

Lemma 2. *Let $f : (J^1T^*)_0\mathbb{R}^m \oplus (T^*T)_0\mathbb{R}^m \to \mathbb{R}^{m*}$ be an G^2_m–equivariant smooth mapping and let $dimM \geq 2$. Then it holds*

$$f_i = \varphi_1(q_iy^i, \omega_iy^i)q_i + \varphi_2(q_iy^i, \omega_iy^i)\omega_i + \varphi_3(q_iy^i, \omega_iy^i)R_{ij}y^j$$

where $\varphi_1, \varphi_2, \varphi_3 : \mathbb{R}^3 \to \mathbb{R}$ are arbitrary smooth functions.

Now we come back to the proof of Proposition 2. By Lemma 1, $\gamma^i = \gamma(I_1, I_2)y^i$. Similarly to the proof of Lemma 1 we deduce that $\delta^i = \delta(I_1, I_2, p_iy^i, S_{ij}y^iy^j)y^i$. By equivariances on the kernel of the jet projection $G^2_m \to G^1_m$ we have $\delta^i = \delta(I_1, I_2)y^i$ and also $\gamma(I_1, I_2) = 0$. Consider now $\beta_i = \beta_i(y^i, p_i, q_i, \omega_i, \omega_{i,j})$. Since $\gamma^i = 0$, then the transformation law of β_i is of the tensorial character $\overline{\beta}_i = \widetilde{a}^j_i\beta_j$. By Lemma 2, $\beta_i = \beta_1(I_1, I_2)q_i + \beta_2(I_1, I_2)\omega_i + \beta_3(I_1, I_2)R_{ij}y^j$. Further, assume α_i in the form $\alpha_i = k_1p_i + k_2S_{ij}y^j + \widetilde{\alpha}_i(y^i, p_i, q_i, \omega_i, \omega_{i,j})$ with undetermined $k_1, k_2 \in \mathbb{R}$. Using equivariance we prove that $\beta_3 = 0$, $\beta_2 = k_2$ and $\delta = k_1 - \beta_1$. Then the full equivariance together with Lemma 2 reads $\widetilde{\alpha}_i = k_3q_i + k_4\omega_i + k_5R_{ij}y^j$. Up till now we have deduced $\gamma^i = 0$, $\delta^i = (\beta_1 - k_1)y^i$, $\beta_i = \beta_1q_i + k_2\omega_i$ and $\alpha_i = k_1p_i + k_2S_{ij}y^j + k_3q_i + k_4\omega_i + k_5R_{ij}y^j$ which can be rewritten in the form

$$\begin{aligned} &\gamma^i = 0, \\ &\delta^i = \beta_1y^i - k_1y^i, \\ &\beta_i = \beta_1q_i + k_2\omega_i + k_3\omega_i, \\ &\alpha_i = k_1p_i + k_2\omega_{i,j}y^j + k_3\omega_{j,i}y^j + k_4q_i + k_5\omega_i. \end{aligned} \tag{14}$$

This is nothing else but the coordinate form of (9).

By [4], every natural operator in question has a finite order. Now we show that the second order natural operators are reduced to the first order ones (the proof for natural operators of the order $r > 2$ is quite similar). The second order natural operators lead to the G^3_m–equivariant maps $(J^2T^*)_0\mathbb{R}^m \oplus (T^*T)_0\mathbb{R}^m \to (T^*T^*T)_0\mathbb{R}^m$. It suffices to prove that all such maps are independent of $\omega_{i,jk}$. The transformation law of $\omega_{i,jk}$ is

$$\overline{\omega}_{i,jk} = \widetilde{a}^\ell_i\widetilde{a}^m_j\widetilde{a}^n_k\omega_{\ell,mn} + \widetilde{a}^\ell_{ik}\widetilde{a}^m_j\omega_{\ell,m} + \widetilde{a}^\ell_i\widetilde{a}^m_{jk}\omega_{\ell,m} + \widetilde{a}^\ell_{ij}\widetilde{a}^m_k\omega_{\ell,m} + \widetilde{a}^\ell_{ijk}\omega_\ell.$$

Put $S_{ijk} = \frac{1}{3}(\omega_{i,jk} + \omega_{j,ik} + \omega_{k,ij})$, $R_{ijk} = \frac{1}{3}(\omega_{i,jk} - \omega_{j,ik})$ and $T_{ijk} = \frac{1}{3}(\omega_{i,jk} - \omega_{k,ij})$. Then $\omega_{i,jk} = S_{ijk} + R_{ijk} + T_{ijk}$, S_{ijk} are symmetric in all indices and on the kernel of the jet projection $G^3_m \to G^1_m$ we have $\overline{S}_{ijk} = S_{ijk} + \widetilde{a}^\ell_{ijk}\omega_\ell$, $\overline{R}_{ijk} = R_{ijk}$ and $\overline{T}_{ijk} = T_{ijk}$. In the case of the second order natural operators the map f from Lemma 1 should be replaced with $f : (J^2T^*)_0\mathbb{R}^m \oplus (T^*T)_0\mathbb{R}^m \to \mathbb{R}^m$, so that we have additional coordinates $\omega_{i,jk}$. Now we show that the function φ from Lemma 1 does not depend on $\omega_{i,jk}$. First, replace $\omega_{i,jk}$ with S_{ijk}, R_{ijk} and T_{ijk}. Since S_{ijk} are symmetric in all indices, the equivariance on the kernel of the jet projection $G^3_m \to G^1_m$ yields that f^i are independent of S_{ijk}. Moreover, since $R_{ijk}y^iy^jy^k = 0$ and $T_{ijk}y^iy^jy^k = 0$, the tensor evaluation theorem leads to the same form of f^i as in the case of the first order operators. The proof that the map f from Lemma 2 is independent of $\omega_{i,jk}$ is quite similar. This completes the proof of Proposition 2. □

Corollary 1. *All natural operators $T^* \rightsquigarrow T^*(T^*T)$ are linear.*

In what follows we will use the concept of a natural 1–form in the sense of the following definition.

Definition 1. A natural 1–form on T^*T is a system of 1–forms $\Omega_M : T^*TM \to T^*T^*TM$ for every m–manifold M satisfying $T^*T^*Tf \circ \Omega_M = \Omega_N \circ T^*Tf$ for all local diffeomorphisms $f : M \to N$.

Definition 2. A natural operator $A : T^* \rightsquigarrow T^*F$ is called absolute, if $A_M\omega_M = A_M O_M$ for every $\omega_M : M \to T^*M$, where O_M means the zero section.

In this way natural 1–forms on T^*TM are exactly the values of absolute natural operators $T^* \rightsquigarrow T^*(T^*T)$.

Corollary 2. *All natural 1–forms on T^*TM are of the form*

$$c_1(I_1)\Omega_1 + c_2(I_1)\Omega_4 + c_3(I_1)\Omega_6 \qquad (15)$$

where the 1–forms Ω_1, Ω_4 and Ω_6 were defined above and $c_1, c_2, c_3 : \mathbb{R} \to \mathbb{R}$ are arbitrary smooth functions of the invariant (3).

Remark 2. It is well-known that if F is a Weil bundle, then all natural operators $T \rightsquigarrow TF$ transforming vector fields on M into vector fields on FM can be constructed from the complete lift $\mathcal{F}$ by applying all natural transformations $TF \to TF$ over the identity of F, [4]. The complete lift $\mathcal{F}$ of a vector field is defined as its flow prolongation. We have proved in [1] that the same holds also for the bundle $F = T^*T$, which does not preserve products. The 1–form $\Omega_3 = \omega_i dy^i + \omega_{i,j}y^j dx^i$can be also considered as a 1–form on TM and is sometimes called the complete lift of $\omega = \omega_i dx^i$ to TM. Using pull-back with respect to the projection $T^*TM \to TM$, we can consider Ω_3 as the complete lift of ω to T^*TM. In this situation we can pose a question what would we obtain after applying all natural transformations $T^*(T^*T) \to T^*(T^*T)$ over the identity of T^*T to the complete lift of ω to T^*T. Using all natural transformations $TTT^* \to TT^*T$ from [1] we easily determine the following equations of all natural transformations $T^*T^*T \to T^*T^*T$ over the identity of T^*T: $\alpha_i = D\alpha_i + Gp_i + Hp_i + K\beta_i + Lq_i$, $\beta_i = D\beta_i + Gq_i$, $\gamma^i = D\gamma^i$, $\delta^i = D\delta^i - Hy^i + K\gamma^i$. Applying this to the complete lift Ω_3, we obtain all 1–forms from the list (9), except $\Omega_2 = \omega_i dy^i + \omega_{i,j}y^i dx^j$. Notice that the 1–form Ω_2 is the only nonabsolute closed 1–form on T^*TM.

Remark 3. The problem of finding all natural operators transforming 1–forms to the bundles TT^* and T^*T^* can be reduced to Proposition 2. This is a simple consequence of well-known natural equivalences $s : TT^* \to T^*T$ and $t : TT^* \to T^*T^*$, [4]. Denoting by $(x^i, u_i, s^i = dx^i, t_i = du_i)$ the coordinates on TT^*M and by $(x^i, v_i, a_i dx^i + b^i dv_i)$ the coordinates on T^*T^*M, the equations of s are $(y^i = s^i, p_i = t_i, q_i = u_i)$ and the equations of t are $(v_i = u_i, a_i = t_i, b^i = -s^i)$.

Remark 4. Let F be a natural bundle and let $\{A_1, \dots, A_p\}$ be a basis of the vector space of all linear natural operators $T^* \rightsquigarrow T^*F$. Using the pull-back with respect to $T^*FM \to FM$, we can consider $\{A_1, \dots, A_p\}$ as a set of linear natural operators $T^* \rightsquigarrow T^*(T^*F)$ transforming 1–forms on M into 1–forms on T^*FM. Further, let $\{B_1, \dots, B_q\}$ be a basis of the vector space of all absolute natural operators $T^* \rightsquigarrow T^*(T^*F)$. Then $\mathcal{S} := \{A_1, \dots, A_p, B_1, \dots, B_q\}$ is certain set of linear natural operators transforming 1-forms to the bundle T^*F. In particular, if $F = T$, then all natural operators $T^* \rightsquigarrow T^*(T^*T)$ are linear and the basis of all such linear operators is exactly the set $\mathcal{S}$ constructed above. Obviously, the 1–forms Ω_2, Ω_3 and Ω_5 can be easily defined also on the tangent bundle (cf. [5] and [8]), so that $A_1 = \Omega_2$, $A_2 = \Omega_3$ and $A_3 = \Omega_5$. By Corollary 2, $B_1 = \Omega_1$, $B_2 = \Omega_4$ and $B_3 = \Omega_6$. In this notation we do not distinguish between the operator and its value. It is our belief that such a construction of linear natural operators $T^* \rightsquigarrow T^*(T^*F)$ has a general character and can be used for other natural bundles F. But this does not work for all natural bundles, the cotangent bundle $F = T^*$ being the simplest example. In this case all linear natural operators $A_i : T^* \rightsquigarrow T^*T^*$ are zero order only, so that the set $\mathcal{S}$ does not describe all natural operators of this type.

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