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Singleton Representations of $U_q(so(3,2))$

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The singleton representations of $U_q(so(3,2))$ (including roots of unity cases) were constructed by us in [1]. Here we give more detailed and explicit results about these representations when q is not a root of unity. In particular, we introduce orthonormal bases in the representation spaces, and we give explicit formulae for the action of the generators of $U_q(so(3,2))$ on these bases.

1. In a previous work [1] we have constructed positive energy representations of the q-deformed anti de Sitter algebra $U_q(so(3,2))$ [2] with $|q| = 1$, from a modern approach [3]. This approach is based on a Verma module construction, and it involves elimination of singular (or null) states associated with the Verma modules, in order to obtain the irreducible and, in some cases, infinitesimally unitarizable subquotient representations [3]. In paper [1] we gave a detailed description of this construction for the singleton representations at both roots of unity and the generic cases (q not a root of unity). Here we obtain additional and more explicit results for the singleton representations when q is not a root of unity.

Our definitions and notation for q numbers are the following: $[m]_q = \frac{q^{m/2}-q^{-m/2}}{q^{1/2}-q^{-1/2}}$. The q-factorial is $[m]_q! = [m]_q\,[m-1]_q \ldots [1]$ and $[0]_q! = [1]_q! = 1$. For the definition of the q-gamma function $\Gamma_q(x)$ see [4]. We use the following facts: $\Gamma_q(n) = [n-1]_q!$ and $\Gamma_q(x+1) = [x]_q\,\Gamma_q(x)$. The q-Pochammer symbol $(a)^q_n = \frac{\Gamma_q(a+n)}{\Gamma_q(a)}$ is also used below.

We recall the q-deformation $U_q(so(5,\mathbb{C}))$ is defined as the associative algebra over $\mathbb{C}$ with Chevalley generators H_j, $X_j^{\pm}$ $(j=1,2)$ and relations [1], [2]:

$$[H_j, H_k] = 0, [H_j, X_k^{\pm}] = \pm a_{jk} X_k^{\pm} \quad (j,k=1,2), \tag{1a}$$

$$[X_j^+, X_k^-] = \delta_{jk}[H_j]_{q_j} \quad (q_1 = q,\ q_2 = q^2) \quad (j,k=1,2), \tag{1b}$$

$$(X_1^{\pm})^3 X_2^{\pm} - [3]_q (X_1^{\pm})^2 X_2^{\pm} X_1^{\pm} + [3]_q X_1^{\pm} X_2^{\pm} (X_1^{\pm})^2 - X_2^{\pm} (X_1^{\pm})^3 = 0, \tag{1c}$$

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$$(X_2^\pm)^2 X_1^\pm \; - \; [2]_{q^2} X_2^\pm X_1^\pm X_2^\pm \; + X_1^\pm (X_2^\pm)^2 \; = \; 0, \qquad (1d)$$

where $(a_{jk}) = 2(\alpha_j, \alpha_k)/(\alpha_j, \alpha_j)$ $(j, k = 1, 2)$ is the Cartan matrix of $so(5, \mathbb{C})$. α_1 and α_2 are the simple roots with products: $(\alpha_1, \alpha_1) = 2$, $(\alpha_2, \alpha_2) = 4$, $(\alpha_1, \alpha_2) = -2$. The non-simple positive roots are $\alpha_3 = \alpha_1 + \alpha_2$ and $\alpha_4 = 2\alpha_1 + \alpha_2$. The Cartan-Weyl generators corresponding to these non-simple roots are [1]:

$$X_3^\pm \; = \; \pm q^{\mp\frac{1}{2}} (q^{\frac{1}{2}} X_1^\pm X_2^\pm \; - \; q^{-\frac{1}{2}} X_2^\pm X_1^\pm) \; , \; [2]_q X_4^\pm \; = \; \pm [X_1^\pm, X_3^\pm] \, . \qquad (2)$$

We define $H_3 = H_1 + 2H_2$, $H_4 = H_1 + H_2$. All commutation relations for $U_q(so(5, \mathbb{C}))$ now follow from the above definitions and relations; in particular,

$$[X_3^+, \; X_3^-] \; = \; [H_3]_q, \; [X_4^+, \; X_4^-] \; = \; [H_4]_{q^2} \, . \qquad (3)$$

We define the real form $U_q(so(3, 2))$ of $U_q(so(5, \mathbb{C}))$ as in the $q = 1$ case [5]. The generators of $U_q(so(3, 2))$ are given by the following expressions:

$$M_{21} = H_1/2, \; M_{31} = \frac{1}{2}(X_1^+ + X_1^-), \; M_{32} = \frac{i}{2}(X_1^+ - X_1^-) \, , \qquad (4a)$$

$$M_{04} = \frac{1}{2}(H_1 + 2H_2), \; M_{30} = -\frac{1}{2}(X_3^+ - X_3^-), \; M_{34} = -\frac{i}{2}(X_3^+ + X_3^-) \, , \qquad (4b)$$

$$M_{10} = \frac{i}{2}(X_2^+ + X_2^- + X_4^+ + X_4^-), \; M_{20} = \frac{1}{2}(X_2^+ - X_2^- - X_4^+ + X_4^-), \qquad (4c)$$

$$M_{41} = \frac{1}{2}(X_4^+ - X_4^- + X_2^+ - X_2^-), \; M_{42} = \frac{i}{2}(X_4^+ + X_4^- - X_2^+ - X_2^-). \qquad (4d)$$

For $|q| = 1$ the generators in (4) are preserved by the following antilinear anti-involution ω of $U_q(so(5, \mathbb{C}))$:

$$\omega(H_j) = H_j \;\; (j = 1, 2), \;\; \omega(X_1^+) = X_1^-, \;\; \omega(X_k^+) = \; - \; X_k^- \;\; (k = 2, \; 3, \; 4). \qquad (5)$$

The center of $U_q(so(3, 2))$ is, in the classical case ($q = 1$), generated by two Casimir operators, and we have explicitly [6]:

$$\begin{aligned} D_2 \; &= \; M_{41}^2 \; + \; M_{42}^2 \; + \; M_{43}^2 \; + \; M_{01}^2 \; + \; M_{02}^2 \; + \; M_{03}^2 \\ &\qquad - \; (M_{40}^2 \; + \; M_{12}^2 \; + \; M_{23}^2 \; + \; M_{31}^2) \\ &= \; - \; M_{40}^2 \; + \; M_{41}^2 \; + \; M_{42}^2 \; + \; M_{43}^2 \; + \; Q_2, \end{aligned} \qquad (6a)$$

$$D_4 \; = \; Q_4 \; - \; \left(\sum_{i\,j\,k\,=\,1}^{3} \; \frac{1}{2} \epsilon_{i\,j\,k} \, M_{4i} \, M_{jk} \right)^2 \; +$$

$$+ \sum_{\substack{i\, j\, k \\ \ell\, m} = 1}^{3} (\epsilon_{i\ j\ k}\{\frac{1}{2}M_{40}\, M_{jk} + M_{4k}\, M_{0j}\})(\epsilon_{i\ \ell\ m}\{\frac{1}{2}M_{40}\, M_{\ell m} + M_{4\ell}\, M_{0m}\}), \quad (6b)$$

where

$$Q_2 \ = \ M_{01}^2 \ + \ M_{02}^2 \ + \ M_{03}^2 \ - \ M^2 \qquad (7a)$$

and

$$Q_4 \ = \ (M_{12}\, M_{30} \ + \ M_{23}\, M_{10} \ + \ M_{31}\, M_{20})^2 . \qquad (7b)$$

$M^2 \ = \ M_{12}^2 \ + \ M_{23}^2 \ + \ M_{31}^2$ is the square of the angular momentum, and D_2 and D_4 are the 2nd order and fourth order Casimir operators of $SO(3,2)$, respectively. (ϵ_{ijk} is the totally antisymmetric symbol.) The q analog of M^2 is well-known[4]: $M_q^2 \ = \ [\frac{1}{2}(H_1 - 1)]_q{}^2 \ + \ X_1^+ X_1^- - \frac{1}{4}$; however, as far as we know, there are no explicit results on q analogs of D_2 and D_4. An expression, which is the analog of D_2 for $U_q(so(5,\mathbb{C}))$ is given in [7], but it does not seem to be of much use for $U_q(so(3,2))$.

2. The singleton representations of $U_q(so(3,2))$, which we have described in [1] are lowest weight representations. The name "singleton" comes from the fact that the reduction of these representations with respect to a Cartan subalgebra (the Cartan subalgebra having basis H_1 and H_2) is multiplicity free. For $q = 1$ there are two such representations called Di and Rac. These representations are unitary, irreducible representations of the univeral covering group $SO_0(3,2)'$ of $SO_0(3,2)$, and both of these representations are characterized by the values $+5/4$ and 0 for D_2 and D_4, respectively. They can be traced back to Majorana [8], and they have also been studied by Dirac [9], Evans[10], Flato and Fronsdal [11] and others. Some authors use singleton representations of $SO_0(3,2)'$ to mean representations of $SO_0(3,2)'$, which have a multiplicity free decomposition with repect to the maximal essentially compact subgroup, $SO(2)'_{M_{40}} \ \times \ SU(2)_{M_{ij}}$ [12], [13]. Many such representations, although unitary, do not have positive energy, and little is known about their q deformations.

Denoting the lowest weight vector by $|\Lambda > = \ |E_0\, m_0 >$, we define the singleton representations as follows:

$$M_{04}|\Lambda> = \ E_0|\Lambda> \ , \ M_{21}|\Lambda> = \ m_0|\Lambda>, \qquad (8a)$$

$$X_k^-|\Lambda> = \ 0 \ \ (k = 1,\ 2,\ 3,\ 4) \ . \qquad (8b)$$

For the q analog of the Rac: $E_0 \ = \ \frac{1}{2}$, $m_0 \ = \ 0$; and for the q analog of the Di: $E_0 \ = \ 1$, $m_0 \ = \ -\frac{1}{2}$. The q deformed Di and Rac are obtained as subquotients of these lowest weight representations and are characterized by the following null state vanishing conditions (for q not a root of unity) [1]:

$$X_1^+|\frac{1}{2},0> = \ 0, \ \ ((X_3^+)^2 - q^{\frac{1}{2}}[2]_q^2 X_2^+ X_4^+)|\frac{1}{2},0> = \ 0 \qquad \text{(Rac)}; \qquad (9a)$$

$$(X_1^+)^2|1,-\frac{1}{2}> = \ 0, \ \ ((X_3^+ - (1 \ + \ q)X_2^+ X_1^+)|1,-\frac{1}{2}> = \ 0 \qquad \text{(Di)}. \qquad (9b)$$

The representation space for the Rac is given by [1]:

$$H_0 \;=\; \ell.s.\{X_4^{+j}X_3^{+\epsilon}X_2^{+k}|\Lambda> \mid j,k=1,\ 2,\ldots,\epsilon=0,\ 1\}; \qquad (10a)$$

and the representation space for the Di is [1]:

$$H_{\frac{1}{2}} \;=\; \ell.s.\{X_4^{+j}X_2^{+k}X_1^{+\epsilon}|\Lambda> \mid j,k=1,\ 2,\ldots,\epsilon=0,\ 1\}. \qquad (10a)$$

$\ell.s.$ means linear span. We refer to the q deformed Di and Rac simply as Di and Rac. Using ω and the X_α^+ it is possible to construct a contravariant hermitian form (,) on each one of these representation spaces [1]. With respect to this form the lowest weight vector has length one:

$$||\ |\Lambda>||^2 \;:=\; (|\Lambda>,\ |\Lambda>) \;=\; 1\ ,$$

and the M_{ij} are hermitian i.e.

$$(M_{ij}v,w) \;=\; (v,M_{ij}w) \;\; (i,j \;=\; 0,\ 1,\ 2,\ 3,\ 4)$$

for all vectors $v,\ w$ in the given representation space.

In [1] we have determined the "norms" of the basic states given in eqns. (10a) and (10b). They are (with correction of some typographical errors):

$$||X_4^{+j}X_3^{+\epsilon}X_2^{+k}|\Lambda>||^2 \;=\; [2k+1]_q^\epsilon[j]_{q^2}![k]_{q^2}!\prod_{a=1}^{j}[a-\frac{1}{2}+\epsilon]_{q^2}\prod_{b=1}^{k}[b-\frac{1}{2}]_{q^2} \;=$$

$$=\; [2k+1]_q^\epsilon[j]_{q^2}![k]_{q^2}!(\frac{1}{2}+\epsilon)_j^{q^2}(\frac{1}{2})_k^{q^2} \qquad \text{(Rac)}; \qquad (11a)$$

and

$$||X_4^{+j}X_2^{+k}X_1^{+\epsilon}|\Lambda>||^2 \;=\; [j]_{q^2}![k]_{q^2}!\prod_{a=1}^{j}[a-\frac{1}{2}+\epsilon]_{q^2}\prod_{b=1}^{k}[b+\frac{1}{2}-\epsilon]_{q^2} \;=$$

$$=\; [j]_{q^2}![k]_{q^2}!(\frac{1}{2}+\epsilon)_j^{q^2}(\frac{3}{2}-\epsilon)_k^{q^2} \qquad \text{(Di)}. \qquad (11b)$$

Note that, although they are never zero, only for $q=1$ are these "norms" positive. (q is not a root of unity!) Hence only for $q=1$ does (,) define a scalar product on the representation space [5].

Now we define $\ell = k+j+\epsilon$, $m = j-k$ for the Rac, and $\ell = k+j+\frac{1}{2}$, $m = j-k+\epsilon-\frac{1}{2}$ for the Di. Then we see that for the Rac ℓ and m take the values $\ell = 0,\ 1,\ 2\ldots, -\ell \leq m \leq \ell$ (m integer), and $j = \frac{1}{2}(\ell+m-\epsilon)$, $k = \frac{1}{2}(\ell-m-\epsilon)$. For the Di we have that ℓ and m take the values $\ell = \frac{1}{2},\ \frac{3}{2},\ldots, -\ell \leq m \leq \ell$ (m half integer), and

$j = \frac{1}{2}(\ell + m - \epsilon)$, $k = \frac{1}{2}(\ell - m - 1 + \epsilon)$. Normalized basis vectors for the Rac are:

$$|\ell m> = \left\{[\ell - m]_q^\epsilon[\frac{1}{2}(\ell + m - \epsilon)]_{q^2}![\frac{1}{2}(\ell - m - \epsilon)]_{q^2}!\right\}^{-\frac{1}{2}} \times$$

$$\times \left\{(\frac{1}{2} + \epsilon)^{q^2}_{\frac{1}{2}(\ell + m - \epsilon)} (\frac{1}{2})^{q^2}_{\frac{1}{2}(\ell - m - \epsilon)}\right\}^{-\frac{1}{2}} X_4^{+j} X_3^{+\epsilon} X_2^{+k}|\Lambda> . \quad (12a)$$

Normalized basis vectors for the Di are:

$$|\ell m> = \left\{[\ell - m]_q^\epsilon[\frac{1}{2}(\ell + m - \epsilon)]_{q^2}![\frac{1}{2}(\ell - m - \epsilon)]_{q^2}!\right\}^{-\frac{1}{2}} \times$$

$$\times \left\{(\frac{1}{2} + \epsilon)^{q^2}_{\frac{1}{2}(\ell + m - \epsilon)} (\frac{3}{2} - \epsilon)^{q^2}_{\frac{1}{2}(\ell - m - 1 + \epsilon)}\right\}^{-\frac{1}{2}} X_4^{+j} X_2^{+k} X_1^{+\epsilon}|\Lambda> . \quad (12b)$$

Theorem: Let $|q| = 1$ and q is not a root of unity, then the action of the generators H_1, H_2, $X_1^\pm$, $X_2^\pm$ in the Rac representation on the basis $|\ell m>$ are as follows:

$$H_1|\ell m> = 2m|\ell m> \quad , \quad H_2|\ell m> = (\frac{1}{2} + \ell - m)|\ell m> , \quad (13a)$$

$$X_1^\pm|\ell m> = \alpha(\ell, m; q^{\pm 1}) ([\ell \mp m]_q[(\ell \pm m + 1)]_q)^{\frac{1}{2}} \quad |\ell m+1> , \quad (13b)$$

$$X_2^\pm|\ell m> = \beta(\ell, m; q^{\pm 1}) \left([\frac{1}{2}(\ell - m \pm 1]_{q^2}[\frac{1}{2}(\ell - (m \mp 1) + 1)]_{q^2}\right)^{\frac{1}{2}}$$
$$|\ell \pm 1 m \mp 1> , \quad (13c)$$

where $\alpha(\ell, m; q^{+1}) = q^{\frac{1}{2}\{\ell + m - [\ell+m] mod 2\}}$, and $\beta(\ell, m; q^{\pm 1})$ is such that

$$\beta(\ell, m; q^{\pm 1}) \to 1 \quad \text{as} \quad q \to 1 .$$

Sketch of proof: (13a) gives the action of a basis (H_1, H_2) for the Cartan subalgebra of $U_q(so(3,2))$ on the $|\ell m>$ basis, and it follows easily from the definitions (eqns. (1a) and (1b)). In order to obtain the other results we establish by induction arguments the following equations:

$$X_1^+ X_2^{+k} = q^{-k} X_2^{+k} X_1^+ + [k]_{q^2} X_2^{+(k-1)} X_3^+ ,$$

$$X_1^+ X_4^+ = q^j X_4^+ X_1^+ , \quad X_3^+ X_2^+ = q^k X_2^+ X_3^+ .$$

We also use

$$X_2^{+k} X_3^{+2}|\Lambda> = [2]_q^2 q^{-2k-\frac{3}{2}} X_4^+ X_2^{+(k-1)}|\Lambda> ,$$

which follows from (1) and the defining relations for the Rac representation (9a). We did not succeed in obtaining a very succinct expression for $\beta(\ell, m; q^{\pm 1})$, and we do not write it down here, since use of it is not made in the following.

3. We have established by the same methods a similar Theorem for the Di. In fact there are equations for the Di representation identical in form to eqns. (13), except that in this case ℓ and m take half integer values in the ranges stated above for the Di, and $\alpha(\ell, m; q^{\pm 1})$ and $\beta(\ell, m; q^{\pm 1})$ are different.

Using the Theorem and the statements just made in the previous paragraph, we readily obtain the results stated at the beginning of Sect. 2 about the eigenvalues of D_2 and D_4 for the Di and the Rac when $q = 1$. With a little more work we are also able to show that Q_2 and Q_4 are also constant operators in these representations, and that the unique eigenvalues of Q_2 and Q_4 are $\frac{3}{4}$ and 0, respectively, in these representations.

We are applying the analysis of the Di and Rac presented here, to a study of the higher spin massless representations of $U_q(so(3,2))$. We are exploiting a q generalization of the the result of Flato and Fronsdal, which states that the tensor product of two singleton representations decomposes into the sum of infinitly many massless representations of the anti de Sitter group [11].

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Notes

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