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NOTES ON CONFORMAL DIFFERENTIAL GEOMETRY

Michael Eastwood†

These notes are in no way meant to be comprehensive, neither in treatment nor in references to the extensive literature. They are merely meant as an introduction to a small selection of topics in the field. They were presented as a series of four lectures at the 15th Winter School on Geometry and Physics, Srní, Czech Republic, January 1995.

Lecture One

Recall that a *Riemannian manifold* M is really a pair (M, g) consisting of a smooth manifold M and a *metric* g, a smooth and everywhere positive definite section of $\bigodot^2 T^*M$, the symmetric tensor product of the cotangent bundle. Following standard practise, we shall usually write g_{ab} instead of g and, more generally, we shall adorn tensors with upper and lower indices in correspondence with the tangent or cotangent bundle. We shall also use the Einstein summation convention to denote the natural pairing of vectors and covectors. Thus, V^a denotes a tangent vector or a vector field and $g_{ab}V^aV^b$ denotes the square of its length with respect to g. We shall often 'raise and lower indices' without comment—if V^a is a vector field then $V_a \equiv g_{ab}V^b$ is the corresponding 1-form. (More precisely, this is Penrose's *abstract index notation*—see [19] for details.)

A *conformal* manifold M is a pair $(M, [g])$ where $[g]$ is a Riemannian metric defined only up to scale. In other words, $[g]$ is a section of $R(\bigodot^2 T^*M)$, the bundle of rays in $\bigodot^2 T^*M$ such that a representative g is positive definite. In yet other words, a conformal manifold is an equivalence class of Riemannian manifolds where two metrics g_{ab} and $\widehat{g}_{ab}$ are said to be equivalent if $\widehat{g}_{ab}$ is a multiple of g_{ab}. In this case it is convenient to write $\widehat{g}_{ab} = \Omega^2 g_{ab}$ for some smooth function Ω. On a conformal manifold, one can measure angles between vectors but not lengths.

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For example, in two dimensions an oriented conformal manifold is precisely a Riemann surface, i.e. a one-dimensional complex manifold. Certainly, if M is a Riemann surface with local coördinate $z = x+iy$, then dx^2+dy^2 is a Riemannian metric and if $w = u+iv$ is another, then the Cauchy Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

imply

$$\begin{aligned} du^2 + dv^2 &= \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)^2 + \left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)^2 \\ &= \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right](dx^2 + dy^2). \end{aligned}$$

Conversely, there is a theorem of Korn and Lichtenstein which says that any Riemannian metric in two dimensions may be written locally in the form

$$\Omega^2(dx^2 + dy^2)$$

for some smooth function Ω and suitable coördinates (x, y) (chosen compatible with the orientation). Taking $z = x + iy$ defines a local complex coördinate and any other choice is holomorphically related. Indeed, the Cauchy Riemann equations are precisely that the Jacobian matrix

$$\begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}$$

is proportional to an orthogonal one.

Notice that a conformal manifold in two dimensions has no local invariants—all Riemann surfaces are locally indistinguishable. The only local invariant of a two-dimensional Riemannian manifold is its scalar curvature and this has been eliminated by the freedom to scale the metric. In higher dimensions it is reasonable to expect some of the rigidity of Riemannian geometry. This is indeed the case and evidence in its favour can be found as follows.

Firstly, let is consider *rigid motions* of $\mathbb{R}^n$, i.e. the connected component of the group of isometries of $\mathbb{R}^n$ with its usual metric. Of course, this is well-known to be generated by translations and rotations. One way of seeing this is to consider an *infinitesimal motion*, i.e. a vector field V^a on $\mathbb{R}^n$ with the property that

$$V^a{}_b \equiv \frac{\partial V^a}{\partial x^b} \in \mathfrak{so}(n), \text{ i.e. } V_{ab} = V_{[ab]}$$

where square brackets around indices denote taking the skew part. Consider now

$$V^a{}_{bc} \equiv \frac{\partial V^a}{\partial x^b \partial x^c} \in \mathfrak{so}(n)^{(1)}, \text{ i.e. } V_{abc} = V_{[ab]c} \text{ and } V_{abc} = V_{a(bc)}$$

where round brackets denote the symmetric part. However, this *first prolongation* $\mathfrak{so}(n)^{(1)}$ vanishes:

$$V_{abc} = -V_{bac} = -V_{bca} = V_{cba} = V_{cab} = -V_{acb} = -V_{abc}. \tag{1}$$

By integration the result concerning rigid motions follows. There are some observations to be made.

- The calculation (1) is the principal ingredient in constructing the Levi-Civita connection on a Riemannian manifold.
- The flat model of Riemannian geometry is $\mathbb{R}^n$ as a homogeneous space
$$\mathbb{R}^n = \frac{\mathbb{R}^n \rtimes \mathrm{SO}(n)}{\mathrm{SO}(n)} = \frac{\text{rigid motions}}{\text{stabiliser of a point}}.$$
- As an alternative proof we could show that geodesics (i.e. straight lines with unit speed parameterisation) are preserved by isometries and, by firing out geodesics, conclude that an isometry fixing a point to first order is necessarily the identity.

Let us try the same technique for the conformal case in an attempt to identify the conformal motions of $\mathbb{R}^n$. An infinitesimal motion is a vector field satisfying

$$V^a{}_b \equiv \frac{\partial V^a}{\partial x^b} \in \mathfrak{co}(n), \text{ i.e. } V_{(ab)} = \lambda g_{ab}.$$

Thus,

$$V^a{}_{bc} \in \mathfrak{co}(n)^{(1)}, \text{ i.e. } V_{(ab)c} = \lambda_c g_{ab} \text{ and } V_{abc} = V_{a(bc)}.$$

This first prolongation $\mathfrak{co}(n)^{(1)}$ is no longer zero. In fact,

$$\lambda_a \leftrightarrow V_{abc} = \lambda_c g_{ab} + \lambda_b g_{ac} - \lambda_a g_{bc}$$

identifies $\mathfrak{co}(n)^{(1)}$ with $\mathbb{R}^n$. However,

$$V^a{}_{bcd} \in \mathfrak{co}(n)^{(2)}, \text{ i.e. } V_{(ab)cd} = \lambda_{cd} g_{ab} \text{ and } V_{abcd} = V_{a(bcd)}$$

and this second prolongation $\mathfrak{co}(n)^{(2)}$ vanishes if the dimension n is greater than 2:

$$\begin{aligned} n\lambda_{cd} &= g^{ab}V_{abcd} = g^{ab}V_{acbd} = g^{ab}(2V_{(ac)bd} - V_{cabd}) \\ &= g^{ab}(2\lambda_{bd}g_{ac} - V_{cdab}) = 2\lambda_{cd} - g^{ab}V_{cdab}. \end{aligned}$$

So,

$$ng^{cd}\lambda_{cd} = 2g^{cd}\lambda_{cd} - g^{ab}g^{cd}\lambda_{ab}g_{cd} = (2-n)g^{cd}\lambda_{cd}$$

and λ_{cd} is trace-free. Since $n \neq 2$, then

$$(2-n)\lambda_{cd} = g^{ab}V_{cdab}$$

implies that the right hand side is symmetric in cd in which case

$$(2-n)\lambda_{cd} = g^{ab}V_{(cd)ab} = g^{ab}\lambda_{ab}g_{cd}$$

which implies that λ_{cd} is pure trace. Now $V_{(ab)cd} = 0$ and we find that V_{abcd} is zero by applying (1) on the first three indices.

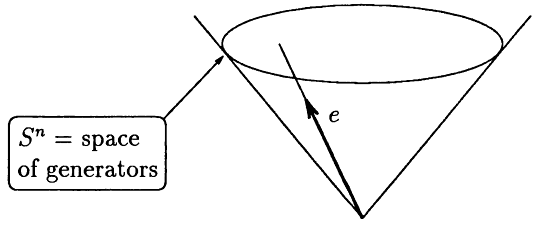
We may conclude that a conformal motion of $\mathbb{R}^n$ for $n \geq 3$ which fixes the origin to second order is necessarily the identity. More precisely, we may identify the group P of conformal motions fixing the identity as matrices of the form

$$\begin{pmatrix} \lambda^{-1} & 0 & 0 \\ r^a & m^a{}_b & 0 \\ -\lambda r^a r_a/2 & -\lambda r_a m^a{}_b & \lambda \end{pmatrix} \quad \text{for } \lambda > 0,\ m^a{}_b \in \mathrm{SO}(n),\ r^a \in \mathbb{R}^n \tag{2}$$

acting on $x^a \in \mathbb{R}^n$ by

$$x^a \longmapsto \frac{4m^a{}_b x^b - 2x^c x_c r^a}{4\lambda - 4\lambda r_a m^a{}_b x^b + \lambda x^c x_c r^b r_b}.$$

It is a good exercise to check that this really is conformal, i.e. that its derivative is everywhere proportional to an orthogonal matrix. Notice that the denominator vanishes when $x^b = 2m_c{}^b r^c / r^a r_a$. On the other hand, this formula is forced by the prolongation argument. In order to allow non-zero r we are therefore obliged to compactify $\mathbb{R}^n$ with a single point to obtain the sphere $S^n = \mathbb{R}^n \cup \{\infty\}$. Further reasoning along these lines identifies the full group of conformal motions of the round sphere S^n (conformally containing Euclidean $\mathbb{R}^n$ via stereographic projection) as the identity connected component G of $\mathrm{SO}(n+1,1)$. The sphere is realised as the space of future pointing null rays



in $\mathbb{R}^{n+1,1}$. The form of P as above is obtained by taking G to preserve the form

$$2x_0 x_{n+1} + {x_1}^2 + {x_2}^2 + \cdots {x_n}^2$$

and the basepoint of S^n to be represented by the null vector

$$e = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

In summary, the flat model of conformal geometry is

$$S^n = G/P$$

for $G = \mathrm{SO}_\circ(n+1,1)$ and P a suitable *parabolic* subgroup. The other two observations in the Riemannian case will turn out to have analogues in the conformal case. We shall find an invariant connection (but not on the tangent bundle). In fact we shall find a somewhat better differential operator (originally found by T.Y. Thomas). We shall also find analogues of geodesics in the conformal case. They are known as *conformal circles* because in the flat case they coincide with the round circles on S^n (equipped with their standard projective parameterisations). Certainly, these circles are preserved by G.

Lecture Two

So much for the flat case. To proceed in general, the naïve approach is to work with a metric in the conformal class and then see how things change when the metric is scaled. If g_{ab} is replaced by $\widehat{g}_{ab} = \Omega^2 g_{ab}$, then the Levi-Civita connection ∇_a, is replaced by the connection $\widehat{\nabla}_a$ acting on 1-forms according to

$$\widehat{\nabla}_a\omega_b = \nabla_a\omega_b - \Upsilon_a\omega_b - \Upsilon_b\omega_a + \Upsilon^c\omega_c g_{ab}$$

where $\Upsilon_a \equiv \Omega^{-1}\nabla_a\Omega$. Indeed, this formula surely defines a torsion-free connection and induces

$$\widehat{\nabla}_a\omega_{bc} = \nabla_a\omega_{bc} - 2\Upsilon_a\omega_{bc} - \Upsilon_b\omega_{ac} - \Upsilon_c\omega_{ba} + \Upsilon^d\omega_{dc}g_{ab} + \Upsilon^d\omega_{bd}g_{ac}$$

on contravariant 2-tensors ω_{ab}, clearly annihilating $\widehat{g}_{ab}$. Correspondingly, the new connection on vector fields is

$$\widehat{\nabla}_a V^b = \nabla_a V^b + \Upsilon_a V^b - \Upsilon^b V_a + \Upsilon_c V^c \delta_a{}^b$$

where $\delta_a{}^b$ is the Kronecker delta. It is convenient to introduce a line bundle $\mathcal{E}[1]$ on M as follows. If a metric g_{ab} in the conformal class is chosen, then $\mathcal{E}[1]$ is identified with the trivial bundle $\mathcal{E}$. Equivalently, a local section ϕ of $\mathcal{E}[1]$ may be regarded as a function, say f. If, however, g_{ab} is replaced by $\widehat{g}_{ab} = \Omega^2 g_{ab}$, then the function $\widehat{f}$ representing ϕ with respect to the metric $\widehat{g}_{ab}$, is given by $\widehat{f} = \Omega f$. The w^{th} power of $\mathcal{E}[1]$ will be denoted by $\mathcal{E}[w]$ and its sections called *conformally weighted* functions of *weight* w. Such a section may be represented by a function scaling according to $\widehat{f} = \Omega^w f$. We shall write $\mathcal{E}^a$ and $\mathcal{E}_a$ for the tangent and cotangent bundle respectively. Other tensor bundles will be denoted by adorning $\mathcal{E}$ with the corresponding indices. The conformal metric may be regarded as an invariantly defined section of $\mathcal{E}_{ab}[2]$. Raising and lowering indices defines a canonical isomorphism of $\mathcal{E}^a[w]$ with $\mathcal{E}_a[w+2]$ for any weight w. Choosing a metric induces a connection on $\mathcal{E}[w]$ which transforms according to

$$\widehat{\nabla}_a\phi = \nabla_a\phi + w\Upsilon_a\phi$$

under scaling of the metric.

The Riemann curvature is defined by

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)V^c = R_{ab}{}^c{}_d V^d$$

and transforms by

$$\widehat{R}_{abcd} = \Omega^2(R_{abcd} - \Xi_{ac}g_{bd} + \Xi_{bc}g_{ad} - \Xi_{bd}g_{ac} + \Xi_{ad}g_{bc}) \tag{3}$$

under scaling of the metric where

$$\Xi_{ab} \equiv \nabla_a\Upsilon_b - \Upsilon_a\Upsilon_b + \tfrac{1}{2}\Upsilon_c\Upsilon^c g_{ab}.$$

In particular, the variation is entirely through traces. The totally trace-free part C_{abcd} of R_{abcd} is therefore invariant and (3) suggests that we write the remaining part in terms of a symmetric tensor P_{ab} according to

$$R_{abcd} = C_{abcd} + \mathrm{P}_{ac}g_{bd} - \mathrm{P}_{bc}g_{ad} + \mathrm{P}_{bd}g_{ac} - \mathrm{P}_{ad}g_{bc}.$$

Then the *Weyl curvature* C_{abcd} has weight 2 and is conformally invariant whilst the *Rho-tensor* P_{ab} has weight 0 and (3) implies that

$$\widehat{\mathrm{P}}_{ab} = \mathrm{P}_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b - \tfrac{1}{2}\Upsilon_c\Upsilon^c g_{ab}.$$

The Rho-tensor is a trace-adjusted multiple of the Ricci tensor $R_{bd} \equiv R_{ab}{}^a{}_d$:

$$\mathrm{P}_{ab} = \frac{1}{n-2}\left(R_{ab} + \frac{R}{2(1-n)}g_{ab}\right)$$

where $R \equiv R^a{}_a$ is the scalar curvature.

Without further ado, we can now introduce the conformally invariant connection alluded to earlier. Further details and motivation can be found in [3]. It is a connection on a vector bundle $\mathcal{E}^A$ which we now define. In the presence of a metric g_{ab} in the conformal class, it may be identified as a direct sum

$$\mathcal{E}^A = \mathcal{E}[1] \oplus \mathcal{E}^a[-1] \oplus \mathcal{E}[-1]$$

but if g_{ab} is replaced by $\widehat{g}_{ab} = \Omega^2 g_{ab}$, then a local section (σ, μ^a, ρ) is identified with its counterpart $(\widehat{\sigma}, \widehat{\mu}^a, \widehat{\rho})$ in the new scale according to

$$\begin{pmatrix} \widehat{\sigma} \\ \widehat{\mu}^a \\ \widehat{\rho} \end{pmatrix} = \begin{pmatrix} \sigma \\ \mu^a + \Upsilon^a\sigma \\ \rho - \Upsilon_b\mu^b - \tfrac{1}{2}\Upsilon_b\Upsilon^b\sigma \end{pmatrix}.$$

It is easy to check that this is an equivalence relation and hence that the bundle $\mathcal{E}^A$ is well-defined. We shall use the term *tractor* for tensor powers of this bundle and their sections (by analogy with the term tensor in Riemannian geomtry). The structure group of this bundle is the group P encountered in the discussion of the flat case. Whereas the tractor bundle corresponds to the standard representation of $G = \mathrm{SO}_o(n+1,1)$ restricted to P, the spin representation (if n is odd) or one of the two spin representations (if n is even) induces the *local twistor* bundles introduced in four dimensions by Penrose (see [20]). Recall that the flat model of conformal geometry is the homogeneous space G/P. The tractor bundle $\mathcal{E}^A$ in this case is the homogeneous bundle induced by restricting the defining representation of G on $\mathbb{R}^{n+2}$ to the subgroup P. It is therefore simply a product $S^n \times \mathbb{R}^{n+2}$.

For a given metric, the *tractor connection* on $\mathcal{E}^A$ is defined by

$$\nabla_b \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_b\sigma - \mu_b \\ \nabla_b\mu^a + \delta_b{}^a\rho + \mathrm{P}_b{}^a\sigma \\ \nabla_b\rho - \mathrm{P}_{ba}\mu^a \end{pmatrix}.$$

This definition is conformally invariant, as can be verified by direct calculation:

$$\hat{\nabla}_b \begin{pmatrix} \hat\sigma \\ \hat\mu^a \\ \hat\rho \end{pmatrix} = \begin{pmatrix} \hat\nabla_b\hat\sigma - \hat\mu_b \\ \hat\nabla_b\hat\mu^a + \delta_b{}^a\hat\rho + \hat{\mathrm{P}}_b{}^a\hat\sigma \\ \hat\nabla_b\hat\rho - \hat{\mathrm{P}}_{ba}\hat\mu^a \end{pmatrix}$$

$$= \begin{pmatrix} \hat\nabla_b\sigma - (\mu_b + \Upsilon_b\sigma) \\ \hat\nabla_b(\mu^a + \Upsilon^a\sigma) + \delta_b{}^a(\rho - \Upsilon_c\mu^c - \frac{1}{2}\Upsilon_c\Upsilon^c\sigma) + (\mathrm{P}_b{}^a - \nabla_b\Upsilon^a + \Upsilon_b\Upsilon^a - \frac{1}{2}\Upsilon_c\Upsilon^c\delta_b{}^a)\sigma \\ \hat\nabla_b(\rho - \Upsilon_c\mu^c - \frac{1}{2}\Upsilon_c\Upsilon^c\sigma) - (\mathrm{P}_{ba} - \nabla_b\Upsilon_a + \Upsilon_b\Upsilon_a - \frac{1}{2}\Upsilon_c\Upsilon^c g_{ba})(\mu^a + \Upsilon^a\sigma) \end{pmatrix}$$

$$= \begin{pmatrix} (\nabla_b + \Upsilon_b)\sigma - (\mu_b + \Upsilon_b\sigma) \\ \nabla_b(\mu^a + \Upsilon^a\sigma) - \Upsilon^a(\mu_b + \Upsilon_b\sigma) + \Upsilon_c(\mu^c + \Upsilon^c\sigma)\delta_b{}^a \\ \qquad + \delta_b{}^a(\rho - \Upsilon_c\mu^c - \Upsilon_c\Upsilon^c\sigma) + (\mathrm{P}_b{}^a - \nabla_b\Upsilon^a + \Upsilon_b\Upsilon^a)\sigma \\ (\nabla_b - \Upsilon_b)(\rho - \Upsilon_c\mu^c - \frac{1}{2}\Upsilon_c\Upsilon^c\sigma) - (\mathrm{P}_{ba} - \nabla_b\Upsilon_a + \Upsilon_b\Upsilon_a - \frac{1}{2}\Upsilon_c\Upsilon^c g_{ba})(\mu^a + \Upsilon^a\sigma) \end{pmatrix}$$

$$= \begin{pmatrix} \nabla_b\sigma - \mu_b \\ \nabla_b\mu^a + \Upsilon^a\nabla_b\sigma - \Upsilon^a\mu_b + \delta_b{}^a\sigma + \mathrm{P}_b{}^a\sigma \\ \nabla_b\rho - \Upsilon_c\nabla_b\mu^c - \frac{1}{2}\Upsilon_c\Upsilon^c\nabla_b\sigma - \Upsilon_b\rho - \mathrm{P}_{ba}\mu^a - \mathrm{P}_{ba}\Upsilon^a\sigma + \frac{1}{2}\Upsilon_c\Upsilon^c\mu_b \end{pmatrix}$$

$$= \widehat{\begin{pmatrix} \nabla_b\sigma - \mu_b \\ \nabla_b\mu^a + \delta_b{}^a\rho + \mathrm{P}_b{}^a\sigma \\ \nabla_b\rho - \mathrm{P}_{ba}\mu^a \end{pmatrix}}.$$

This definition is due to T.Y. Thomas [23]. It is equivalent to E. Cartan's conformal connection on the associated frame bundle. Thomas's discovery was slightly later than though independent of Cartan's. The connection induced on the bundle of local twistors is called *local twistor transport* (see [8, 20] for explicit formulae in dimension four).

In the flat case $M = S^n = G/P$, the tractor connection is the flat connection on the product bundle $S^n \times \mathbb{R}^{n+2}$. In general, the curvature is given by

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\begin{pmatrix} \sigma \\ \mu^c \\ \rho \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 2\nabla_{[a}\mathrm{P}_{b]}{}^c & C_{ab}{}^c{}_d & 0 \\ 0 & -2\nabla_{[a}\mathrm{P}_{b]d} & 0 \end{pmatrix}\begin{pmatrix} \sigma \\ \mu^d \\ \rho \end{pmatrix}. \qquad (4)$$

For $n \geq 4$, the Bianchi identity $\nabla_{[a}R_{bc]}{}^d{}_e = 0$ implies that

$$2\nabla_{[a}\mathrm{P}_{b]}{}^c = \tfrac{1}{3-n}\nabla^d C_{ab}{}^c{}_d$$

so the tractor curvature is equivalent to the Weyl curvature. When $n = 3$, the Weyl curvature vanishes (by symmetry considerations) and so we may conclude that $\nabla_{[a}\mathrm{P}_{b]c}$ is conformally invariant. This is known as the *Cotton-York* tensor. In any case, it is straightforward to deduce that these tensors are precisely the obstruction to a given conformal manifold being locally equivalent to the flat model.

The tractor bundle carries a non-degenerate symmetric form, the *tractor metric*, a Lorentzian metric characterised by

$$\|(\sigma, \mu^a, \rho)\|^2 = 2\sigma\rho + \mu^a\mu_a.$$

It is conformally invariant and is preserved by the tractor connection. In the flat case it coincides with the Lorentzian metric on $\mathbb{R}^{n+1,1}$.

We are now in a position to define the conformal analogues of geodesics, the so-called conformal circles. Though it is possible to proceed directly (as in [2]), it is convenient to use tractors. More details can be found in [3].

Suppose γ is a smooth curve in M parameterised by t, a smooth function on γ with nowhere-vanishing derivative. This determines the *velocity vector* U^a along γ by requiring that it be tangent to γ and that $U^a\nabla_a t = 1$. Define $u \equiv \sqrt{U^a U_a}$, a function of conformal weight 1. Define the *acceleration vector* $A^b \equiv U^a\nabla_a U^b$. (A unit speed geodesic is defined by $u = 1$ and $A^b = 0$.) Of course, the acceleration vector is not conformally invariant (in fact $\widehat{A}^b = A^b - u^2\Upsilon^b + 2(U^c\Upsilon_c)U^b$). Define the *velocity tractor* U^B and *acceleration tractor* A^B by

$$U^B \equiv U^a\nabla_a \begin{pmatrix} 0 \\ 0 \\ u^{-1} \end{pmatrix} \quad \text{and} \quad A^B \equiv U^a\nabla_a U^B.$$

These are manifestly conformally invariant definitions. Calculation yields

$$\|A\|^2 = 2u^{-2}U_bU^a\nabla_a A^b + 3u^{-2}A_bA^b - 6u^{-4}(U_bA^b)^2 + 2\mathrm{P}_{ab}U^aU^b,$$

automatically invariant! Its vanishing may be regarded as a 3rd order ordinary differential equation along γ for the function t. This gives a preferred family of local parameterisations of γ which we call *projective*. If s and t are projective parameters, then the third order ODE relating them reduces to the Schwarzian

$$2\frac{ds}{dt}\frac{d^3s}{dt^3} - 3\left(\frac{d^2s}{dt^2}\right)^2 = 0 \quad \text{whence} \quad s = \frac{at+b}{ct+d}.$$

The curve γ is called a projectively parameterised conformal circle if and only if

$$\|A\|^2 = 0 \quad \text{and} \quad U^a\nabla_a A^B = 0.$$

Certainly, this is a conformally invariant system. In fact, it reduces to a third order ODE with leading term $U^a\nabla_a A^b$. See [3] for more details. The upshot of this discussion is that for every velocity and acceleration vector at a point, there is a unique parameterised conformal circle with these as initial conditions. In the flat case, it may be verified that these are the round circles with standard projective parameterisations. As indicated earlier, this gives an alternative approach to the conformal motions of $\mathbb{R}^n$. The details are left to the reader.

Lecture Three

Recall from Lecture One, that the flat model of conformal geometry is the sphere S^n as a homogeneous space G/P where $G = \mathrm{SO}_\mathrm{o}(n+1,1)$ and P is the subgroup consisting of matrices of the form (2). It is an exercise in the theory of *Verma modules* to classify the G-invariant differential operators on S^n and this lecture will mostly be devoted to indicating this theory and how it applies. Of course, it is important to understand the flat model, else we cannot hope to understand the *curved*, i.e. general, case. More importantly, it will turn out that various elements of the flat theory generalise to the curved case.

For the following discussion, G and P can be arbitrary Lie groups. A *homogeneous* vector bundle on G/P is one whose total space is equipped with an action of G which is compatible with the action on G/P and which is linear on the fibres. Such a bundle may be reconstructed from its fibre over the identity coset. In other words, suppose

$$\rho : P \longrightarrow \mathrm{Aut}(\mathbb{E})$$

is a finite-dimensional representation of P. Then

$$E \equiv G \times_P \mathbb{E} \equiv \frac{G \times \mathbb{E}}{(g,e) \sim (gp, \rho(p^{-1})e)}$$

is homogeneous and this provides a 1–1 correspondence between the finite dimensional representations of P and the homogeneous vector bundles on G/P. (For more details, see, for example, [17].) For an arbitrary smooth vector bundle E on a smooth manifold, one has the jet bundles

$$J^\infty E \twoheadrightarrow \cdots \twoheadrightarrow J^3 E \twoheadrightarrow J^2 E \twoheadrightarrow J^1 E \twoheadrightarrow E$$

where $J^\infty E$ is defined as the inverse limit (of course, allowing an infinite-dimensional vector bundle in this instance). For a homogeneous vector bundle, there is the equivalent diagram of P-modules

$$J^\infty \mathbb{E} \twoheadrightarrow \cdots \twoheadrightarrow J^3 \mathbb{E} \twoheadrightarrow J^2 \mathbb{E} \twoheadrightarrow J^1 \mathbb{E} \twoheadrightarrow \mathbb{E}.$$

A linear differential operator of order $\leq k$ between arbitrary vector bundles E and F may be defined (see, for example, [22]) as a homomorphism of vector bundles

$D : J^kE \to F$. In the homogeneous case, the G-invariant such operators correspond to homomorphisms of P-modules $D : J^k\mathbb{E} \to \mathbb{F}$.

For $\mathbb{E}$, a representation of P, define the *Verma module* $V(\mathbb{E})$ to be the $\mathfrak{g}$-module

$$\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{p}} \mathbb{E}^* \equiv \frac{\mathfrak{U}(\mathfrak{g}) \otimes \mathbb{E}^*}{\mathfrak{U}(\mathfrak{g})\text{-submodule generated by } \{p \otimes f - 1 \otimes \dot{\rho}^*(p)f\}}.$$

where $\dot{\rho}^*$ is the derivative of the dual representation, $\mathfrak{p}$ is the Lie algebra of P, and $\mathfrak{g}$ is the Lie algebra of G with universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$. (Warning: this is somewhat non-standard terminology—see the parenthetical remark starting at the bottom of this page. The term *induced module* is also used (see, for example, [25]).) In fact, P also acts on $V(\mathbb{E})$ in a compatible way. In the homogeneous case observe that the action of G on E induces an action of $\mathfrak{g}$ on $J^\infty\mathbb{E}$. It is straightforward to verify that there is a natural isomorphism $V(\mathbb{E})^* = J^\infty\mathbb{E}$ as $(\mathfrak{g}, P)$-modules. The universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ is filtered by degree

$$\mathbb{R} = \mathfrak{U}_0(\mathfrak{g}) \subset \mathfrak{U}_1(\mathfrak{g}) \subset \cdots \subset \mathfrak{U}_k(\mathfrak{g}) \subset \mathfrak{U}_{k+1}(\mathfrak{g}) \subset \cdots \subset \mathfrak{U}(\mathfrak{g})$$

with corresponding gradings

$$\frac{\mathfrak{U}_k(\mathfrak{g})}{\mathfrak{U}_{k-1}(\mathfrak{g})} = \bigodot^k\mathfrak{g}.$$

This induces a filtration of the Verma module

$$\mathbb{R} = V_0(\mathbb{E}) \subset V_1(\mathbb{E}) \subset \cdots \subset V_k(\mathbb{E}) \subset V_{k+1}(\mathbb{E}) \subset \cdots \subset V(\mathbb{E})$$

which is preserved by the action of P. The corresponding gradings may be viewed as an exact sequences of P-modules

$$0 \longrightarrow V_{k-1}(\mathbb{E}) \longrightarrow V_k(\mathbb{E}) \longrightarrow \bigodot^k(\mathfrak{g}/\mathfrak{p}) \otimes \mathbb{E}^* \longrightarrow 0$$

whose dual induces the jet exact sequence (see, for example, [22])

$$0 \longrightarrow \bigodot^kT^*M \otimes E \longrightarrow J^kE \longrightarrow J^{k-1}E \longrightarrow 0.$$

In this way, Verma modules build the P-modules $J^k\mathbb{E}$ which arise in constructing G-invariant linear differential operators. However, it is usually more convenient to use the Verma modules as a whole—a G-invariant linear differential operator between the bundles on G/P induced from representations $\mathbb{E}$ and $\mathbb{F}$ is equivalent to a homomorphism of $(\mathfrak{g}, P)$-modules

$$V(\mathbb{E}) \longleftarrow V(\mathbb{F}).$$

This is easily shown from a geometric point of point; in representation theory, it is known as *Frobenius reciprocity* (see, for example, [25]).

To proceed further it is necessary to be more specific concerning G, P, and $\mathbb{E}$. Matters are especially congenial if G is semisimple, P a parabolic subgroup, and $\mathbb{E}$ an irreducible representation. (The term generalised Verma module is often used for this case (see, for example, [18]) with the term Verma module reserved for the case when P is Borel (in which case $\mathbb{E}$, being irreducible, is necessarily one-dimensional).)

It is illuminating to consider the following simple example:

$$G = \mathrm{SL}(2,\mathbb{R}) \qquad P = \left\{ \begin{pmatrix} \lambda & r \\ 0 & \lambda^{-1} \end{pmatrix} \text{ s.t. } \lambda > 0,\ r \in \mathbb{R} \right\} \qquad \mathbb{E} = \mathbb{R}_w = \mathbb{R}$$

with an element of P of the form shown acting on $\mathbb{R}_w$ as multiplication by λ^{-w} for some $w \in \mathbb{R}$. The corresponding homogeneous bundle is denoted

$$\mathcal{E}(w) \qquad \text{(as in T.N. Bailey's lectures [1])}.$$

To investigate the Verma module $V(w)$, introduce the standard generators of $\mathfrak{sl}(2,\mathbb{R})$:

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We may use the standard commutation relations to put elements of $\mathfrak{U}(\mathfrak{sl}(2,\mathbb{R}))$ into a standard order (i.e. apply the Poincaré-Birkhoff-Witt procedure) and hence identify

$$V(w) = \mathbb{R}[y]\alpha \quad \text{where } x\alpha = 0 \text{ and } h\alpha = w\alpha.$$

In other words, the action of y on this polynomial algebra is by left multiplication, whilst the actions of x and h are obtained by repeated commutation to bring them to the right whereupon they act on α as specified. Thus, $V(w)$ is a highest weight module generated by α. The following calculations are typical. Consider the case $w = 1$. Then

$$\begin{aligned} xy\alpha &= [x,y]\alpha + yx\alpha = h\alpha + 0 = \alpha \\ hy\alpha &= [h,y]\alpha + yh\alpha = -2y\alpha + y\alpha = -y\alpha \\ xy^2\alpha &= [x,y]y\alpha + yxy\alpha = hy\alpha + y\alpha = -y\alpha + y\alpha = 0 \\ hy^2\alpha &= [h,y]y\alpha + yhy\alpha = -2y^2\alpha - y^2\alpha = -3y^2\alpha. \end{aligned}$$

This shows that $y^2\alpha$ is a *maximal* (sometimes called *singular*) weight vector in $V(1)$ (i.e. an eigenvector for h that is annihilated by y). Apart from α itself, it is easily verified by similar calculations, that, up to scale, there are no other maximal weight vectors. As the weight of $y^2\alpha$ is -3, it follows that there is a homomorphism of Verma modules

$$V(-3) = \mathbb{R}[y]\beta \longrightarrow \mathbb{R}[y]\alpha = V(1)$$

with $\beta \mapsto y^2\alpha$ (and, therefore, $f(y)\beta \mapsto y^2 f(y)\alpha$). Equivalently, there is a second order linear differential operator

$$\eth^2 : \mathcal{E}(1) \longrightarrow \mathcal{E}(-3).$$

More generally, these calculations show that, up to scale, the only non-trivial linear differential operators on the circle which are *projectively* invariant (i.e. invariant under the action of $\mathrm{SL}(2,\mathbb{R})$), are

$$\eth^{w+1} : \mathcal{E}(w) \longrightarrow \mathcal{E}(-w-2) \quad \text{for } w \in \mathbb{Z}_{\geq 0}$$

where $\eth^k$ is a k^{th}-order differential operator (cf. [9] for the complex case). The *non-linear* differential operators in this situation are the subject of [1]. The operator $\eth : \mathcal{E} \to \mathcal{E}(-2)$ is the exterior derivative.

Leaving this example for the moment, it is useful to note how much of this applies in general, i.e. when G is semisimple, P parabolic, and $\mathbb{E}$ irreducible. Certainly, the search for G-invariant linear differential operators between irreducible bundles on G/P, is equivalent to the search for maximal weight vectors in $V(\mathbb{E})$. When $G/P = S^n$, the flat model of conformal geometry, these Verma modules are of the form

$$V(\mathbb{E}) = \mathbb{R}[y_1, y_2, \ldots, y_n] \otimes \mathbb{E}^*$$

where $y_1, y_2, \ldots, y_n$ are 'lowering' operators in $\mathfrak{g}$ corresponding to the n positions in the top row opposite the vector r^a in (2). The representation $\mathbb{E}^*$ of P, being irreducible, is generated by a highest weight vector α, unique up to scale. The entire Verma module is then generated by applying lowering operators from $\mathfrak{g}$ to α (in particular, the operators $y_1, y_2, \ldots, y_n$, which together with lowering operators from $\mathfrak{p}$, span the lowering operators of $\mathfrak{g}$). In this generality, the search for maximal weight vectors in $V(\mathbb{E})$ is not so amenable to direct computation. Instead, the Jantzen-Zuckerman *translation functor* (see, for example, [25]) avoids these calculations and also provides an inductive method of constructing the whole family of operators from their simplest members (such as the exterior derivatives). They key to the translation functor is to consider the action of the centre of $\mathfrak{U}(\mathfrak{g})$.

Returning to our example, the centre of $\mathfrak{U}(\mathfrak{sl}(2, \mathbb{R}))$ contains the element

$$C \equiv h^2 + 4yx + 2h$$

(and, in fact, consists of polynomials in C). Applying C to $\alpha \in V(w)$ gives $w(w+2)\alpha$. Since $V(w)$ is generated by α and C commutes with the action of $\mathfrak{g}$, it follows that C acts on all of $V(w)$ by multiplication by $w(w+2)$. This already restricts the possibilities for non-trivial homomorphisms $V(w') \to V(w)$ for, in this case, clearly one must have $w'(w'+2) = w(w+2)$. The action of C also plays a rôle as follows. The standard representation of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R}^2$ gives rise to a Verma module $V(\mathbb{R}^2)$. Generally, if $\rho : G \to \mathrm{Aut}(\mathbb{E})$ is restricted to P, then the corresponding homogeneous vector bundle E is canonically trivial:

$$\begin{array}{ccc} E & = & G/P \times \mathbb{E} \\ & & \downarrow \\ & & G/P\,. \end{array}$$

Thus, on the level of Verma modules,

$$V(\mathbb{R}^2) = V(0) \otimes (\mathbb{R}^2)^*$$

and, more generally,

$$V(\mathbb{R}^2 \otimes \mathbb{R}_w) = V(w) \otimes (\mathbb{R}^2)^*.$$

The short exact sequence of P-modules

$$0 \to \mathbb{R}_{w-1} \longrightarrow \mathbb{R}^2 \otimes \mathbb{R}_w \longrightarrow \mathbb{R}_{w+1} \to 0$$

gives rise to a short exact sequence of $\mathfrak{g}$-modules

$$0 \leftarrow V(w-1) \longleftarrow V(w) \otimes (\mathbb{R}^2)^* \longleftarrow V(w+1) \leftarrow 0.$$

For $w \neq -1$, notice that C acts by different scalars on $V(w-1)$ and $V(w+1)$. In these cases, therefore, $V(w) \otimes (\mathbb{R}^2)^*$ decomposes as a $\mathfrak{g}$-module into a direct sum of the two eigenspaces of C. Bearing in mind that a homomorphism of Verma modules corresponds to a G-invariant differential operator, this shows that there are invariant differential splittings, for example

$$0 \to \mathcal{E}(-1) \underset{D_A}{\rightleftarrows} \mathcal{E}^A \underset{D^A}{\rightleftarrows} \mathcal{E}(1) \to 0$$

where $\mathcal{E}^A$ is the trivial bundle with $\mathbb{R}^2$ as fibre. In homogeneous coördinates,

$$D_A = -\partial / \partial x^A.$$

The exterior derivative $\eth : \mathcal{E} \to \mathcal{E}(-2)$ also acts on this trivial bundle to give the diagram

$$\begin{tikzcd}
\mathcal{E}(-1) \arrow[r] & \mathcal{E}^A \arrow[r, shift left] \arrow[d, "\eth"] & \mathcal{E}(1) \arrow[l, shift left, "D^A"] . \\
\mathcal{E}(-3) \arrow[r, shift left] & \mathcal{E}^A(-2) \arrow[l, shift left, "D_A"] \arrow[r] & \mathcal{E}(-1) .
\end{tikzcd}$$

The composition $D^A \circ \eth \circ D_A$ is then $\eth^2 : \mathcal{E}(1) \to \mathcal{E}(-3)$. Notice that it has been constructed in a manifestly invariant manner. Iterating this procedure generates all the invariant operators $\eth^k$ inductively.

In the flat conformal case, there are similar invariant differential splittings of the tractor bundle. Their existence is guaranteed abstractly by analysing the action of the centre of $\mathfrak{U}(\mathfrak{g})$ (see [8] for corresponding splittings of the twistor bundles in the four-dimensional case). This analysis not only gives a complete classification of the conformally invariant linear differential operators between irreducible conformally weighted tensor bundles in the flat case, but also gives, via the translation functor, a way of building families of invariant operators from their simplest members. For the purposes of this article, the results of this classification are not so important (see, for example, [15] or [8] for the four-dimensional case, [4] for the general case, and [7] for generalisations).

The rest of this article is concerned with the extent to which the translation functor, as a method for generating invariant operators, extends to the curved case. The main point is that the differential splittings mentioned above exist in the curved case too. In fact, the basic examples were already written down in 1932 by Thomas [24]. As motivation, he noted that the tractor connection is an inadequate substitute for the Levi-Civita connection. In the Riemannian case one can apply the Levi-Civita connection repeatedly. Indeed, in some well-defined sense, which we won't go into here, this captures all the invariant calculus that is present on a Riemannian manifold. In the conformal case, having formed $\nabla_b V^A$ for a tractor field V^A, there is no invariant connection on the cotangent bundle so one is at a loss for a second derivative. Thomas suggested the following replacement. Define $D^B : \mathcal{E}[w] \to \mathcal{E}^B[w-1]$ by

$$D^B f = \left(\begin{array}{c} w(n+2w-2)f \\ (n+2w-2)\nabla^b f \\ (\Delta - w\mathrm{P})f \end{array}\right)$$

where Δ denotes the Laplacian $-\nabla^a\nabla_a$ and $\mathrm{P} \equiv \mathrm{P}^a{}_a$, a multiple of the scalar curvature. As usual, this definition is written with respect to a particular choice of metric in the conformal class. It is, however, conformally invariant. To see this, consider how the Laplacian changes under scaling of the metric $g_{ab} \mapsto \widehat{g}_{ab} = \Omega^2 g_{ab}$.

$$\begin{aligned} \widehat{\nabla}_b\widehat{\nabla}_a f &= \widehat{\nabla}_b(\nabla_a f + w\Upsilon_a f) \\ &= \nabla_b(\nabla_a f + w\Upsilon_a f) + (w-1)\Upsilon_b(\nabla_a f + w\Upsilon_a f) \\ &\qquad - \Upsilon_a(\nabla_b f + w\Upsilon_b f) + \Upsilon^c(\nabla_c f + w\Upsilon_c f)g_{ab}. \end{aligned} \tag{5}$$

Thus,

$$\begin{aligned} \widehat{\nabla}^a\widehat{\nabla}_a f &= \nabla^a(\nabla_a f + w\Upsilon_a f) + (n+w-2)\Upsilon^a(\nabla_a f + w\Upsilon_a f) \\ &= \nabla^a\nabla_a f + w(\nabla^a\Upsilon_a)f + (n+2w-2)\Upsilon^a\nabla_a f + w(n+w-2)\Upsilon^a\Upsilon_a f \\ &= \nabla^a\nabla_a f + (n+2w-2)\Upsilon^a\nabla_a f + w(\nabla^a\Upsilon_a + (n+w-2)\Upsilon^a\Upsilon_a)f. \end{aligned}$$

On the other hand,

$$\widehat{\mathrm{P}} = \mathrm{P} - \nabla^a\Upsilon_a + \left(1 - \tfrac{n}{2}\right)\Upsilon^a\Upsilon_a.$$

Hence,

$$(\widehat{\Delta} - w\widehat{\mathrm{P}})f = (\Delta - w\mathrm{P})f - (n+2w-2)\Upsilon_b\nabla^b f - \tfrac{1}{2}\Upsilon_b\Upsilon^b w(n+2w-2)f$$

whilst

$$(n+2w-2)\widehat{\nabla}^b f = (n+2w-2)\nabla^b f + \Upsilon^b w(n+2w-2)f.$$

These are exactly the transformations of a tractor, as required.

Notice that the argument is unaffected if f is replaced by a tractor field of conformal weight w (where, of course, the Laplacian is replaced by the tractor Laplacian (an explicit formula for which is quite complicated)). In other words, the operator $D^B : \mathcal{E}^A[w] \to \mathcal{E}^{BA}[w-1]$ makes perfectly good sense. We can continue in this way to form

$$D^B V^A \quad D^C D^B V^A \quad \cdots$$

for any conformally weighted tractor field V^A.

The D-operator combines several features of conformal geometry. If $w = 0$, then the first component of $D^B V^A$ vanishes and so the second component is conformally invariant. This is the tractor connection. If $w = 1 - \frac{n}{2}$, then both the first and second components vanish. The third component is therefore conformally invariant. In other words, the differential operator

$$f \mapsto (\Delta - w\mathrm{P})f = (\Delta + \frac{n-2}{4(n-1)}R)f \tag{6}$$

is conformally invariant when acting on functions or tractor fields of weight $1 - \frac{n}{2}$. In the flat model, D^A is closely related to differentiation with respect to the coördinates on $\mathbb{R}^{n+2}$. Indeed, this aspect may be exploited in the curved case with D^A finding interpretation in the *ambient metric* construction of Fefferman and Graham [10].

Lecture Four

The investigation of invariant differential operators on a conformal manifold is an active area of research (see, for example, [5, 6, 15, 16, 21, 26]). The conformally invariant Laplacian or Yamabe operator (6) is an example of such an operator. More generally, by a *conformally invariant* differential operator, we shall mean any polynomial expression in the Levi-Civita connection and its curvature acting between conformally weighted tensor bundles, which is unchanged when the metric is scaled. Some examples should suffice to make clear what is meant here. The exterior derivative

$$\nabla_a : \mathcal{E}_{[b\ldots d]} \longrightarrow \mathcal{E}_{[ab\cdots d]}$$

is certainly invariant. Consider the operator

$$\begin{array}{ccc} \mathcal{E}[1] & \longrightarrow & \text{trace free part of } \mathcal{E}_{(ab)}[1] \\ \Cup & & \Cup \\ f & \longmapsto & \nabla_{(a}\nabla_{b)}f + \mathrm{P}_{ab}f - \frac{1}{n}g_{ab}(\nabla^c\nabla_c f + \mathrm{P}f). \end{array} \tag{7}$$

From (5)

$$\widehat{\nabla}_{(a}\widehat{\nabla}_{b)}f = \nabla_{(a}\nabla_{b)}f + (\nabla_{(a}\Upsilon_{b)} - \Upsilon_a\Upsilon_b)f + \text{ trace terms}$$

whilst

$$\widehat{\mathrm{P}}_{ab} - \tfrac{1}{n}\widehat{\mathrm{P}}g_{ab} = \mathrm{P}_{ab} - \tfrac{1}{n}\mathrm{P}g_{ab} - \nabla_{(a}\Upsilon_{b)} + \Upsilon_a\Upsilon_b.$$

The operator (7) is therefore invariant. A more exotic invariant operator is given by

$$f \longmapsto \text{totally trace free part of } (\nabla_{(a}\nabla_b\nabla_{c)}f + 4\mathrm{P}_{(ab}\nabla_{c)}f + 2[\nabla_{(a}\mathrm{P}_{bc)}]f)$$

acting on conformally weighted functions of weight 2. In fact, all these operators are *strongly* invariant in the sense that they are also invariant when acting on tractor fields rather than just scalar functions. This is because the transformations under scaling of the metric simply do not distinguish between pure tensors and tensor fields coupled to the tractor bundle. However, it is not the case that all invariant operators are strongly invariant. Consider, for example, the operator $L : \mathcal{E} \to \mathcal{E}[-4]$ in dimension four given by

$$Lf = \nabla_b[\nabla^b\nabla^a + 4\mathrm{P}^{ba} - 2\mathrm{P}g^{ba}]\nabla_a f. \tag{8}$$

Calculation yields

$$\hat{L}f = Lf + 2\Upsilon^b\nabla^a(\nabla_a\nabla_b - \nabla_b\nabla_a)f + 2\Upsilon^b(\nabla_a\nabla_b - \nabla_b\nabla_a)\nabla^a f - 2\left[(\nabla_a\nabla_b - \nabla_b\nabla_a)\Upsilon^a\right]\nabla^b f.$$

The curvature terms cancel so L is conformally invariant. However, if f is replaced by f^C, a section of $\mathcal{E}^C$, then, writing $\Omega_{ab}{}^C{}_D$ for the tractor curvature (cf. (4)),

$$\hat{L}f^C = Lf^C + 4\Upsilon^b\Omega_{ab}{}^C{}_D\nabla^a f^D + 2\Upsilon^b[\nabla^a\Omega_{ab}{}^C{}_D]f^D$$

so conformal invariance is lost.

We shall now describe a procedure (the *curved translation principle*) for generating conformally invariant operators. This procedure takes a strongly invariant operator and yields new strongly invariant operators from it. (We should remark, however, that though this is a generally applicable procedure, there are much more efficient means for calculating certain series of operators in particular cases (see, for example, [12]). To proceed, we need to generalise the D-operator so that it acts on arbitrary conformally weighted irreducible tensor bundles. Thus, $D^B : \mathcal{F}[w] \to \mathcal{F}^B[w-1]$ for any irreducible tensor bundle $\mathcal{F}$. The general existence of these operators follows from their existence in the flat case and this, as indicated in Lecture Three, follows from the general theory of Verma modules. The formulae become quite complicated. For example, $D^B : \mathcal{E}_a[w] \to \mathcal{E}_a{}^B[w-1]$ sends a conformally weighted 1-form ϕ_a of weight w to

$$\left(\begin{array}{c} (n+2w-4)(n+w-2)(w-2)w\phi_a \\ (n+2w-4)[(n+w-2)(w-1)\nabla^b\phi_a + (n+w-2)\nabla_a\phi^b - w\delta_a{}^b\nabla^c\phi_c] \\ (n+w-2)(w-1)(\Delta-(w-1)\mathrm{P})\phi_a - (n-2)[\nabla_a\nabla_b + (n+w-2)\mathrm{P}_{ab}]\phi^b \end{array}\right).$$

However, these operators only of second order. It is this fact that allows a general existence argument. In fact, there is an alternative approach using twistors instead of tractors where the corresponding operators are only of first order. This makes their existence more straightforward (cf. [11]). A general formula in four dimensions is given in [8]. As with $D^B : \mathcal{E}[w] \to \mathcal{E}^B[w-1]$, the same formulae with the tractor connection define conformally invariant operators when acting on tractor-valued conformally weighted tensors, i.e. these operators are strongly invariant. (Note, however, that there is some choice here—the same operator on tensors may be defined by two different formulae (when derivatives are commuted at the expense of curvature) in which case these different formulae may give rise to genuinely different operators on tractors. This is another reason why the twistor approach may be preferred.)

There is also a series $E^{bC} : \mathcal{F}_b[w] \to \mathcal{F}^C[w-1]$ of strongly invariant first order operators for any irreducible tensor bundle $\mathcal{F}_b$. For example,

$$\phi_b \mapsto \left(\begin{array}{c} 0 \\ (n+w-2)\phi^c \\ -\nabla^b\phi_b \end{array}\right)$$

for ϕ_a a local section of $\mathcal{E}_a[w]$ and

$$\psi_{ab} \mapsto \begin{pmatrix} 0 \\ (n+w-2)\psi_a{}^c \\ -\nabla^b\psi_{ab} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ (w-2)\delta_a{}^c\psi_d{}^d \\ -n\nabla^b\psi_{ab} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ (n+w-4)\psi_a{}^c \\ -\nabla^b\psi_{ab} \end{pmatrix}$$

for ψ_{ab} a local section of $\mathcal{E}_{ab}[w]$ and either symmetric trace free, pure-trace, or skew, respectively.

Finally, there is a series of zeroth order invariant operators $F^B : \mathcal{F}[w] \to \mathcal{F}^B[w+1]$ for any irreducible tensor bundle $\mathcal{F}$ whose definition is tautological:

$$\phi_{a\cdots c} \mapsto \begin{pmatrix} 0 \\ 0 \\ \phi_{a\cdots c} \end{pmatrix}.$$

Dually, there are srongly invariant operators

$$D_B : \mathcal{F}^B[w] \to \mathcal{F}[w-1] \qquad E_{bC} : \mathcal{F}^C[w] \to \mathcal{F}_b[w+1] \qquad F_B : \mathcal{F}^B[w] \to \mathcal{F}[w+1].$$

For example,

$$\begin{array}{ccl} E_{dC} : \mathcal{E}_{[ab]}{}^C[w] & \longrightarrow & (\text{irreducible decomposition of } \mathcal{E}_{[ab]d}[w+1]) \\ \Psi & & \qquad\qquad \Psi \\ \begin{pmatrix} \sigma_{ab} \\ \mu_{ab}{}^c \\ \rho_{ab} \end{pmatrix} & \longmapsto & \left\{ \begin{array}{l} (w-2)\left[\mu_{abd} + \frac{2}{n-1} g_{d[a}\mu_{b]e}{}^e - \mu_{[abd]}\right] \\ \qquad + \frac{2}{3}\left[\nabla_{[a}\sigma_{b]d} - \nabla_d\sigma_{ab}\right] - \frac{2}{n-1} g_{d[a}\nabla^e\sigma_{b]e} \end{array} \right\} \\ & & + \left\{(n+w-3)g_{d[a}\mu_{b]e}{}^e - g_{d[a}\nabla^e\sigma_{b]e}\right\} \\ & & + \left\{(w+1)\mu_{[abd]} - \nabla_{[a}\sigma_{bd]}\right\} \end{array}$$

The curved translation principle is obtained simply by combining these operators as follows. Suppose $K : \mathcal{F}[w] \to \mathcal{G}[w']$ is a strongly invariant linear differential operator for irreducible tensor bundles $\mathcal{F}$ and $\mathcal{G}$ and conformal weights w and w'. Strong invariance implies that the same formula defines:

$$K : \mathcal{F}^B[w] \to \mathcal{G}^B[w']$$

which we may compose with the D, E, and F-operators to obtain new strongly invariant operators between conformally weighted irreducible tensor bundles. The following example illustrates this procedure. Let K be the exterior derivative from 1-forms to 2-forms. Then $K : \mathcal{E}_b{}^C \to \mathcal{E}_{[ab]}{}^C$ is given by

$$\begin{pmatrix} \sigma_b \\ \mu_b{}^c \\ \rho_b \end{pmatrix} \longmapsto \begin{pmatrix} \nabla_{[a}\sigma_{b]} + \mu_{[ab]} \\ \nabla_{[a}\mu_{b]}{}^c + \delta_{[a}{}^c\rho_{b]} + \mathrm{P}_{[a}{}^c\sigma_{b]} \\ \nabla_{[a}\rho_{b]} - \mathrm{P}_{c[a}\mu_{b]}{}^c \end{pmatrix}.$$

We may compose with E^{aC} : (trace-free part of $\mathcal{E}_{(ab)}[1]) \to \mathcal{E}_b{}^C$ given by

$$\psi_{ab} \mapsto \begin{pmatrix} 0 \\ (n-1)\psi_b{}^c \\ -\nabla^d\psi_{bd} \end{pmatrix}$$

and E_{dC} : $\mathcal{E}_{[ab]}{}^C \to$ (irreducible decomposition of $\mathcal{E}_{[ab]d}[1]$) to obtain, after some calculation,

$$(n-1)\nabla_{[a}\psi_{b]c} - g_{c[a}\nabla^d\psi_{b]d},$$

another strongly invariant operator, this one acting on symmetric trace-free ψ_{ab} of weight 1.

This is still first order. However, the translation principle can also increase the order by 1 (or decrease by 1 or more). For example,

$$\begin{array}{ccc} \mathcal{E}[1] \xrightarrow{D^C} \mathcal{E}^C \xrightarrow{\nabla_a} \mathcal{E}_a{}^C \xrightarrow{E_{bC}} & \text{(trace-free part of } \mathcal{E}_{(ab)}[1]) \\ \Cup & \Cup \\ f \longmapsto & n\nabla_{(a}\nabla_{b)}f + n\mathrm{P}_{ab}f + g_{ab}\Delta f - g_{ab}\mathrm{P}f \end{array}$$

yields a second order operator from a first order operator whilst

$$\begin{array}{ccc} \mathcal{E}_b[1] \xrightarrow{D^C} \mathcal{E}_b{}^C \xrightarrow{\nabla_a} \mathcal{E}_{[ab]}{}^C \xrightarrow{E_{dC}} & \text{(highest weight part of } \mathcal{E}_{[ab]d}[1]) \\ \Cup & \Cup \\ \phi_a \longmapsto & (n-1)C_{abd}{}^c\phi_c \end{array}$$

yields a zeroth order operator from a first order operator.

To summarize: The existence of the D and E-operators derives from the flat case where this translation principle coincides with the Jantzen-Zuckerman translation functor. The resulting classification in the flat case yields various series of operators and the curved translation principle shows that most of these series admit 'curved analogues,' i.e. invariant operators with the same symbol as an invariant operator in the flat case. Indeed, in the odd-dimensional case, all the flat operators admit curved analogues and the even dimensional case most can be obtained by translating the exterior derivative. There are, however, some exceptions and in dimension four, Graham has shown that the flat invariant operator

$$\Delta^3 : \mathcal{E}[1] \longrightarrow \mathcal{E}[-5] \tag{9}$$

has no curved analogue [13]. In the flat case, this operator may be obtained by translating (8) but, in the curved case, this breaks down since, as we observed, (8) is not strongly invariant.

There are many questions yet to be answered.

- Precisely which flat operators have curved analogues? There is a clear conjecture, knowing where the curved translation principle breaks down but (9) is the only explicitly known example.
- Further investigate the curved analogue of $\Delta^{n/2} : \mathcal{E} \to \mathcal{E}[-n]$ for n even. This operator has been shown to exist [14] but Branson has asked, for example, whether it can be chosen to be self-adjoint.
- Find more explicit formulae for the curved analogues (cf. [6, 12]). (They are not, in general, unique).
- Generalise this theory to other geometries: AHS (see [6]), CR, ...

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