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1 Introduction

Generalizations of complex analysis to higher dimensions are studied for a long time already. The golden period of the biggest stream - the theory of several complex variables - came after the World War Two and the field is nowadays a well established and flourishing part of mathematics.

The works of Moisil, Théodorescu and Fueter ([8, 9, 15, 20]) opened the way to the second natural generalization which attracted a broader interest later then the theory of several complex variables. It studies the Dirac equation as a natural generalization of the Cauchy-Riemann equations to higher dimensions. It has a local part (an analogue of the function theory of one complex variable) which is commonly known as Clifford analysis (see [4, 6, 10, 11]) and a global part which was very important in the development of the index theory for elliptic operators in the sixties (see [13, 2, 3]) and which is very active field of study ever since. The terminology is not still unified, solutions of the Dirac equation in Clifford analysis are traditionally called monogenic functions, while in global analysis they are usually called harmonic spinors ([17]). The next natural question is what is a suitable generalization of holomorphic forms to higher dimensions. A suitable answer was recently described (under the name of monogenic forms) in [6, 18]. The generalization was based on the following main features of holomorphic forms.

Firstly, analogues of Cauchy-Riemann equations should be invariant with respect to the orthogonal (resp. Spin) group. This property alone restricts possibilities enormously, there is only a finite number of possible generalizations having this property. A wish to have a suitable generalization of the Cauchy theorem available is the second important restriction. To get a third one, let us recall that it is possible to describe topological properties of domains in the complex plane using holomorphic functions and forms. The definition of monogenic forms in domains in $\mathbf{R}_m$ given in [6, 18] preserves this property. The fact that the definition of monogenic forms is invariant with respect to the Spin group is the key property making possible a generalization of the

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definition of monogenic forms to Riemannian manifolds with a given spin structure (see [19]) .

To come to the main topic of the paper, one of the most useful tools in complex analysis is the residue theory for meromorphic forms. A broad generalization of residues for monogenic forms in domains in $\mathbf{R}_m$ was introduced and studied in [5, 6]. The aim of the paper is to show that the theory of residues for monogenic forms can be naturally extended to monogenic forms on Riemannian manifolds.

Let us now describe in more details the main idea behind the suggested generalization of residues. The theory of pointwise residues is a fully satisfactory analogue of the one complex variable theory and reduce to it for $m = 2$ (see [4, 22]). Let $\mathbf{S}^+$ be a basic spinor representation of the group Spin(n) and let f be a smooth map from a domain $\Omega \subset \mathbf{R}_m$ to $\mathbf{S}^+$. If f is a solution of the Dirac equation with an isolated singularity in a point $P \in \Omega$, then there are two definitions of the residue $\mathrm{res}_P f$ which coincide. The first one uses the decomposition of f into Taylor and Laurent series (see [4, 6] for more details). The other one is given by a suitable integral. Let $e_j, j = 1, \dots, m$, be a canonical basis of $\mathbf{R}_m$ interpreted as elements of the corresponding Clifford algebra and let $x_j; j = 1, \dots, m$, be the corresponding coordinates. Let us consider a spinor-valued differential form ω of degree $m-1$ given by $\omega = D\sigma.f$, where

$$D\sigma = \sum_1^m (-1)^{j+1} e_j dx_1 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_n \tag{1}$$

and where the multiplication means the action of elements of the Clifford algebra on spinors. Then $\mathrm{res}_P\, \omega$ is defined by

$$\mathrm{res}_P\, \omega = \int_{S^{m-1}(P)} \omega \in \mathbf{S}^+,$$

where $S^{m-1}(P)$ is a sphere of dimension $m-1$ of a small radius around P.

A special feature of Clifford analysis (in comparison with the theory of several complex variables) is that even if it is "one variable" analogue of the complex function theory, there is more space than in the complex plane. Hence it is interesting to study monogenic functions with singularities on higher dimensional compact submanifolds. This generalization was studied in [5, 6]. The basic tool needed here (as well as in the generalization to manifolds) is the Leray-Norguet theory of residues, see [16].

To recall it shortly, let M be a smooth oriented manifold of dimension m and let Σ be its oriented submanifold of dimension $k, 1 \le k \le m-2$. Let q be a positive integer, $q \ge m-k-1$. Then there are two maps (dual to each other)

$$\mathrm{Res}_\Sigma : H^q(M \setminus \Sigma) \mapsto H^{q-(m-k-1)}(\Sigma)$$

and

$$\delta : H^c_{q-(m-k-1)}(\Sigma) \mapsto H^c_q(M \setminus \Sigma).$$

To indicate a geometrical meaning of the cobord map δ, let us first consider the projection $p : U \mapsto \Sigma$ of a tubular neighbourhood U onto Σ and its restriction

$p' = p|_{\partial U}$ to the boundary of U. Then it is possible to represent the class of homology of the cobord $\delta\gamma$ of a closed $q-(m-k-1)$-dimensional submanifold $\gamma \subset \Sigma$ by its preimage $(p')^{-1}(\gamma)$ in $M \setminus \Sigma$. Hence every point of γ is substituted by a copy of the $(m-k-1)$-dimensional sphere going into a transverse direction.

The map $\mathrm{Res}_{\,\Sigma}$, called the Leray-Norguet residue, can be interpreted then as an integration over fibers. The duality is expressed by the theorem (or definition) saying that

$$\int_{\delta\gamma} \omega = \int_{\gamma} \mathrm{Res}_{\,\Sigma}\, \omega$$

for all $\omega \in H^q(M \setminus \Sigma)$ and $\gamma \in H^c_{q-(m-k-1)}(\Sigma)$.

Now, for a monogenic $(m-1)$-form on $\Omega \subset \mathbf{R}_m$ with a singularity on a submanifold $\Sigma \subset \Omega$, i.e. for a form $\omega = D\sigma.f$, where f is a monogenic function on $\Omega \setminus \Sigma$, we can define the residue $\mathrm{res}_{\,\Sigma}\, \omega$ by

$$\mathrm{res}_{\,\Sigma}\, \omega = \int_{\delta\Sigma} \omega = \int_{\Sigma} \mathrm{Res}_{\,\Sigma}\, \omega \in \mathbf{S}^+.$$

Such a generalization of pointwise residues was introduced and studied in [5, 18]. There are two main reasons why this definition of the residue is not fully satisfactory. Firstly, the residue is a number (i.e. an element in $\mathbf{S}^+$, hence it carries a very small information on the behaviour of the form ω near a higher -dimensional singularity submanifold. Secondly, it is a definition which is not suitable for generalization to the curved situation. Indeed, if a form ω is a monogenic $(m-1)$-form with a pointwise singularity on a spin manifold (for a definition seee below), a natural generalization given by

$$res_{\,P}\, \omega = \int_{\delta P} \omega$$

has no sense (integration of a form with values in a nontrivial bundle is not well-defined).

An interesting and useful way how to define a more general notion of residue was first introduced in a flat situation by F.Somme(see e.g. [5, 18]). Due to non-commutativity of Clifford-valued functions, a suitable formulation of the Cauchy theorem in Clifford analysis is the property that if f, g are two Clifford-valued functions and if D denotes the Dirac operator, then the form $f.D\sigma.g$ is closed iff $f.D = 0 = D.g$ (i.e. f, resp. g, are right, resp. left monogenic). Then f can be considered as a "test" function and the residue can be defined for a (left) monogenic form $\omega = d\sigma.f$ with a pointwise singularity in a point P as a functional $\mathrm{Rs_P}$ on the space of all (right) monogenic functions in a neighborhood of P given by

$$\mathrm{Rs_P\,(f)} = \int_{\delta \mathrm{P}} \mathrm{f}\omega.$$

The residue $\mathrm{Rs_P}$ carries much more comprehensive information on the behaviour near the singularity (in the simplest case of complex analysis it amounts to a knowledge of all coefficients in the Laurent series of the function with a pointwise singularity). It is also possible to prove (using a quite bit of analysis) that for "flat" singularities, it is possible to relate this residue to the corresponding coefficients in suitably generalized Laurent series. An important feature of this approach is that the product $f\omega$ is (in spinor-valued case,where the product is substituted by a Hermitean scalar product) an ordinary form with valued in $\mathbf{C}$. Hence the integral of it over a manifold is well defined. It makes possible to generalize the definition of the residue Rs_{P} for monogenic forms on manifolds.

We are describing below this type of residue in full details and we prove analogues of basic facts known for residues of meromorphic forms on Riemannian surfaces (the residue theorem, the sum of residues on a closed surface is zero). The whole theory is applicable for more general forms than those defined in [18, 6, 5]. The necessary property needed is a version of the Cauchy theorem, the discussion of allowed possibilities is given in Section 2. Then the residue is defined and the main properties are proved (Sect.3.).

2 The Cauchy theorem for monogenic forms

Let us first recall what the definition of monogenic forms is. Let (M, g) be an oriented Riemannian manifold of dimension m. Let us choose a spin structure on M, i.e. let us suppose that we have chosen a principal fibre bundle $\tilde{P}$ over M with the group $G = \mathrm{Spin(m)}$ together with the corresponding 2:1 covering map $\tilde{P} \to P$ onto the bundle P of oriented orthonormal frames. The Levi-Civita connection on P induces then a covariant derivative ∇ on the spinor bundle $S^+ = \tilde{P} \times_{Spin} \mathbf{S}^+$ associated to the basic half-spinor representation $\mathbf{S}^+$ (in odd dimension there is only one basic irreducible representation $\mathbf{S}^+ \simeq \mathbf{S}$).

Let us denote the space of smooth S^+-valued forms of degree j on M by $\mathcal{E}^j(S^+)$. The covariant derivative ∇ maps $\mathcal{E}^0(S^+)$ to $\mathcal{E}^1(S^+)$ and can be extended to the maps $\nabla : \mathcal{E}^k(S^+) \mapsto \mathcal{E}^{k+1}(S^+)$ for all $k = 1, \ldots, n-1$ (for more details see [21]).

An invariant Hermitian scalar product on $\mathbf{S}^+$ induces the Hermitian product $< ., . >_x$, in each fiber $\mathrm{S}^+{}_x, x \in M$, so that we can define a map $< ., . >$ from $\mathcal{E}^j(S^+) \times \mathcal{E}^k(S^+)$ into the space $\mathcal{E}^{j+k}$ of $\mathbf{C}$-valued forms by

$$< \omega \otimes s, \omega' \otimes s' >_x = \omega \wedge \omega' < s, s' >_x, \omega \in \mathcal{E}^j{}_x, \omega \in \mathcal{E}^k{}_x; s, s' \in \mathrm{S}^+{}_x;\ x \in M.$$

The covariant derivative ∇ induced by the Levi-Civita connection is compatible with the Hermitean structure, i.e. we have

$$d < \omega, \tau > = < \nabla\omega, \tau > + (-1)^j < \omega, \nabla\tau >;\ \omega \in \mathcal{E}^j(S^+), \tau \in \mathcal{E}^k(S^+)$$

(for 0-forms it is proved e.g. in [13] and it can be checked that it is true for general forms).

The definition of monogenic differential forms is based on a choice of a splitting of S^+-valued k-forms, $k = 1, \ldots, m-1$, on M into two parts

$$\mathcal{E}^k(S^+) = \mathcal{E}^{k'} \oplus \mathcal{E}^{k''}. \qquad (2)$$

Using it, we get the diagram

$$\begin{array}{ccccccccc} & \overset{d''}{\nearrow} & \mathcal{E}^{1''} & \longrightarrow & \mathcal{E}^{2''} & \cdots & \mathcal{E}^{(m-1)''} & \searrow & \\ \mathcal{E}^0 & & \oplus & \overset{d''}{\times} & \oplus & & \oplus & & \mathcal{E}^m \\ & \searrow & \mathcal{E}^{1'} & \longrightarrow & \mathcal{E}^{2'} & \cdots & \mathcal{E}^{(m-1)'} & \overset{d''}{\nearrow} & \end{array}$$

The operators d'' are defined as the composition of d with the projection onto $\mathcal{E}^{k''}$. Monogenic forms are defined then as elements of the kernels of the operators d''. The space of all monogenic-k forms will be denoted by $\mathcal{M}^k(S^+)$.

The definition clearly depends on a choice of the splitting (2), which is far from being unique. It is necessary (and possible) to consider several requirements retricting the choice substantially.

The most important requirement is the invariance with respect to Spin group. Spinor-valued differential forms on a spin-manifold M are section of the bundle $\Lambda^*(T_c^*) \otimes S^+$, where T_c^* is the complexified cotangent bundle. It is the vector bundle associated to the representation $\Lambda^*(\mathbf{C}_m^*) \otimes \mathbf{S}^+$.

The representation $\Lambda^j(\mathbf{C}_m^*) \otimes \mathbf{S}^+ \cong \Lambda^{m-j}(\mathbf{C}_m^*) \otimes \mathbf{S}^+, j = 1, \ldots, [m/2]$ is not irreducible, but it decomposes into $j+1$ irreducible parts

$$\mathbf{E}_{\mu_0} \oplus \ldots \oplus \mathbf{E}_{\mu_j}, \tag{3}$$

the summands in the decomposition being characterized by their highest weights $\mu_l, l = 0, \ldots, j$ (for details see [7, 19]). Let us denote the part $\mathbf{E}_{\mu_l}$ in the decomposition of the product $\Lambda^j(\mathbf{C}_m^*) \otimes \mathbf{S}^+$, resp. $\Lambda^{m-j}(\mathbf{C}_m^*) \otimes \mathbf{S}^+$, for $j = 1, \ldots, [m/2]$ by $\mathbf{E}^{j,l}$, resp. $\mathbf{E}^{m-j,l}$; the associated bundles will be denoted by $\mathcal{E}^{j,l}$, resp. $\mathcal{E}^{m-j,l}$.

The splitting (2) is invariant if and only if the both pieces are sums of $\mathcal{E}^{k,j}$. But it still leaves a finite number of different possibilities for the splitting.

The next requirement which restricts possibilities further is the need to have an analogue of the Cauchy theorem. In its standard setting (see e.g. [4, 6]), it says that the product $\omega = fD\sigma g$ is closed, where $D\sigma$ is the standard Clifford algebra valued volume form of degree $m-1$ defined by (1) and f, resp. g are a left (resp. right) monogenic functions.

A suitable generalization of the Cauchy theorem to our setting is the following one.

Definition 1 *Let $\mathcal{M}^k$ and $\widetilde{\mathcal{M}^k}$ be subspaces of $\mathcal{E}^k(S^+)$; $k = 1, \ldots, m-1$. We say that the sequences $\{\mathcal{M}^k\}_{k=0}^m$ and $\{\widetilde{\mathcal{M}^k}\}_{k=0}^m$ are complementary to each other, if*

$$d(\omega \wedge \tau) = 0$$

for all $\omega \in \mathcal{M}^k, \tau \in \mathcal{M}^{m-k-1}$.

Complementary spaces of forms are generalizations of the space $\mathcal{M}^0(S^+)$ of left monogenic functions, resp. of the space $\widetilde{\mathcal{M}}^{n-1}(S^+)$ of all forms given by $\omega = D\sigma g$, where g is a right monogenic function and the property that the product of any couple of forms in complementary dimensions is necessarily closed is a generalization of the classical Cauchy theorem in Clifford analysis (see [4, 6]).

Note that the property of being complementary implies a bound on the size of the subspaces $\mathcal{M}^k$ and $\widetilde{\mathcal{M}^k}$; it is impossible, for example, to set both of them equal to $\mathcal{E}^k(S^+)$. On the other hand, it gives no restriction from below; we could, of course, define $\mathcal{M}^k = \widetilde{\mathcal{M}^k} = 0$.

In the flat case, an additional criterion preventing that and giving a lower bound on the size of spaces $\mathcal{M}^k$ and $\widetilde{\mathcal{M}^k}$ was the condition that the restriction of the de Rham complex to $\{\mathcal{M}^k\}$, resp. $\{\widetilde{\mathcal{M}^k}\}$, gives the standard homology of a domain under consideration. It was shown (see [18, 6]) that there exists complementary spaces of forms both having this property; in the construction given there $\mathcal{M}^k = \widetilde{\mathcal{M}^k}$ for all k with only one exception (m odd, $k = m/2$). The residue theory, however, can be formulated for any pair of complementary spaces of forms, so we shall do it.

To construct some interesting examples, let us note that for each $\omega \in \mathcal{E}^{k,i}$ and $\tau \in \mathcal{E}^{m-k,j}, i \neq j$, we have

$$< \omega, \tau >= 0. \tag{4}$$

This is the consequence of the fact that different pieces in the decomposition (3) are not isomorphic as Spin modules, hence they are orthogonal to each other with respect to the invariant scalar product.

Hence we can prove the following lemma:

Lemma 1 *Let $\{\mathcal{M}^k\}_{k=0}^m$ and $\{\widetilde{\mathcal{M}^k}\}_{k=0}^m$ are two sequences of subspaces of the space $\mathcal{E}^k(S^+)$ which are defined by two splittings*

$$\mathcal{E}^k(S^+) = \mathcal{E}^{k'} \oplus \mathcal{E}^{k''},\ \tilde{\mathcal{E}}^k(S^+) = \tilde{\mathcal{E}}^{k'} \oplus \tilde{\mathcal{E}}^{k''},$$

which are invariant with respect to the action of the Spin group (i.e. $\mathcal{E}^{k'}$, resp. $\tilde{\mathcal{E}}^{k'}$ are sums of irreducible parts in the decomposition (3)). Suppose further that the spaces $\mathcal{E}^{k'}$ and $\tilde{\mathcal{E}}^{(m-k)'}$ have no common irreducible pieces.

Then the sequences $\{\mathcal{M}^k\}_{k=0}^m$ and $\{\widetilde{\mathcal{M}^k}\}_{k=0}^m$ are complementary to each other.

Proof.

The definition of the complementary spaces $\mathcal{M}^k$ and $\widetilde{\mathcal{M}^k}$ implies that for all forms $\omega \in \mathcal{M}^j, \tau \in \widetilde{\mathcal{M}}^{m-j-1}$, we have $\nabla\omega \in \mathcal{E}^{(j+1)'}$ and $\nabla\tau \in \tilde{\mathcal{E}}^{(m-j)'}$. The assumptions of the lemma hence imply that $< \nabla\omega, \tau >= 0, < \omega, \nabla\tau >= 0$. The compatibility of the connection ∇ then implies that the form $< \omega, \tau >$ is closed. ∎

Example.

As an illustration, let us consider the case of dimension $m = 7$. We shall consider the identical splittings $\mathcal{E}^{k'} = \widetilde{\mathcal{E}^{k'}}$ shown in the picture below (the numbers indicate the dimension of $E^{k,j}$ counted in multiples of dim S^+, the pieces, belonging to $\mathcal{E}^{k'}$, are indicated by boxes. In the top row, the dimension of the full spaces $\mathcal{E}^k$ is written.

1	7	21	35	35	21	7	1
$\boxed{1}$	1	$\boxed{1}$	1	$\boxed{1}$	1	$\boxed{1}$	1
	$\boxed{6}$	6	$\boxed{6}$	6	$\boxed{6}$	6	
		$\boxed{14}$	14	$\boxed{14}$	14		
			$\boxed{14}$	14			

The condition of the lemma above is clearly satisfied, so the sequences of spaces $\mathcal{M}^k = \widetilde{\mathcal{M}^k}$ are complementary to each other.

3 The residue theory

As discussed in the introduction, we want now to generalize the residue Rs_Σ to the case of monogenic forms on manifolds.

Definition 2 *Let M be an oriented Riemannian manifold with a chosen spin structure and let Σ be a compact oriented submanifold of M. Let us supppose that $\dim M = m,\ \dim \Sigma = j;\ j \in \{0, \ldots, m-2\}$.*

Let $\mathcal{M}^k$ and $\widetilde{\mathcal{M}^k}$ be subspaces of $\mathcal{E}^k(S^+); k = 1, \ldots, m-1$ such that the sequences $\{\mathcal{M}^k\}_{k=0}^m$ and $\{\widetilde{\mathcal{M}^k}\}_{k=0}^m$ are complementary to each other.

Let us define the space $\widetilde{\mathcal{M}}_k(\Sigma)$ of monogenic forms of degree k on Σ as the set of all form which are restrictions of monogenic k-forms from a neighborhood U of Σ, i.e.

$$\widetilde{\mathcal{M}}_k(\Sigma) = \lim \mathrm{ind}_{U \supset \Sigma} \widetilde{\mathcal{M}}_k(U).$$

Then for every form $\omega \in \mathcal{M}^j(M \setminus \Sigma)$, the functional $\mathrm{Rs}_\Sigma\, \omega \in \widetilde{\mathcal{M}}^{\mathrm{m-j-1}}(\Sigma)'$ given by

$$\mathrm{Rs}_\Sigma\, \omega[\tau] := \int_{\delta\Sigma} < \omega, \tau > = \int_\Sigma \mathrm{Res}_{\,\Sigma} (< \omega, \tau >);\ \tau \in \widetilde{\mathcal{M}}^{\mathrm{m-j-1}}(\Sigma),$$

will be called the grand residue of ω at Σ.

Let us prove now the two basic theorem on the grand residue which give analogues of the residue theorem and the theorem stating that the sum of residues of a holomorphic differential on a compact Riemann surface is equal to zero.

Theorem 1 (Residue theorem)

Let M be an oriented Riemannian manifold of dimension m with a given spin structure and let $\Sigma_i, i = 1, \ldots, l$, be a finite set of compact oriented submanifolds of M, with $1 \leq \dim \Sigma_i \leq m-2; i = 1, \ldots, l$.

Let $\mathcal{M}^k$ and $\widetilde{\mathcal{M}^k}$ be subspaces of $\mathcal{E}^k(S^+), k = 1, \ldots, m-1$ such that the sequences $\{\mathcal{M}^k\}_{k=0}^m$ and $\{\widetilde{\mathcal{M}^k}\}_{k=0}^m$ are complementary to each other.

Let $\omega \in \mathcal{M}^j(M \setminus (\cup_{i=1}^l \Sigma_i)$. Then for every compact oriented submanifold $M' \subset M$ of dimension m with boundary $\partial M'$ (with the induced orientation) such that $\Sigma_i \subset (M' \setminus \partial M'), i = 1, \dots l$ and for every test form $\nu \in \widetilde{\mathcal{M}}^{n-1-j}(M)$ we have

$$\int_{\partial M'} < \omega, \nu >= \sum_{i=1}^{l} \mathrm{Rs}_{\Sigma_j}\, \omega(\nu).$$

Proof.

Let us consider sufficiently small tubular neighbourhoods $U_i \subset M'$ of submanifolds $\Sigma_i, i = 1 \dots, k$ such that their closures are pairwise disjoint and contained in M'. Then the boundary $\partial M''$ of the manifold $M'' = M' \setminus \left(\cup_{i=1}^l U_i\right)$ consists of $\partial M'$ (with the induced orientations) and ∂U_i, $i = 1, \dots, l$ (equipped with the orientations opposite to the induced ones). The assumptions of the theorem imply that the form $< \omega, \nu >$ is closed, hence the Stokes theorem gives us that

$$\begin{aligned}\int_{\partial M'} < \omega, \nu > &= \sum_{i=1}^{l} \int_{\partial U_i} < \omega, \nu >= \sum_{i=1}^{l} \int_{\delta \Sigma_i} < \omega, \nu >= \\ &= \sum_{i=1}^{l} \mathrm{Rs}_{\Sigma_j}\, \omega(\nu).\end{aligned}$$

■

Theorem 2 *Let M be a compact oriented Riemannian manifold of dimension m with a given spin structure and let $\Sigma_i, i = 1, \dots, l$ be a finite set of compact oriented submanifolds of M, $1 \leq \dim \Sigma_i \leq m - 2$; $i = 1, \dots, l$.*

Let $\mathcal{M}^k$ and $\widetilde{\mathcal{M}}^k$ be subspaces of $\mathcal{E}^k(S^+), k = 1, \dots, m-1$ such that the sequences $\{\mathcal{M}^k\}_{k=0}^m$ and $\{\widetilde{\mathcal{M}}^k\}_{k=0}^m$ are complementary to each other. Let us consider a differential form $\omega \in \mathcal{M}^j(M \setminus (\cup_{i=1}^l \Sigma_i)$.

Then for every test form $\nu \in \widetilde{\mathcal{M}}^{m-1-j}(M)$ we have

$$\sum_{i=1}^{l} \mathrm{Rs}_{\Sigma_i}\, \omega(\nu) = 0.$$

Proof.

Let $U_i \subset M$ be sufficiently small tubular neighbourhoods of submanifolds $\Sigma_i, i = 1 \dots, l$ such that their closures are pairwise disjoint. Then the boundary $\partial M'$ of the manifold $M' = M \setminus \left(\cup_{i=1}^l U_i\right)$ consists of l pieces ∂U_i, $i = 1, \dots, l$ (equipped with the orientations opposite to the induced ones). Due to the fact that the form $< \omega, \nu >$ is closed, we have

$$0 = \int_{\partial M'} < \omega, \nu >= \sum_{i=1}^{l} \int_{\partial U_i} < \omega, \nu >=$$

$$= \sum_{i=1}^{l} \int_{\delta\Sigma_i} < \omega, \nu > = \sum_{i=1}^{l} \mathrm{Rs}_{\Sigma_i} \omega(\nu).$$

■

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References

[1] R.Baston, M.Eastwood: *The Penrose Transform: its interaction with representation theory,* Oxford Univ. Press, Oxford, 1989

[2] N.Berline, E.Getzler, M.Vergne: *Heat kernels and Dirac operators,* Springer, 1992

[3] B.Booß-Bavnbek, K.P.Wojciechowski: Elliptic Boundary Problems for Dirac operators, Birkhäuser, Basel, 1993

[4] F.Brackx, R.Delanghe, F.Sommen: *Clifford analysis,* Research Notes in Math. 76, Pitman, Boston, 1982

[5] R.Delanghe, F.Sommen, V.Souček: Residues in Clifford analysis, in H. Begehr, A. Jeffrey (Eds.): Partial differential equations with complex analysis, Pitman Research Notes in Math. 262, Longman, 1992, 61-92

[6] R.Delanghe, F.Sommen, V.Souček: 1992, *Clifford Algebra and Spinor-Valued Functions,* Mathematics and its Applications 53, Kluwer, Dordrecht

[7] R.Delanghe, V.Souček: On the structure of spinor-valued differential forms, Complex Variables, 18, 1992, 223-236

[8] R.Fueter: Analytische Funktionen einer Quaternionen-variable, Comm. Math.Helv. 4, 1932, 9-20

[9] R. Fueter: Die Funktionentheorie der Differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit vier Reallen Variablen, Com.Math.Helv. I, 1934-35, 307-330

[10] J.E.Gilbert, M.A.M.Murray: *Clifford algebras and Dirac operators in harmonic analysis,* Cambridge University Press, Cambridge, 1991

[11] K. Gürlebeck, W. Sprössig: *Quaternionic analysis and elliptic boundary value problems,* Math. Research 56, Akademie-Verlag, Berlin, 1989, 253 pp.

[12] N.J.Hitchin: Harmonic spinors, Adv. in Math., 14, 1974, 1-55,

[13] H.B.Lawson,jr., M.-L.Michelsohn: *Spin Geometry,* Princeton Univ. Press, Princeton, 1989

[14] J.Leray: Le calcul differentiel et intégral sur une variéte analytique complexe (Probleme de Cauchy III), Bull. Soc. Math. France, 87(1959), 81-180

[15] G.C.Moisil: Sur les systémes d'equations de M. Dirac, du type elliptique, Comptes Rendus des séances de l'Académie des Sciences, 191, 1930, p.1292

[16] F.Norguet: Sur la théorie des résidus, C. R. Acad. Sci., sér.A, 248, 14(1959), 2057-2059

[17] M.W.Hirsch: *Differential topology,* GTM 33, Springer, New York 1976

[18] F.Sommen, V.Souček: Monogenic differential forms, to be published in Applicable Analysis

[19] V.Souček: Monogenic forms on manifolds, in Z.Oziewicz et al. (Eds.): Spinors, Twistors, Clifford Algebras and Quantum Deformations, Kluwer, 1993, 159-166

[20] N.Théodoresco: Quelques pas dans une théorie des fonctions de variable complexe au sens général, deux Notes dans les Rendiconti della R. Acc. Naz. dei Lincie, XI, 1930

[21] R.O.Wells, jr.: *Differential analysis on complex manifolds*, Prentice Hall, 1973

[22] G.Zöll: Regular n-forms in Clifford analysis, their behaviour under change of variables and their residues, Complex Variables, 11(1989), 25-38

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