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PROPERTIES OF PRODUCT PRESERVING FUNCTORS

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0. Introduction.

In this paper we collect properties of product preserving functors. A product preserving functor is a covariant functor $\mathcal{F}$ from the category of manifolds into the category of fibered manifolds such that $\mathcal{F}(M_1 \times M_2)$ is equivalent to $\mathcal{F}(M_1) \times \mathcal{F}(M_2)$. The tangent bundle, the tangent bundle of p^r-velocities are well-known examples of product preserving functors. The most important and general examples of product preserving functors are so-called *Weil functors*.

Let $\mathbb{R}[p] = \mathbb{R}[[X_1, ..., X_p]]$ be the algebra of all formal power series of p indeterminates $X_1, ..., X_p$ and let $\mathfrak{a}$ be an ideal of $\mathbb{R}[p]$ such that $\dim \mathbb{R}[p]/\mathfrak{a} < \infty$. The algebra $A = \mathbb{R}[p]/\mathfrak{a}$ defines a product preserving functor T^A called *Weil functor*. If M is a manifold, then $T^A M$ is the set of equivalence classes of smooth mappings $\mathbb{R}^p \to M$, where $\varphi, \varphi' : \mathbb{R}^p \to M$ are equivalent if and only if for every smooth function $f : M \to \mathbb{R}$ the formal Taylor series at 0 of $f \circ \varphi$ and $f \circ \varphi'$ are equal modulo $\mathfrak{a}$.

G. Kainz, P. Michor [6], O. O. Luciano [11] and D. J. Eck [2] have given characterization of product preserving functors (see also [9]). Namely, they have proved

[0]) This paper is in final form and no versionof it will be submitted for publication elsewhere.

that any product preserving functor $\mathcal{F}$ is equivalent to T^A with some algebra A. It is an answer for the Morimoto's conjecture [13].

Usually properties of product preserving functors are proved as follows: firstly they are proved for Weil functors of type T^A and next they are extended to an arbitrary product preserving functor using the classification theorem of Kainz, Michor, Luciano and Eck. In this paper we will prove all presented properties of product preserving functor $\mathcal{F}$ directly from the functoriality of $\mathcal{F}$.

We suppose always that all manifolds, mappings, vector fields and so on are of class C^∞.

1. Weil algebra associated with a product preserving functor.

First we recall the definition.

A product preserving functor is a covariant functor $\mathcal{F}$ from the category of all manifolds and all smooth mappings into the category of fibered manifolds satisfying the following conditions:

(1) For every manifold M, the space $\mathcal{F}(M)$ is a fibered manifold over M with a projection $\pi = \pi_M : \mathcal{F}M \longrightarrow M$. For a point $x \in M$ we denote by $\mathcal{F}_x(M) = \pi_M^{-1}(x)$ the fibre over x.

(2) *The naturality condition.* For every mapping $\varphi : M \to N$ of two manifolds M, N, for the induced mapping $\mathcal{F}(\varphi) : \mathcal{F}(M) \to \mathcal{F}(N)$ the following diagram

$$\begin{array}{ccc} \mathcal{F}(M) & \xrightarrow{\mathcal{F}(\varphi)} & \mathcal{F}(N) \\ {\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{\varphi} & N \end{array}$$

commutes.

(3) If $\varphi : M \to N$ is an embedding, where $\dim M = \dim N$, then for every $x \in M$ the restriction $\mathcal{F}(\varphi)_{|\mathcal{F}_x(M)} : \mathcal{F}_x(M) \to \mathcal{F}_{\varphi(x)}(N)$ is a diffeomorphism. [1]

(4) *The regularity condition.* If $\varphi_t : M \to N$ is a differentiable family of mappings, then $\mathcal{F}(\varphi_t) : \mathcal{F}(M) \to \mathcal{F}(N)$ is a differentiable family of mappings.

(5) For two manifolds M_1, M_2, if $\pi_i : M_1 \times M_2 \to M_i$ denotes the standard

[1]) This condition is equivalent to so-called *the locality condition* saying: if $\varphi_1, \varphi_2 : M \to N$ are two smooth mappings such that $\varphi_1|U = \varphi_2|U$ for an open subset $U \subset M$, then $\mathcal{F}(\varphi_1)|\pi_M^{-1}(U) = \mathcal{F}(\varphi_2)|\pi_M^{-1}(U)$.

projection on the i-th factor, where $i = 1, 2$, then the mapping

$$(\mathcal{F}(\pi_1), \mathcal{F}(\pi_2)) : \mathcal{F}(M_1 \times M_2) \longrightarrow \mathcal{F}(M_1) \times \mathcal{F}(M_2)$$

is a diffeomorphism.

I. Kolář and J. Slovák have proved that the regularity condition is a consequence of conditions (1) and (2) of the definition (see [10]). Let us observe that for every fixed natural number n the restriction of a product preserving functor to the category of n-dimensional manifolds and their embeddings is *a natural bundle* (see [16]).

The definition immediately implies:

(1) if $U \subset M$ is an open subset then we can identify $\mathcal{F}(U)$ with $\mathcal{F}(M)_{|U}$ by $\mathcal{F}(i) : \mathcal{F}(U) \to \mathcal{F}(M)_{|U}$, where $i : U \to M$ is the inclusion;
(2) $\mathcal{F}(\mathbb{R}^n)$ is isomorphic with the trivial bundle $\mathbb{R}^n \times F$, where $F = \mathcal{F}_0(\mathbb{R}^n)$. The isomorphism $\Psi : \mathbb{R}^n \times F \to \mathcal{F}(\mathbb{R}^n)$ is given by $\Psi(x, y) = \mathcal{F}(\tau_x)(y)$, where $\tau_x : \mathbb{R}^n \to \mathbb{R}^n$ is the translation.
(3) every product preserving functor transforms immersions, submersions and embeddings into immersions, submersions and embeddings respectively (see [10]).

To see (3) we observe that $f : M \to N$ is a submersion (respectively an immersion) if and only if for any point $x \in N$ (respectively $x \in M$) there exists $J : N \to M$ (respectively $J : M \to N$) such that $f \circ J = id$ (respectively $J \circ f = id$) on some neighborhood U of x. By the functoriality of $\mathcal{F}$ we obtain $\mathcal{F}(f) \circ \mathcal{F}(J) = id$ (respectively $\mathcal{F}(J) \circ \mathcal{F}(f) = id$) over U.

For a product preserving functor $\mathcal{F}$ we will always identify $\mathcal{F}(M_1 \times M_2)$ with $\mathcal{F}(M_1) \times \mathcal{F}(M_2)$ by the diffeomorphism from the definition. After this identification we have

$$\mathcal{F}(f_1 \times f_2) = \mathcal{F}(f_1) \times \mathcal{F}(f_2) \tag{1.1}$$

$$\mathcal{F}(f, g) = (\mathcal{F}(f), \mathcal{F}(g)) \tag{1.2}$$

for all mappings $f_1 : M_1 \to N_1$, $f_2 : M_2 \to N_2$, $f : M \to N_1$ and $g : M \to N_2$.

From the definition we obtain that a product preserving functor $\mathcal{F}$ has *the point-property*, i.e. $\mathcal{F}(\text{point}) = \text{point}$. This implies that for a constant mapping $\varphi : M \to N$ the induced mapping $\mathcal{F}(\varphi)$ is also constant.

The tangent bundle TM and the tangent bundle of p^r-velocities $T^r_p M = J^r_0(\mathbb{R}^p, M)$ (see [13], [14]) are important examples of product preserving functors. The most general examples of product preserving functors are so-called *Weil functors* (see [15]). We give some remarks on these functors on the end of the section. At first we prove:

PROPOSITION 1.1. *If $\mathcal{F}$ is a product preserving functor then $A = \mathcal{F}(\mathbb{R})$ is a real, associative, commutative and finite dimensional algebra.*

If $+, \cdot : \mathbb{R}^2 \to \mathbb{R}$ are the addition and the multiplication on $\mathbb{R}$ and $m_\alpha : \mathbb{R} \to \mathbb{R}$ is the multiplication by $\alpha \in \mathbb{R}$, then $\mathcal{F}(+)$, $\mathcal{F}(\cdot)$, $\mathcal{F}(m_\alpha)$ are the operations in A, $\mathcal{F}(0)$ and $\mathcal{F}(1)$ are the zero and the unity in A (²).

The set $N = \mathcal{F}_0(\mathbb{R})$ is the ideal of nilpotent elements of A. We have $A = \mathbb{R}\cdot 1 \oplus N$.

PROOF: $A = \mathcal{F}(\mathbb{R})$ is an algebra by the functoriality of $\mathcal{F}$. For instance, to show the associativity of $\mathcal{F}(+)$ we apply $\mathcal{F}$ to the formula $+ \circ (+ \times id) = + \circ (id \times +)$.

To prove the properties of N we observe that the restriction of $\mathcal{F}$ to the category of 1-dimensional manifolds is a natural bundle, and by [17] it is of finite order h. Let $q(t) = t + t^{h+1}$. Since $j_0^h q = j_0^h id$, thus for $a \in N$ we have

$$a + a^{h+1} = \mathcal{F}_0(q)(a) = a\,.$$

It implies $a^{h+1} = 0$. □

The algebra $A = \mathcal{F}(\mathbb{R})$ constructed in Proposition 1.1 is called *Weil algebra* of a product preserving functor $\mathcal{F}$.

Let us observe that natural transformations of product preserving functors are determined by their values on their Weil algebras. We recall that for two product preserving functors $\mathcal{F}, \mathcal{G}$ *a natural transformation* of $\mathcal{F}$ into $\mathcal{G}$ is a family of smooth mappings $\Psi_M : \mathcal{F}(M) \to \mathcal{G}(M)$ such that the following diagram

$$\begin{array}{ccc} \mathcal{F}(M) & \xrightarrow{\Psi_M} & \mathcal{G}(M) \\ {\scriptstyle \pi_M^{\mathcal{F}}}\downarrow & & \downarrow{\scriptstyle \pi_M^{\mathcal{G}}} \\ M & \xrightarrow{id_M} & M \end{array}$$

commutes and for each smooth mapping $f : M \to N$ the diagram

$$\begin{array}{ccc} \mathcal{F}(M) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(N) \\ {\scriptstyle \Psi_M}\downarrow & & \downarrow{\scriptstyle \Psi_N} \\ \mathcal{G}(M) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(N) \end{array} \tag{1.3}$$

also commutes. We can prove the following proposition:

²) We always identify constant mappings with their values.

PROPOSITION 1.2. *Let $\mathcal{F}, \mathcal{G}$ be two product preserving functors and let $A = \mathcal{F}(\mathbb{R})$ and $B = \mathcal{G}(\mathbb{R})$ be their Weil algebras.*

If $\Psi = \{\Psi_M\}$ is a natural transformation of $\mathcal{F}$ into $\mathcal{G}$, then $\Psi_{\mathbb{R}} : A \to B$ is a homomorphism of algebras.

If $\Psi = \{\Psi_M\}$ and $\Psi' = \{\Psi'_M\}$ are two natural transformations of $\mathcal{F}$ into $\mathcal{G}$ such that $\Psi_{\mathbb{R}} = \Psi'_{\mathbb{R}}$, then $\Psi = \Psi'$.

If $\psi : A \to B$ is a homomorphism of algebras, then there is one and only one natural transformation $\Psi = \{\Psi_M\}$ of $\mathcal{F}$ into $\mathcal{G}$ such that $\Psi_{\mathbb{R}} = \psi$.

If $\Psi = \{\Psi_M\}$ is a natural transformation of $\mathcal{F}$ into $\mathcal{G}$ such that $\Psi_{\mathbb{R}} : A \to B$ is an isomorphism (respectively a monomorphism, an epimorphism), then for each manifold M the mapping Ψ_M is a diffeomorphism (respectively an embedding, a surjective submersion).

PROOF: Let $\Psi = \{\Psi_M\}$ be a natural transformation.

At first we observe that from (1.3) applying to the natural projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ we obtain

$$\begin{array}{ccc}
\mathcal{F}(M_1 \times M_2) & \xrightarrow{(\mathcal{F}(\pi_1),\mathcal{F}(\pi_2))} & \mathcal{F}(M_1) \times \mathcal{F}(M_2) \\
\Psi_{M_1\times M_2}\downarrow & & \downarrow \Psi_{M_1}\times\Psi_{M_2} \\
\mathcal{G}(M_1 \times M_2) & \xrightarrow{(\mathcal{G}(\pi_1),\mathcal{G}(\pi_2))} & \mathcal{G}(M_1) \times \mathcal{G}(M_2)
\end{array}$$

It means that after the identification $\mathcal{F}(M_1 \times M_2)$ with $\mathcal{F}(M_1) \times \mathcal{F}(M_2)$ we have

$$\Psi_{M_1\times M_2} = \Psi_{M_1} \times \Psi_{M_2}\,. \tag{1.4}$$

Now from (1.4) and (1.3) applying to $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ we obtain the commutative diagrams

$$\begin{array}{ccc}
A \times A & \xrightarrow{\mathcal{F}(+)} & A \\
\Psi_{\mathbb{R}}\times\Psi_{\mathbb{R}}\downarrow & & \downarrow\Psi_{\mathbb{R}} \\
B \times B & \xrightarrow{\mathcal{G}(+)} & B
\end{array}
\qquad
\begin{array}{ccc}
A \times A & \xrightarrow{\mathcal{F}(\cdot)} & A \\
\Psi_{\mathbb{R}}\times\Psi_{\mathbb{R}}\downarrow & & \downarrow\Psi_{\mathbb{R}} \\
B \times B & \xrightarrow{\mathcal{G}(\cdot)} & B
\end{array}$$

which means that $\Psi_{\mathbb{R}}$ is a homomorphism of algebras.

If $\Psi = \{\Psi_M\}$ and $\Psi' = \{\Psi'_M\}$ are two natural transformation of $\mathcal{F}$ into $\mathcal{G}$ such that $\Psi_{\mathbb{R}} = \Psi'_{\mathbb{R}}$, then by (1.4) we have $\Psi_{\mathbb{R}^n} = \Psi'_{\mathbb{R}^n}$. Using an atlas on M we deduce that $\Psi_M = \Psi'_M$ for each manifold M.

Let $\psi : A \to B$ be a homomorphism of algebras. According to the previous part it is sufficient to show the existence of a natural transformation $\Psi = \{\Psi_M\}$ such that $\Psi_{\mathbb{R}} = \psi$.

We set $\Psi_{\mathbb{R}^n} = \psi \times \cdots \times \psi$ (n times). Next we verify that the diagram (1.3) commutes for every mapping $f : \mathbb{R}^n \to \mathbb{R}^m$. To see this without loss of generality we suppose $m = 1$. Since $\mathcal{F}$ is locally of a finite order (see [12]) we can assume that f is a polynomial $f(x) = \sum a_\alpha x^\alpha$, $x \in \mathbb{R}^n$. Then according to the definitions of the operations in A an B we deduce that $\mathcal{F}(f) : A^n \to A$ is given by $\mathcal{F}(f)(x) = \sum a_\alpha x^\alpha$ and $\mathcal{G}(f) : B^n \to B$ by $\mathcal{G}(f)(x) = \sum a_\alpha x^\alpha$. Since ψ is an algebra homomorphism, then $\psi \circ \mathcal{F}(f) = \mathcal{G}(f) \circ (\psi \times \cdots \times \psi)$.

Now using an atlas we define Ψ_M for every manifold M such that the diagram (1.3) commutes.

If $\phi = \Psi_{\mathbb{R}}$ is an isomorphism, then this construction implies that Ψ_M is a diffeomorphism for each M. □

Now we consider an algebra $A = \mathbb{R} \cdot 1 \oplus N$, where N is an ideal of nilpotent elements. We can construct a product preserving functor such that its Weil algebra is isomorphic to A. In order of this we use a result of Weil [19] which says that for some natural number p, the algebra A is isomorphic with an algebra constructed as follows.

Let $\mathbb{R}[p] = \mathbb{R}[[X_1, ..., X_p]]$ be the algebra of all formal power series of p indeterminates $X_1, ..., X_p$ and let $\mathfrak{m}_p$ be the maximal ideal of $\mathbb{R}[p]$ containing all formal power series without constant terms. Let $\mathfrak{a}$ be an ideal of $\mathbb{R}[p]$ such that $\dim \mathbb{R}[p]/\mathfrak{a} < \infty$. The algebra $A = \mathbb{R}[p]/\mathfrak{a}$ has the unique maximal ideal $\mathfrak{m} = \mathfrak{m}_p/\mathfrak{a}$.

We construct a product preserving functor T^A.

Let $\xi_A : \mathbb{R}[p] \to A$ be the natural projection. We denote by $\tau : \mathcal{C}^\infty(\mathbb{R}^p) \to \mathbb{R}[p]$ the formal Taylor expansion at the origin $t = 0$, i.e. for $f \in \mathcal{C}^\infty(\mathbb{R}^p)$ we have

$$\tau(f) = \sum_\nu \frac{1}{\nu!}[(\frac{\partial}{\partial t})^\nu f]_{t=0} X^\nu .$$

Now we define an equivalence relation in the set $\mathcal{C}^\infty(\mathbb{R}^p, M)$ of smooth mappings $\mathbb{R}^p \to M$ (similar to the relation of jets) as follows: $\gamma, \gamma' : \mathbb{R}^p \to M$ are A-equivalent if

$$\xi_A(\tau(f \circ \gamma)) = \xi_A(\tau(f \circ \gamma'))$$

for every $f \in \mathcal{C}^\infty(M)$. We denote by $j^A\gamma$ the equivalence class of $\gamma : \mathbb{R}^p \to M$, by $T^A M$ the set of all equivalence classes and by $\pi_A : T^A M \to M$ the natural projection $\pi_A(j^A\gamma) = \gamma(0)$.

For a smooth mapping $\varphi : M \to N$ we define $T^A\varphi : T^AM \to T^AN$ by

$$T^A\varphi(j^A\gamma) = j^A(\varphi \circ \gamma).$$

If (U, φ) is a chart on M, then $(T^AU, T^A\varphi)$ is a chart on T^AM. It is easy to observe that T^A is a product preserving functor.

In 1986 Eck [2], Kainz, Michor [6] and Luciano [11] have proved independently that any product preserving functor is in fact equivalent to some Weil functor.

THEOREM 1.3. *If $\mathcal{F}$ is a product preserving functor, then there is an algebra $A = \mathbb{R}[p]/\mathfrak{a}$ such that $\mathcal{F}(M) = T^A(M)$ for every manifold M.*

In the paper we do not use the above theorem.

2. Product preserving functors and algebraic structures.

Product preserving functors have many interesting properties. In this section we transform manifolds with some algebraic structures as groups, vector spaces, algebras and so on by a product preserving functor.

We start from vector spaces. We have

PROPOSITION 2.1. *Let $\mathcal{F}$ be a product preserving functor and let $A = \mathcal{F}(\mathbb{R})$ be its Weil algebra.*

If V is a finite dimensional vector space, then $\mathcal{F}(V)$ is a finite dimensional vector space. If $+ : V \times V \to V$ is the sum mapping in V and $d_\alpha : V \to V$ is the multiplication by a scalar $\alpha \in \mathbb{R}$, then $\mathcal{F}(+) : \mathcal{F}(V) \times \mathcal{F}(V) \to \mathcal{F}(V)$ is the sum mapping in $\mathcal{F}(V)$ and $\mathcal{F}(d_\alpha) : \mathcal{F}(V) \to \mathcal{F}(V)$ is the multiplication by α in $\mathcal{F}(V)$. The zero of $\mathcal{F}(V)$ is $\mathcal{F}(0)$, where $0 : V \to V$ is the constant zero mapping.

If V is a finite dimensional vector space, then $\mathcal{F}(V)$ is an A-module. If $m : \mathbb{R} \times V \to V$ is the multiplication, then the induced mapping $\mathcal{F}(m) : A \times \mathcal{F}(V) \to \mathcal{F}(V)$ defines the action of A on $\mathcal{F}(V)$.

If $v_1, \dots, v_n$ is a basis of a vector space V, then $\mathcal{F}(v_1), \dots, \mathcal{F}(v_n)$ is a basis of the A-module $\mathcal{F}(V)$. Furthermore, if $a_1, \dots, a_K$ is a basis of A over $\mathbb{R}$, then all products $a_\nu\mathcal{F}(v_i)$, where $i = 1, \dots, n$ and $\nu = 1, \dots, K$, form a basis of $\mathcal{F}(V)$ over $\mathbb{R}$.

PROOF: Exactly as in the proof of Proposition 1.1 we verify that $\mathcal{F}(V)$ is a vector space and an A-module.

To show the last part of our proposition we apply $\mathcal{F}$ to the linear isomorphism

$$f : \mathbb{R}^n \ni (t_1, \dots, t_n) \longrightarrow \sum_{i=1}^{n} t_i v_i \in V.$$

We obtain a diffeomorphism $\mathcal{F}(f) : A^n \to \mathcal{F}(V)$. According to the definition of A-module structure on $\mathcal{F}(V)$ it is given by

$$\mathcal{F}(f)(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i \mathcal{F}(v_i).$$

Hence $\mathcal{F}(v_1), \dots, \mathcal{F}(v_n)$ is a basis of A-module $\mathcal{F}(V)$. Since the multiplication by scalars on $\mathcal{F}(V)$ is the restriction of the action of A on $\mathcal{F}(V)$, where $\mathbb{R}$ is contained in A via the inclusion $\mathbb{R} \ni t \to t \cdot 1 \in A = \mathbb{R} \cdot 1 + N$, thus $a_\nu \mathcal{F}(v_i)$, where $i = 1, \dots, n$ and $\nu = 1, \dots, K$, form a basis of $\mathcal{F}(V)$ over $\mathbb{R}$. □

If V is a vector space then $\mathcal{F}(V)$ is always considered as a vector space or as A-module with the structures defined in Proposition 2.1. For induced mappings by linear mappings we have

PROPOSITION 2.2. *Let $\mathcal{F}$ be a product preserving functor and let $A = \mathcal{F}(\mathbb{R})$ be its Weil algebra.*

If $f : V \to W$ is a linear mapping of two finite dimensional vector spaces, then $\mathcal{F}(f) : \mathcal{F}(V) \to \mathcal{F}(W)$ is also linear over A and over $\mathbb{R}$.

If $f : V_1 \times \dots \times V_k \to W$ is a k-linear mapping, then $\mathcal{F}(f) : \mathcal{F}(V_1) \times \dots \times \mathcal{F}(V_k) \to \mathcal{F}(W)$ is also k-linear over A and over $\mathbb{R}$.

PROOF: Let $m : \mathbb{R} \times V \to V$ be the multiplication by scalars and let $+ : V \times V$ be the sum in V. The linearity of f means that we have $f \circ m = m \circ (id_{\mathbb{R}} \times f)$ and $f \circ + = + \circ (f \times f)$. This implies that

$$\mathcal{F}(f) \circ \mathcal{F}(m) = \mathcal{F}(m) \circ (id_A \times \mathcal{F}(f))$$
$$\mathcal{F}(f) \circ \mathcal{F}(+) = \mathcal{F}(+) \circ (\mathcal{F}(f) \times \mathcal{F}(f)),$$

i.e. $\mathcal{F}(f)$ is linear over A and in consequence, linear over $\mathbb{R}$.

Analogously we verify the second part of the proposition. □

Next we prove properties of direct sums as well as kernels and images of induced mappings. Namely, we have

PROPOSITION 2.3. *Let $\mathcal{F}$ be a product preserving functor.*

If $V = U_1 \oplus U_2$ is a direct sum of subspaces U_1, U_2, then we have $\mathcal{F}(V) = \mathcal{F}(U_1) \oplus \mathcal{F}(U_2)$.

If $f : V \to W$ is linear, then we have

1) $$\ker \mathcal{F}(f) = \mathcal{F}(\ker f), \qquad \operatorname{im} \mathcal{F}(f) = \mathcal{F}(\operatorname{im} f)$$

PROOF: V is a direct sum of U_1 and U_2 means that the mapping $\Phi : U_1 \times U_2 \to V$ given by $\Phi(u_1, u_2) = u_1 + u_2$ is an isomorphism. According to Proposition 2.2 $\mathcal{F}(\Phi) : \mathcal{F}(U_1) \times \mathcal{F}(U_2) \to \mathcal{F}(V)$ is an isomorphism. By the definition of sum in $\mathcal{F}(V)$ we obtain that $\mathcal{F}(\Phi)$ is given by $\mathcal{F}(\Phi)(x_1, x_2) = x_1 + x_2$. It means that $\mathcal{F}(V) = \mathcal{F}(U_1) \oplus \mathcal{F}(U_2)$.

To show (2.1) we consider a subspace U such that $V = \ker f \oplus U$ and we denote by $i : \ker f \to V$ and $j : U \to V$ the inclusions. If we apply $\mathcal{F}$ to the equality $f \circ i = 0$ and to the isomorphism $f \circ j : U \to \operatorname{im} f$ we obtain $\mathcal{F}(f)_{|\mathcal{F}(\ker f)} = 0$ and the isomorphism $\mathcal{F}(f)_{|\mathcal{F}(U)} : \mathcal{F}(U) \to \mathcal{F}(\operatorname{im} f)$. This implies immediately (2.1). □

Now we formulate properties of prolongations of Lie groups and their actions on manifolds by a product preserving functor. Analogously we can prove:

PROPOSITION 2.4. *Let $\mathcal{F}$ be a product preserving functor.*

If G is a Lie group, then $\mathcal{F}(G)$ is also a Lie group. If $m : G \times G \to G$ is the product in G and 1 is the unit of G, then $\mathcal{F}(m)$ is the product in $\mathcal{F}(G)$ and $\mathcal{F}(1)$ is the unit of $\mathcal{F}(G)$.

If $f : G \to G'$ is a Lie group homomorphism, then $\mathcal{F}(f) : \mathcal{F}(G) \to \mathcal{F}(G')$ is a Lie group homomorphism. Particularly, if $H \subset G$ is a Lie subgroup, then $\mathcal{F}(H)$ is a Lie subgroup of $\mathcal{F}(G)$.

PROPOSITION 2.5. *Let $\mathcal{F}$ be a product preserving functor.*

If a Lie group G acts on a manifold M and $\Lambda : G \times M \to M$ is the action, then $\mathcal{F}(G)$ acts on $\mathcal{F}(M)$ and $\mathcal{F}(\Lambda) : \mathcal{F}(G) \times \mathcal{F}(M) \to \mathcal{F}(M)$ is the action.

In particular, if $ad : G \times G \to G$ is the adjoint action of G on G, then $\mathcal{F}(ad) : \mathcal{F}(G) \times \mathcal{F}(G) \to \mathcal{F}(G)$ is the adjoint action of $\mathcal{F}(G)$ on $\mathcal{F}(G)$.

If $\rho : GL(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n$ is the standard action, then $\mathcal{F}(\rho)$ gives an action of A-linear transformations on $\mathcal{F}(\mathbb{R}^n)$ and we have a Lie group monomorphism

$$I : \mathcal{F}(GL(\mathbb{R}^n)) \longrightarrow GL_A(\mathcal{F}(\mathbb{R}^n)) \subset GL(\mathcal{F}\mathbb{R}^n)$$

given by $I(X)(y) = \mathcal{F}(\rho)(X, y)$ for $X \in \mathcal{F}(GL(\mathbb{R}^n))$ and $y \in \mathcal{F}(\mathbb{R}^n)$.

PROOF: The unique nonstandard part of the proof is the injectivity of I. Applying $\mathcal{F}$ to

$$\mathbb{R}^{n^2} \times \mathbb{R}^n \supset GL(\mathbb{R}^n) \times \mathbb{R}^n \xrightarrow{\ \rho\ } \mathbb{R}^n$$

we obtain that the induced mapping

$$A^{n^2} \times A^n = \mathcal{F}(\mathbb{R}^{n^2}) \times \mathcal{F}(\mathbb{R}^n) \supset \mathcal{F}(GL(\mathbb{R}^n)) \times \mathcal{F}(\mathbb{R}^n) \xrightarrow{\ \mathcal{F}(\rho)\ } \mathcal{F}(\mathbb{R}^n)$$

is given by $\mathcal{F}(\rho)([x^i_j], (x^k)) = (\sum_{j=1}^n x^i_j\, x^j)$. It implies the injectivity of I. □

We finish this section by remarks on Lie algebras. Using the standard methods we obtain

PROPOSITION 2.6. *Let $\mathcal{F}$ be a product preserving functor.*

If $\mathfrak{g}$ is a Lie algebra, then $\mathcal{F}(\mathfrak{g})$ is also a Lie algebra with the Lie bracket $\mathcal{F}([\ ,\ ])$, where $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the Lie bracket in $\mathfrak{g}$.

If $f : \mathfrak{g} \to \mathfrak{g}'$ is a Lie algebra homomorphism, then $\mathcal{F}(f) : \mathcal{F}(\mathfrak{g}) \to \mathcal{F}(g')$ is also a Lie algebra homomorphism.

In Section 4 we will verify

PROPOSITION 2.7. *Let $\mathcal{F}$ be a product preserving functor. If G is a Lie group and $\mathcal{L}(G)$ is its Lie algebra, then there exists a canonical Lie algebra isomorphism $\eta_G : \mathcal{F}(\mathcal{L}(G)) \to \mathcal{L}(\mathcal{F}(G))$.*

3. Product preserving functors and fibered manifolds.

Let $\mathcal{F}$ be a product preserving functor. If $\pi : Y \to X$ is a fibered manifold, i.e. π is a surjective submersion, then $\mathcal{F}(\pi)$ is also a surjective submersion. Therefore, $\mathcal{F}(\pi) : \mathcal{F}(Y) \to \mathcal{F}(X)$ is a fibered manifold. In particular cases of vector bundles and principal fibre bundles we can show very interesting properties. We start with vector bundles. We have

PROPOSITION 3.1. *Let $\mathcal{F}$ be a product preserving functor.*

If $\pi : E \to M$ is a vector bundle, then $\mathcal{F}(\pi) : \mathcal{F}(E) \to \mathcal{F}(M)$ is a vector bundle too. If V is the standard fibre of E and $\varphi : E_{|U} \to U \times V$ is a trivialization over an open subset $U \subset M$, then $\mathcal{F}(V)$ is the standard fibre of $\mathcal{F}(E)$ and $\mathcal{F}(\varphi) : \mathcal{F}(E)_{|\mathcal{F}(U)} \to \mathcal{F}(U) \times \mathcal{F}(V)$ is a trivialization over $\mathcal{F}(U) \subset \mathcal{F}(M)$.

If $\Psi : E \to E'$ is a vector bundle homomorphism, then $\mathcal{F}(\Psi) : \mathcal{F}(E) \to \mathcal{F}(E')$ is also a vector bundle homomorphism.

Let $E_1, \dots, E_k$ be vector bundles over the same base M and E be a vector bundle over N. If $\Psi : E_1 \times_M \cdots \times_M E_k \to E$ is a k-linear mapping covering $\psi : M \to N$ ([3]), then $\mathcal{F}(\Psi) : \mathcal{F}(E_1) \times_{\mathcal{F}(M)} \cdots \times_{\mathcal{F}(M)} \mathcal{F}(E_k) \to \mathcal{F}(E)$ is a k-linear mapping covering $\mathcal{F}(\psi) : \mathcal{F}(M) \to \mathcal{F}(N)$.

PROOF: Let $\varphi_i : E_{|U_i} \to U_i \times \mathbb{R}^k$ for $i = 1,2$ be two trivializations of E. Then $\varphi_1 \circ \varphi_2^{-1} : (U_1 \cap U_2) \times \mathbb{R}^k \to (U_1 \cap U_2) \times \mathbb{R}^k$ is given by $(\varphi_1 \circ \varphi_2^{-1})(x, v) = (x, L(x)\, v)$,

[3]) It means that for each point $x \in M$ Ψ transforms $(E_1)_x \times \cdots \times (E_k)_x$ into $E_{\psi(x)}$ and $\Psi_x = \Psi_{|(E_1)_x \times \cdots \times (E_k)_x} : (E_1)_x \times \cdots \times (E_k)_x \to E_{\psi(x)}$ is k-linear.

where $L(x)$ is a matrix. It implies that

$$\mathcal{F}(\varphi_1)\circ\mathcal{F}(\varphi_2)^{-1} : (\mathcal{F}(U_1)\cap\mathcal{F}(U_2))\times\mathcal{F}(\mathbb{R}^k)\to(\mathcal{F}(U_1)\cap\mathcal{F}(U_2))\times\mathcal{F}(\mathbb{R}^k)$$

is given by $(\mathcal{F}(\varphi_1)\circ\mathcal{F}(\varphi_2)^{-1})(x,v)=(x,\mathcal{F}(L)(x)\,v)$. Let us observe that by Proposition 2.5 $\mathcal{F}(L)(x)\in GL(\mathcal{F}(\mathbb{R}^k))$.

Similarly we prove other parts. □

Similarly as in Proposition 2.3 we deduce

PROPOSITION 3.2. *Let $\mathcal{F}$ be a product preserving functor.*

If $E=E_1\oplus E_2$ is a direct sum of two vector bundles E_1 and E_2, then $\mathcal{F}(E)=\mathcal{F}(E_1)\oplus\mathcal{F}(E_2)$.

If $f : E\to E'$ is a vector bundle homomorphism such that the function $x\to\dim\ker f_x$ is constant on the base of E (4), then

$$\ker\mathcal{F}(f)=\mathcal{F}(\ker f),\qquad \operatorname{im}\mathcal{F}(f)=\mathcal{F}(\operatorname{im} f).$$

In the case of principal fibre bundles we have:

PROPOSITION 3.3. *Let $\mathcal{F}$ be a product preserving functor.*

If $P(M,G,\pi)$ is a principal fibre bundle with base M, structure group G and projection π, then $\mathcal{F}(P)(\mathcal{F}(M),\mathcal{F}(G),\mathcal{F}(\pi))$ is a principal fibre bundle with base $\mathcal{F}(M)$, structure group $\mathcal{F}(G)$ and projection $\mathcal{F}(\pi)$. If $\varphi : P_{|U}\to U\times G$ is a trivialization over U, then $\mathcal{F}(\varphi):\mathcal{F}(P)_{|\mathcal{F}(U)}\to\mathcal{F}(U)\times\mathcal{F}(G)$ is a trivialization over $\mathcal{F}(U)$.

If $f : P(M,G)\to P'(M',G')$ is a homomorphism of principal fibre bundles covering $\varphi : M\to M'$ with an induced Lie group homomorphism $\rho_f : G\to G'$, then $\mathcal{F}(f):\mathcal{F}(P)\to\mathcal{F}(P')$ is a homomorphism of principal fibre bundles covering $\mathcal{F}(\varphi):\mathcal{F}(M)\to\mathcal{F}(M')$ and $\mathcal{F}(\rho_f):\mathcal{F}(G)\to\mathcal{F}(G')$ is the induced Lie group homomorphism.

The proof is standard.

In section 4 we will prove:

PROPOSITION 3.4. *Let $\mathcal{F}$ be a product preserving functor. There is a canonical monomorphism $I_M : \mathcal{F}(LM)\longrightarrow L(\mathcal{F}M)$ of principal fibre bundles, where LM denotes the linear frame bundle.*

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(4) This assumption gives a sufficient and necessary condition under which $\operatorname{im} f\subset E'$ and $\ker f\subset E$ are vector subbundles.

4. The commutativity of product preserving functors.

In this section we prove a very interesting property saying that for two product preserving functors $\mathcal{F}_1, \mathcal{F}_2$ there exists a natural diffeomorphim $\mathcal{F}_1(\mathcal{F}_2(M)) \to \mathcal{F}_2(\mathcal{F}_1(M))$. For particular functors $\mathcal{F}_1$ and $\mathcal{F}_2$ we can show some supplementary properties of this functor transformation.

We start our considerations from the following remarks:

Let $\mathcal{F}_1, \mathcal{F}_2$ be two product preserving functors and let $A_1 = \mathcal{F}_1(\mathbb{R}), A_2 = \mathcal{F}_2(\mathbb{R})$ be their Weil algebras. We denote by $\pi^1_M : \mathcal{F}_1(M) \to M$ and $\pi^2_M : \mathcal{F}_2(M) \to M$ the bundle projections for $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively.

The projection $\pi^1_{\mathbb{R}} : A_1 \to \mathbb{R}$ is an algebra homomorphism and in consequence the induced mapping

$$\mathcal{F}_2(\pi^1_{\mathbb{R}}) : \mathcal{F}_2(A_1) = \mathcal{F}_2(\mathcal{F}_1(\mathbb{R})) \to A_2$$

is also an algebra homomorphism. Since the projection $\pi^2_{A_1} : \mathcal{F}_2(\mathcal{F}_1(\mathbb{R})) \to A_1$ is an algebra homomorphism, thus

$$\rho^{\mathcal{F}_2,\mathcal{F}_1} = s_{A_1,A_2} \circ (\pi^2_{A_1}, \mathcal{F}_2(\pi^1_{\mathbb{R}})) : \mathcal{F}_2(\mathcal{F}_1(\mathbb{R})) \longrightarrow A_1 \otimes A_2 \tag{4.1}$$

is an algebra homomorphism, where $s_{A_1,A_2} : A_1 \times A_2 \to A_1 \otimes A_2$ is the canonical bilinear mapping. We have the following proposition:

PROPOSITION 4.1. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two product preserving functors. If $A_1 = \mathcal{F}_1(\mathbb{R})$ and $A_2 = \mathcal{F}_2(\mathbb{R})$ are their Weil algebras, then the mapping $\rho^{\mathcal{F}_2,\mathcal{F}_1}$ defined by (4.1) is an algebra isomorphism.*

PROOF: Let $a^1_1, \ldots, a^1_{K_1}$ be a basis of A_1 and $a^2_1, \ldots, a^2_{K_2}$ be a basis of A_2. Now $\{ a^1_\nu \otimes a^2_\mu \ : \ \nu = 1, \ldots, K_1, \mu = 1, \ldots, K_2 \}$ is a basis of $A_1 \otimes A_2$. We can assume that $\pi^1_{\mathbb{R}}(a^1_\nu) = 1$ and $\pi^2_{\mathbb{R}}(a^2_\mu) = 1$.

Using the algebra monomorphism $i_1 : \mathbb{R} \to A_1$ given by $i_1(t) = t \cdot 1$ we obtain an inclusion $\mathcal{F}_2(i_1) : A_2 \to \mathcal{F}_2(\mathcal{F}_1(\mathbb{R}))$ and in consequence $\mathcal{F}_2(i_1)(a^2_\mu)$ belongs to $\mathcal{F}_2(\mathcal{F}_1(\mathbb{R}))$. On the other hand, identifying a^1_ν with the constant mapping $A_1 \to A_1$, the induced constant mapping $\mathcal{F}_2(a^1_\nu)$ is identified with an element of $\mathcal{F}_2(\mathcal{F}_1(\mathbb{R}))$.

Now we can verify

$$\begin{aligned}
&\rho^{\mathcal{F}_2,\mathcal{F}_1}(\mathcal{F}_2(a^1_\nu)\mathcal{F}_2(i_1)(a^2_\mu)) \\
&\quad = s_{A_1,A_2}\Big(\pi^2_{A_1}(\mathcal{F}_2(a^1_\nu)\mathcal{F}_2(i_1)(a^2_\mu)), \mathcal{F}_2(\pi^1_{\mathbb{R}})(\mathcal{F}_2(a^1_\nu)\mathcal{F}_2(i_1)(a^2_\mu))\Big) \\
&\quad = s_{A_1,A_2}(\pi^2_{A_1}(\mathcal{F}_2(a^1_\nu)), \mathcal{F}_2(\pi^1_{\mathbb{R}})(\mathcal{F}_2(i_1)(a^2_\mu))) \\
&\quad = s_{A_1,A_2}(a^1_\nu, a^2_\mu) \\
&\quad = a^1_\nu \otimes a^2_\mu \,.
\end{aligned}$$

because $\pi^2_{A_1}(\mathcal{F}_2(i_1)(a^2_\mu)) = 1$ and $\mathcal{F}_2(\pi^1_{\mathbb{R}})(\mathcal{F}_2(a^1_\nu)) = 1$ It implies that $\rho^{\mathcal{F}_2,\mathcal{F}_1}$ is an epimorphism. In consequence $\rho^{\mathcal{F}_2,\mathcal{F}_1}$ is an isomorphism because $\dim \mathcal{F}_2(\mathcal{F}_1(\mathbb{R})) = \dim A_1 \otimes A_2$. □

Propositions 4.1 and 1.2 imply

THEOREM 4.2. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two product preserving functors. If $A_1 = \mathcal{F}_1(\mathbb{R})$ and $A_2 = \mathcal{F}_2(\mathbb{R})$ are their Weil algebras, then there is one and only one family $\eta_M : \mathcal{F}_2(\mathcal{F}_1(M)) \to \mathcal{F}_1(\mathcal{F}_2(M))$ of diffeomorphisms such that*

(1) *for every manifold M the following diagram*

$$\begin{array}{ccc} \mathcal{F}_2(\mathcal{F}_1(M)) & \xrightarrow{\eta_M} & \mathcal{F}_1(\mathcal{F}_2(M)) \\ {\scriptstyle \pi^{\mathcal{F}_1}_M \circ \pi^{\mathcal{F}_2}_{\mathcal{F}_1(M)}} \downarrow & & \downarrow {\scriptstyle \pi^{\mathcal{F}_2}_M \circ \pi^{\mathcal{F}_1}_{\mathcal{F}_2(M)}} \\ M & \xrightarrow{id_M} & M \end{array} \tag{4.2}$$

commutes;

(2) *for every smooth mapping $\varphi : M \to N$ the diagram*

$$\begin{array}{ccc} \mathcal{F}_2(\mathcal{F}_1(M)) & \xrightarrow{\mathcal{F}_2(\mathcal{F}_1(\varphi))} & \mathcal{F}_2(\mathcal{F}_1(N)) \\ {\scriptstyle \eta_M} \downarrow & & \downarrow {\scriptstyle \eta_N} \\ \mathcal{F}_1(\mathcal{F}_2(M)) & \xrightarrow{\mathcal{F}_1(\mathcal{F}_2(\varphi))} & \mathcal{F}_1(\mathcal{F}_2(N)) \end{array} \tag{4.3}$$

commutes.

(3) *if $\rho^{\mathcal{F}_2,\mathcal{F}_1} : \mathcal{F}_2(\mathcal{F}_1(\mathbb{R})) \to A_1 \otimes A_2$ and $\rho^{\mathcal{F}_1,\mathcal{F}_2} : \mathcal{F}_1(\mathcal{F}_2(\mathbb{R})) \to A_2 \otimes A_1$ are the algebra isomorphisms defined by (4.1), then*

$$\eta_{\mathbb{R}} = (\rho^{\mathcal{F}_1,\mathcal{F}_2})^{-1} \circ \alpha \circ \rho^{\mathcal{F}_2,\mathcal{F}_1} : \mathcal{F}_2(\mathcal{F}_1(\mathbb{R})) \to \mathcal{F}_1(\mathcal{F}_2(\mathbb{R})), \tag{4.4}$$

where $\alpha : A_1 \otimes A_2 \to A_2 \otimes A_1$ is the algebra isomorphism verifying the condition $\alpha(a_1 \otimes a_2) = a_2 \otimes a_1$.

Furthermore, for two manifolds M, N we have $\eta_{M\times N} = \eta_M \times \eta_N$.

PROOF: Let us observe that $\eta_{\mathbb{R}}$ is an algebra isomorphism. Now by Proposition 1.2 the proof is finished. □

In the case of the tangent bundle $\mathcal{F}_1(M) = TM$ we have

PROPOSITION 4.3. *Let $\mathcal{F}$ be a product preserving functor. There exists one and*

$$\begin{array}{ccc} \mathcal{F}(TM) & \xrightarrow{\eta_M} & T(\mathcal{F}M) \\ \mathcal{F}(p_M)\searrow & & \swarrow p_{\mathcal{F}(M)} \\ & \mathcal{F}(M) & \end{array}$$

(where $p_M : TM \to M$ is the projection) such that the following conditions hold:

(1) *for every smooth mapping $\varphi : M \to N$ the following diagram*

$$\begin{array}{ccc} \mathcal{F}(TM) & \xrightarrow{\mathcal{F}(d\varphi)} & \mathcal{F}(TN) \\ {\scriptstyle \eta_M}\downarrow & & \downarrow{\scriptstyle \eta_N} \\ T\mathcal{F}(M) & \xrightarrow{d\mathcal{F}(\varphi)} & T\mathcal{F}(N) \end{array}$$

commutes;

(2) *if $\Psi_{\mathbb{R}} : T\mathbb{R} \to \mathbb{R} \times \mathbb{R}$ and $\Psi_{\mathcal{F}(\mathbb{R})} : T\mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R})$ are the standard trivializations, then*

$$\eta_{\mathbb{R}} : \mathcal{F}(T\mathbb{R}) \xrightarrow{\mathcal{F}(\Psi_{\mathbb{R}})} \mathcal{F}(\mathbb{R} \times \mathbb{R}) = \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \xrightarrow{\Psi^{-1}_{\mathcal{F}(\mathbb{R})}} T(\mathcal{F}(\mathbb{R})).$$

Furthermore, for two manifolds M, N we have $\eta_{M\times N} = \eta_M \times \eta_N$;

PROOF: It is sufficient to verify that in this case $\eta_{\mathbb{R}} = (\rho^{T,\mathcal{F}})^{-1} \circ \alpha \circ \rho^{\mathcal{F},T}$ and it is a vector bundle isomorphism covering the identity on $\mathcal{F}(\mathbb{R})$ and next we apply Theorem 4.2. □

Let G be a Lie group. We denote by $\mathcal{L}(G)$ the Lie algebra of G. For a Lie group homomorphism $f : G \to G'$ we denote by $\mathcal{L}(f) : \mathcal{L}(G) \to \mathcal{L}(G')$ the induced Lie algebra homomorphism. After identification $\mathcal{L}(G)$ with the tangent space T_eG, where e is the unit of G, the mapping $\mathcal{L}(f)$ is identified with the differential $d_ef : T_eG \to T_{e'}G'$. The functor $\mathcal{L}(G)$ is a product preserving functor from the category of Lie groups and their homomorphism into the category of Lie algebras and their homomorphisms. For this functor we have:

PROPOSITION 4.4. *Let $\mathcal{F}$ be a product preserving functor. If G is a Lie group and $\mathcal{L}(G)$ is its Lie algebra, then the restriction $(\eta_G)_{|\mathcal{F}(\mathcal{L}(G))} : \mathcal{F}(\mathcal{L}(G)) \to \mathcal{L}(\mathcal{F}(G))$ is a Lie algebra isomorphism, where η_G is from Proposition 4.3.*

The restriction $(\eta_G)_{|\mathcal{F}(\mathcal{L}(G))}$ will be denoted also by η_G.

Before the proof of this proposition we recall the definitions and properties of lifts of vector fields to a product preserving functor $\mathcal{F}$.

The standard example is so-called *complete lift.* If $X : M \to TM$ is a vector field on M, then we define $X^C = \eta_M \circ \mathcal{F}(X)$, where η_M is the isomorphism from Proposition 4.3. If φ_t is a local flow of X, then $\mathcal{F}(\varphi_t)$ is a local flow of X^C. It implies that the complete lift has the following properties

$$(4.5) \qquad (\alpha X + \beta Y)^C = \alpha X^C + \beta Y^C, \qquad [X,Y]^C = [X^C, Y^C]$$

for all vector fields X, Y on M and all reals α, β.

In order to define other examples of lifts of vector fields to $\mathcal{F}$ we consider the mapping $\Psi : \mathbb{R} \times TM \to TM$ given by $\Psi(t,v) = t\,v$. Using the natural isomorphism $\eta_M : \mathcal{F}(TM) \longrightarrow T(\mathcal{F}M)$, the induced mapping $\mathcal{F}(\Psi) : A \times \mathcal{F}(TM) \to \mathcal{F}(TM)$ determines

$$\widetilde{\Psi} = \eta_M \circ \mathcal{F}(\Psi) \circ (id_A \times \eta_M^{-1}) : A \times T(\mathcal{F}M) \to T(\mathcal{F}M).$$

For an element $a \in A$ and a vector $\widetilde{v} \in T(\mathcal{F}M)$ we define

$$(4.6). \qquad a \cdot \widetilde{v} = \widetilde{\Psi}(a, \widetilde{v})$$

Now for a vector field X on M and an element $a \in A$ we define

$$(4.7) \qquad X^{(a)} = a \cdot X^C = \widetilde{\Psi}(a, X^C).$$

$X^{(a)}$ is a vector field on $\mathcal{F}(M)$ called *a-lift* of X. This a-lift was introduced by Kolář [8]. We have $X^C = X^{(1)}$, where 1 is the unit of A. These a-lifts have the following properties (see [5]):

(i) If X, Y are vector fields on M, α, β are reals and $a, b \in A$, then

$$\begin{aligned}(\alpha X + \beta Y)^{(a)} &= \alpha X^{(a)} + \beta Y^{(a)} \\ X^{(\alpha a + \beta b)} &= \alpha X^{(a)} + \beta X^{(b)} \\ [X^{(a)}, Y^{(b)}] &= [X,Y]^{(ab)}\end{aligned}$$

(ii) If X is a left invariant vector field on a Lie group G and $a \in A$, then $X^{(a)}$ is a left invariant vector field on $\mathcal{F}(G)$.

PROOF OF PROPOSITION 4.4: Let $E_1, \ldots, E_N$ be a basis of $\mathcal{L}(G)$ and $a_1, \ldots, a_K$ be a basis of $A = \mathcal{F}(\mathbb{R})$. By Proposition 2.1 $\mathcal{F}(E_1), \ldots, \mathcal{F}(E_N)$ is a basis of $\mathcal{F}(\mathcal{L}(G))$ over A. On the other hand by (ii) $E_j^{(a_\nu)}$ belongs to $\mathcal{L}(\mathcal{F}(G))$ for $j = 1, \ldots, N$ and

$\nu = 1, \dots, K$. Of course, $\eta_G(a\mathcal{F}(X)) = X^{(a)}$ for all $a \in A$ and $X \in \mathcal{L}(G)$. By Propositions 2.2 and 2.6

$$[a_\nu \mathcal{F}(E_j), a_\mu \mathcal{F}(E_i)] = a_\nu a_\mu [\mathcal{F}(E_j), \mathcal{F}(E_i)] = a_\nu a_\mu \mathcal{F}([E_j, E_i]).$$

Thus

$$\begin{aligned}\eta_G([a_\nu \mathcal{F}(E_j), a_\mu \mathcal{F}(E_i)]) &= \eta_G(a_\nu a_\mu \mathcal{F}([E_j, E_i])) \\ &= [E_j, E_i]^{(a_\nu a_\mu)} \\ &= [E_j^{(a_\nu)}, E_i^{(a_\mu)}] \\ &= [\eta_G(a_\nu \mathcal{F}(E_j)), \eta_G(a_\mu \mathcal{F}(E_i))].\end{aligned}$$

Since by Proposition 4.3 η_G is a linear isomorphism, thus the proof is finished. □

We finish this paper by the following proposition announced in Section 3.

PROPOSITION 4.5. *Let $\mathcal{F}$ be a product preserving functor, $A = \mathcal{F}(\mathbb{R})$ be its Weil algebra and let LM be the linear frame bundle over M.*

For every manifold M there exists one and only one monomorphism

$$I_M : \mathcal{F}(LM) \to L(\mathcal{F}(M))$$

of principal fibre bundles covering the identity on $\mathcal{F}(M)$ and with the inclusion $I : \mathcal{F}(GL(\mathbb{R}^n)) \to GL(\mathcal{F}(\mathbb{R}^n))$ given in Proposition 2.5 such that for each chart (U, φ) on M we have $I_M \circ \mathcal{F}(\sigma_\varphi) = \sigma_{\mathcal{F}(\varphi)}$, where $\sigma_\varphi : U \to LM$ and $\sigma_{\mathcal{F}(\varphi)} : \mathcal{F}(U) \to L(\mathcal{F}M)$ are local sections associated with φ and $\mathcal{F}(\varphi)$ respectively.

The family $\{I_M\}$ is natural, i.e. for every embedding $\varphi : M \to N$ of two n-dimensional manifolds M, N the diagram

$$\begin{array}{ccc} \mathcal{F}(LM) & \xrightarrow{I_M} & L(\mathcal{F}(M)) \\ {\scriptstyle \mathcal{F}(L(\varphi))}\downarrow & & \downarrow{\scriptstyle L(\mathcal{F}(\varphi))} \\ \mathcal{F}(LN) & \xrightarrow{I_M} & L(\mathcal{F}(N)) \end{array}$$

commutes, where $L(\varphi) : LM \to LM$ is the induced mapping.

PROOF: We choose the canonical mapping $K_M : LM \times \mathbb{R}^n \to TM$, $K_M(l, v) = l(v)$. Let us define $I_M : \mathcal{F}(LM) \to L(\mathcal{F}M)$ by

$$I_M(\bar{l})(\bar{v}) = (\eta_M \circ \mathcal{F}(K_M))(\bar{l}, \bar{v}), \tag{4.8}$$

where $\bar{l} \in \mathcal{F}(LM)$, $\bar{v} \in A^n$ and η_M is defined in Proposition 4.3. Since $K_M(l\,X, v) = K_M(l, X\,v)$ we obtain

$$\mathcal{F}(K_M)(\bar{l}\bar{X}, \bar{v}) = \mathcal{F}(K_M)(\bar{l}, \bar{X}\bar{v})$$

for $\bar{l} \in \mathcal{F}(LM)$, $\bar{X} \in \mathcal{F}(GL(\mathbb{R}^n)) \subset GL(\mathcal{F}(\mathbb{R}^n))$ and $\bar{v} \in A^n$, where the inclusion I is described in Proposition 2.5. Therefore

$$(I_M(\bar{l}\bar{X}))(\bar{v}) = (I_M(\bar{l}))(\bar{X}\bar{v}) = (I_M(\bar{l})\bar{X})(\bar{v}),$$

i.e. I_M is a principal fibre bundle homomorphism. Since the corresponding Lie group homomorphism is the inclusion I, I_M is a principal fibre bundle monomorphism.

If φ is a chart on M, then $K_M(\sigma_\varphi(x), v) = d\varphi^{-1}(\varphi(x), v)$ after the standard identification $T\mathbb{R}^n$ with $\mathbb{R}^n \times \mathbb{R}^n$. Using $\mathcal{F}$ and (4.8) we obtain

$$\begin{aligned} I_M(\mathcal{F}(\sigma_\varphi)(\bar{x}))(\bar{v}) &= \big(\eta_M \circ \mathcal{F}(K_M)\big)(\mathcal{F}(\sigma_\varphi)(\bar{x}), \bar{v}) \\ &= \big(\eta_M \circ \mathcal{F}(d\varphi^{-1})\big)(\mathcal{F}(\varphi)(\bar{x}), \bar{v}) \\ &= d\mathcal{F}(\varphi^{-1})(\mathcal{F}(\varphi)(\bar{x}), \bar{v}) \\ &= \sigma_{\mathcal{F}(\varphi)}(\bar{x})(\bar{v}). \end{aligned}$$

Thus $I_M \circ \mathcal{F}(\sigma_\varphi) = \sigma_{\mathcal{F}(\varphi)}$. □

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