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ON THE CLASSIFICATION OF ORIENTED VECTOR BUNDLES OVER 9-COMPLEXES

Martin Čadek, Jiří Vanžura

Abstract. Necessary and sufficient conditions on a CW-complex X of dimension ≤ 9 which allow to classify oriented stable vector bundles and oriented 9-dimensional vector bundles over X in terms of characteristic classes are presented. The complete classification of these bundles is described.

1. Introduction. In this paper we present a characterization of oriented stable vector bundles over a CW-complex of dimension ≤ 9 via their characteristic classes. Since oriented stable vector bundles over a CW-complex X are in one-to-one correspondence with homotopy classes in $[X, BSO]$, we can consider a mapping γ which assigns characteristic classes to every homotopy class from $[X, BSO]$. Now, our task is twofold: (i) To describe the image of γ. (ii) To find conditions under which γ is injective. In Theorem 1 we describe the image and give necessary and sufficient conditions for the injectivity of the mapping γ. In the corollary which follows we present an estimate for the number of oriented stable vector bundles with prescribed characteristic classes. Under some conditions which are not very restrictive they are at most two if X is a compact manifold.

Some results in this direction were already known. L. M. Woodward in [W], Theorem 3, solved the problems (i) and (ii) in the case $\dim X \leq 8$ with sufficient conditions for the injectivity of γ. For $\dim X \leq 9$ sufficient conditions for the injectivity of γ can also be obtained using [T2], Theorem 4.3. But these conditions require even more than our "stronger conditions"(A), (B), (C') and (D').

Along the same lines as above we can consider a CW-complex X with $\dim X \leq 8$ (resp. $\dim X \leq 9$) and investigate oriented vector bundles of dimension 8 (resp. 9) over X, i. e. the set of homotopy classes $[X, BSO(8)]$ (resp. $[X, BSO(9)]$). Again it is possible to introduce a mapping γ assigning characteristic classes to every element of $[X, BSO(8)]$ (resp. $[X, BSO(9)]$). In both these cases we describe the image of γ and give necessary and sufficient conditions for the injectivity of γ. These results are presented without proofs.

2. Notation and preliminaries. All vector bundles will be considered over a connected CW-complex X and will be oriented. The mappings $i_* : H^*(X; \mathbb{Z}_2) \to H^*(X; \mathbb{Z}_4)$ and $\rho_k : H^*(X; \mathbb{Z}) \to H^*(X; \mathbb{Z}_k)$ are induced from the inclusion $\mathbb{Z}_2 \to \mathbb{Z}_4$ and reduction mod k, respectively.

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We will use $w_s(\xi)$ for the s-th Stiefel–Whitney class of the vector bundle ξ, $p_s(\xi)$ for the s-th Pontrjagin class, and $e(\xi)$ for the Euler class. The classifying spaces for the groups SO and $SO(n)$ will be denoted by BSO and $BSO(n)$, respectively. The letters w_s and p_s will stand for the characteristic classes of the universal bundle over the classifying space BSO. Our results given below are based on the following relations among the characteristic classes

$$\rho_4 p_1(\xi) = \mathfrak{P} w_2(\xi) + i_* w_4(\xi), \tag{1}$$

$$\rho_4 p_2(\xi) = \mathfrak{P} w_4(\xi) + i_*\{w_8(\xi) + w_2(\xi)w_6(\xi)\}, \tag{2}$$

$$P_3^1 \rho_3(p_1(\xi)) = \rho_3(2p_2(\xi) - p_1^2(\xi)), \tag{3}$$

$$w_6(\xi) = Sq^2 w_4(\xi) + w_2(\xi)w_4(\xi), \tag{4}$$

where $\mathfrak{P}$ is the Pontrjagin square, a cohomology operation from $H^{2k}(X;\mathbb{Z}_2)$ into $H^{4k}(X;\mathbb{Z}_4)$, P_3^1 is the first Steenrod operation for $\mathbb{Z}_3$-cohomology from $H^k(X;\mathbb{Z}_3)$ into $H^{k+4}(X;\mathbb{Z}_3)$, and Sq^2 is the usual Steenrod square. See [BS], [T1] and [SE].

We say that $x \in H^*(X;\mathbb{Z})$ is an element of order r ($r = 2,3,4,\dots$) if and only if $x \neq 0$ and r is the least positive integer such that $rx = 0$ (if it exists).

The Eilenberg–MacLane space with n-th homotopy group G will be denoted $K(G,n)$ and $\iota_n(G)$ will stand for the fundamental class in $H^n(K(G,n);G)$. When writing only ι_n for the fundamental class it will be always clear which group G we have in mind.

In every fibration $\Omega K \to E \to B$ induced from the path fibration $\Omega K \to PK \to K$ we can consider the natural multiplication $m : \Omega K \times E \to E$.

In the proof of Theorem 1 we will need suspension. Being defined for every fibration $F \xrightarrow{j} E \xrightarrow{p} B$, it is a natural mapping from a subgroup of $H^{k+1}(B)$ into $H^k(F)/\text{im } j^*$ which commutes with the Steenrod squares and i_* (see [MT]).

3. Classification theorems. Our main result reads as

Theorem 1. *Let X be a connected CW-complex of dimension ≤ 9 and suppose*

$$\gamma : [X, BSO] \to H^2(X;\mathbb{Z}_2) \oplus H^4(X;\mathbb{Z}_2) \oplus H^8(X;\mathbb{Z}_2) \oplus H^4(X;\mathbb{Z}) \oplus H^8(X;\mathbb{Z})$$

is defined by $\gamma(\xi) = (w_2(\xi), w_4(\xi), w_8(\xi), p_1(\xi), p_2(\xi))$. Then

(i) *$\text{im } \gamma = \{(a,b,c,x,y) \mid \rho_4 x = \mathfrak{P}a + i_*b,\ \rho_4 y = \mathfrak{P}b + i_*(c + a^2b + aSq^2b),$ $P_3^1\rho_3 x = \rho_3(2y - x^2)\}$*

(ii) *γ is injective if and only if the following conditions are satisfied*

(A) $H^4(X;\mathbb{Z})$ has no element of order 4.

(B) $H^8(X;\mathbb{Z})$ has no element of order 4.

(C) $H^7(X;\mathbb{Z}_3) = \rho_3 H^7(X;\mathbb{Z}) + P_3^1\rho_3 H^3(X;\mathbb{Z})$.

(D) $H^9(X;\mathbb{Z}_2) = Sq^2 H^7(X;\mathbb{Z}_2) + Sq^4 Sq^2 H^3(X;\mathbb{Z}_2)$.

Remark. We shall introduce also stronger, but more practical conditions

(C') $H^8(X;\mathbf{Z})$ has no element of order 3.

(D') $H^9(X;\mathbf{Z}_2) = Sq^2H^7(X;\mathbf{Z}_2)$.

Obviously (C') implies (C) and (D') implies (D).

An important example of CW-complex which satisfies Condition (D') is an 9-dimensional oriented manifold M with $w_2(M) \neq 0$. The Poincaré duality and the fact that the second Wu class is equal to $w_2(M)$ yields

$$Sq^2H^7(M;\mathbf{Z}_2) = w_2(M)H^7(M;\mathbf{Z}_2) = H^9(M;\mathbf{Z}_2).$$

In the case of 9-complexes the application of the general result from [T2] requires stronger conditions for the injectivity of γ, namely $H^4(X;\mathbf{Z})$ and $H^8(X;\mathbf{Z})$ have no element of order 2, $H^8(X;\mathbf{Z})$ has no element of order 3 and $Sq^2\rho_2H^7(X;\mathbf{Z}) = H^9(X;\mathbf{Z}_2)$.

Corollary. *Let X be a connected CW-complex of dimension 9 satisfying Conditions (A), (B) and (C). Then for every element (a,b,c,x,y) in $im\ \gamma$ there are at most card $H^9(X;\mathbf{Z}_2)$ oriented stable vector bundles over M with the characteristic classes $w_2 = a$, $w_4 = b$, $w_8 = c$, $p_1 = x$ and $p_2 = y$. (card denotes the cardinality.)*

For completeness we present other two similar results on the classification of oriented n-plane bundles over an n-complex for $n = 8$ and 9. They can be obtained in an analogous fashion as Theorem 1 and so we restrict ourselves only to the proof of Theorem 1 and its Corollary.

Theorem 2. *Let X be a connected CW-complex of dimension ≤ 8 and suppose*

$$\gamma : [X, BSO(8)] \to H^2(X;\mathbf{Z}_2) \oplus H^4(X;\mathbf{Z}_2) \oplus H^8(X;\mathbf{Z}) \oplus H^4(X;\mathbf{Z}) \oplus H^8(X;\mathbf{Z})$$

is defined by $\gamma(\xi) = (w_2(\xi), w_4(\xi), e(\xi), p_1(\xi), p_2(\xi))$. Then

(i) $im\ \gamma = \{(a,b,e,x,y) \mid \rho_4 x = \mathfrak{P}a + i_*b,\ \rho_4 y = \mathfrak{P}b + 2\rho_4 e + i_*(a^2b + aSq^2b),\ P_3^1\rho_3 x = \rho_3(2y - x^2)\}$

(ii) *γ is injective if and only if Conditions (A), (B) and (C) are satisfied.*

Theorem 3. *Let X be a connected CW-complex of dimension ≤ 9 and suppose*

$$\gamma : [X, BSO(9)] \to H^2(X;\mathbf{Z}_2) \oplus H^4(X;\mathbf{Z}_2) \oplus H^8(X;\mathbf{Z}_2) \oplus H^4(X;\mathbf{Z}) \oplus H^8(X;\mathbf{Z})$$

is defined by $\gamma(\xi) = (w_2(\xi), w_4(\xi), w_8(\xi), p_1(\xi), p_2(\xi))$. Then

(i) $im\ \gamma = \{(a,b,c,x,y) \mid \rho_4 x = \mathfrak{P}a + i_*b,\ \rho_4 y = \mathfrak{P}b + i_*(c + a^2b + aSq^2b),\ P_3^1\rho_3 x = \rho_3(2y - x^2)\}$

(ii) *γ is injective if and only if Conditions (A), (B), (C), (D') together with (D") $H^9(X;\mathbf{Z}_2) = Sq^4Sq^2H^3(X;\mathbf{Z}_2)$ are satisfied.*

Proof of Theorem 1. Denote $K = K(\mathbf{Z}_2,2)\times K(\mathbf{Z}_2,4)\times K(\mathbf{Z}_2,8)\times K(\mathbf{Z},4)\times K(\mathbf{Z},8)$ and $C = K(\mathbf{Z}_4,4)\times K(\mathbf{Z}_4,8)\times K(\mathbf{Z}_3,8)$. Up to homotopy there is just one mapping $k = (k_1,k_2,k_3) : K \to C$ such that

$$k_1^*(\iota_4(\mathbf{Z}_4)) = 1\otimes 1\otimes 1\otimes \rho_4\iota_4\otimes 1 - \mathfrak{P}\iota_2\otimes 1\otimes 1\otimes 1\otimes 1 - 1\otimes i_*\iota_4\otimes 1\otimes 1\otimes 1$$

$$k_2^*(\iota_8(\mathbf{Z}_4)) = 1\otimes 1\otimes 1\otimes 1\otimes \rho_4\iota_8 - 1\otimes \mathfrak{P}\iota_4\otimes 1\otimes 1\otimes 1 - 1\otimes 1\otimes i_*\iota_8\otimes 1\otimes 1 - i_*(\iota_2\otimes Sq^2\iota_4)\otimes 1\otimes 1\otimes 1 - i_*(\iota_2^2\otimes \iota_4)\otimes 1\otimes 1\otimes 1$$

$$k_3^*(\iota_8(\mathbf{Z}_3)) = 1\otimes 1\otimes 1\otimes P_3^1\rho_3\iota_4\otimes 1 + 1\otimes 1\otimes 1\otimes \rho_3\iota_4^2\otimes 1 + 1\otimes 1\otimes 1\otimes 1\otimes \rho_3\iota_8.$$

Let us consider $k : K \to C$ as a fibration with a fibre F. Let $\alpha : BSO \to K$ be defined by $(w_2, w_4, w_8, p_1, p_2)$ as a fibration as well. Since $k\circ\alpha : BSO \to C$ is homotopic to zero, there is a mapping $l : BSO \to F$ such that the following diagram is commutative

$$\begin{array}{ccccc} F & \xrightarrow{\ j\ } & K & \xrightarrow{\ k\ } & C \\ & \nwarrow^{l} & \uparrow{\scriptstyle\alpha} & & \\ & & BSO & & \end{array}$$

We are going to show that l is a 9-equivalence. The long exact homotopy sequence for fibration $F \to K \to C$ gives immediately $\pi_4(F) = 0$, $\pi_2(F) = \mathbf{Z}_2$. Further,

$$0 \longrightarrow \pi_4(F) \xrightarrow{j_*} \pi_4(K) \cong \mathbf{Z}_2 \oplus \mathbf{Z} \xrightarrow{k_*} \pi_4(C) \cong \mathbf{Z}_4 \longrightarrow \pi_3(F) \longrightarrow 0.$$

Since $k_*(b, x) = \rho_4 x - i_* b$ is surjective, we get $\pi_3(F) = 0$, $\pi_4(F) \cong \mathbf{Z}$ and $j_*(z) = (\rho_2 z, 2z)$. Next $\pi_5(F) = 0$, $\pi_6(F) = 0$, and

$$0 \longrightarrow \pi_8(F) \xrightarrow{j_*} \pi_8(K) \cong \mathbf{Z}_2 \oplus \mathbf{Z} \xrightarrow{k_*} \pi_8(C) \cong \mathbf{Z}_4 \oplus \mathbf{Z}_3 \longrightarrow \pi_7(F) \longrightarrow 0.$$

Here $k_{2*}(c, y) = \rho_4 y - i_* c$, $k_{3*}(c, y) = \rho_3 y$ is surjective. Hence $\pi_7(F) \cong 0$, $\pi_8(F) \cong \mathbf{Z}$ and $j_*(z) = (\rho_2 z, 6z)$. Now we compare j_* with α_*

$\alpha_* : \pi_2(BSO) \to \mathbf{Z}_2$ is equal to $w_{2*} = id$
$\alpha_* : \pi_4(BSO) \to \mathbf{Z}_2 \oplus \mathbf{Z}$ is equal to $(w_{4*}, p_{1*}) : z \mapsto (\rho_2 z, -2z)$
$\alpha_* : \pi_8(BSO) \to \mathbf{Z}_2 \oplus \mathbf{Z}$ is equal to $(w_{8*}, p_{1*}) : z \mapsto (\rho_2 z, 6z)$

(see [BS], Th. 26.5 and [W] for p_{1*} and p_{2*}). That is why $l_* : \pi_i(BSO) \to \pi_i(F)$ is an isomorphism for $i \le 8$ and epimorphism for $i = 9$ ($\pi_9(F) \cong 0$, $\pi_9(BSO) \cong \mathbf{Z}_2$). It means that l is a 9-equivalence.

Now, let us consider the fibration $\Omega C \xrightarrow{j_1} E_1 \xrightarrow{\pi_1} K$ which is induced from the fibration $\Omega C \to PC \to C$ by the mapping k. E_1 is homotopy equivalent to F, and consequently there is a 9-equivalence $\beta_1 : BSO \to E_1$ such that $\alpha = \pi_1 \circ \beta_1$. In fact, E_1 is the first stage in the Postnikov tower for the fibration $\alpha : BSO \to K$, where the first invariant is given by the mapping $k : K \to C$. Denote the new mappings according to the following diagram.

$$\begin{array}{ccccccc} \bar{F}_1 & \longrightarrow & V & \xrightarrow{\ \bar{\beta}_1\ } & K(\mathbf{Z}_4,3)\times K(\mathbf{Z}_4,7)\times K(\mathbf{Z}_3,7) & & \\ & & \downarrow & & \downarrow{\scriptstyle j_1} & & \\ F_1 & \longrightarrow & BSO & \xrightarrow{\ \beta_1\ } & E_1 & & \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\pi_1} & & \\ & & K & = & K & \xrightarrow{\ k\ } & K(\mathbf{Z}_4,4)\times K(\mathbf{Z}_4,8)\times K(\mathbf{Z}_3,8) \end{array}$$

Consider β_1 as a fibration with a fibre F_1. This fibre is homotopy equivalent to the homotopy fibre $\bar{F}_1$ of the mapping $\bar{\beta}_1$ (see [T3]). Since β_1 is a 9-equivalence and $\pi_9(V) \cong \pi_9(BSO) \cong \mathbf{Z}_2$, F_1 is 8-connected and $\pi_9(F_1) \cong \mathbf{Z}_2$.

The next invariant $\varphi \in H^{10}(E_1, \mathbf{Z}_2)$ is the transgression of the generator of $H^9(F_1, \mathbf{Z}_2)$ in the Serre exact sequence for the fibration β_1. So, it is sufficient to find a non-zero element of $H^{10}(E_1, \mathbf{Z}_2)$ such that β_1^* vanishes on it. We will show that $\varphi = \pi_1^* v$, where $v = 1 \otimes 1 \otimes Sq^2\iota_8 \otimes 1 \otimes 1 + 1 \otimes Sq^4Sq^2\iota_4 \otimes 1 \otimes 1 \otimes 1 + 1 \otimes \iota_4 Sq^2\iota_4 \otimes 1 \otimes 1 \otimes 1 + Sq^1\iota_2 \otimes Sq^3\iota_4 \otimes 1 \otimes 1 \otimes 1 + \iota_2^2 \otimes Sq^2\iota_4 \otimes 1 \otimes 1 \otimes 1$. First we get

$$\begin{aligned}\beta_1^*\pi_1^* v = \alpha^* v &= Sq^2 w_8 + Sq^4Sq^2 w_4 + w_4 Sq^2 w_4 + Sq^1 w_2 Sq^3 w_4 + w_2^2 Sq^2 w_4 = \\ &= (Sq^2 w_8 + w_2 w_8) + (w_2 w_8 + Sq^4(w_2 w_4 + Sq^2 w_4) + \\ &\quad w_4(w_2 w_4 + Sq^2 w_4)) = w_{10} + (w_2 w_8 + Sq^4 w_6 + w_4 w_6) = \\ &= w_{10} + w_{10} = 0\end{aligned}$$

To prove that $\varphi \neq 0$ we find a mapping $H : \bar{E} \to E_1$ such that $H^*(\varphi) \neq 0$. Let $h : K(\mathbf{Z}_2, 8) \to K$ be the inclusion. We define $K(\mathbf{Z}_4, 7) \to \bar{E} \xrightarrow{\bar{\pi}} K(\mathbf{Z}_2, 8)$ as the principal fibration induced from the fibration $\Omega K(\mathbf{Z}_4, 8) \to PK(\mathbf{Z}_4, 8) \to K(\mathbf{Z}_4, 8)$ by the mapping $k_2 \circ h$. Since $(k_2 \circ h)^*(\iota_8(\mathbf{Z}_4)) = i_*\iota_8(\mathbf{Z}_2)$, the Serre exact sequence for the fibration $K(\mathbf{Z}_4, 7) \to \bar{E} \xrightarrow{\bar{\pi}} K(\mathbf{Z}_2, 8)$ yields that the transgression $\tau(\iota_7(\mathbf{Z}_4)) = i_*\iota_8(\mathbf{Z}_2)$ and $\tau(Sq^2\rho_2\iota_7(\mathbf{Z}_4)) = 0$, and consequently $\bar{\pi}^* Sq^2\iota_8(\mathbf{Z}_2) \in H^{10}(\bar{E}, \mathbf{Z}_2)$ is not zero. Further, we have $k \circ h \circ \bar{\pi} = 0$, and hence there is a mapping $H : \bar{E} \to E_1$ such that the following diagram is commutative.

$$\begin{array}{ccccc} \bar{E} & \xrightarrow{H} & E_1 & & \\ \downarrow{\scriptstyle \bar{\pi}} & & \downarrow{\scriptstyle \pi_1} & & \\ K(\mathbf{Z}_2, 8) & \xhookrightarrow{h} & K & \xrightarrow{k} & C \end{array}$$

Now we obtain

$$H^*(\varphi) = H^* \circ \pi_1^*(v) = \bar{\pi}^* \circ h^*(v) = \bar{\pi}^*(Sq^2\iota_8(\mathbf{Z}_2)) \neq 0.$$

Having the second invariant we can build the second stage E_2 of our Postnikov tower.

$$\begin{array}{ccccccc} \bar{F}_2 & \longrightarrow & F_1 & \xrightarrow{\bar{\beta}_2} & K(\mathbf{Z}_2, 9) & & \\ & & \downarrow & & \downarrow{\scriptstyle j_2} & & \\ F_2 & \longrightarrow & BSO & \xrightarrow{\beta_2} & E_2 & & \\ & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \pi_2} & & \\ & & E_1 & = & E_1 & \xrightarrow{\varphi} & K(\mathbf{Z}_2, 10) \end{array}$$

Let the notation of the new mappings accord with this diagram. We can consider β_2 to be a fibration with a fibre F_2. Similarly as for the first stage, we can compute the homotopy groups of F_2. We find that F_2 is 9-connected and β_2 is a 10-equivalence.

Up to homotopy there is just one mapping $p = (k, v) : K \to L = C \times K(\mathbf{Z}_2, 10)$. Due to Lemma 8.1 in [T4], there is a homeomorphism $g : E_2 \to E$, where $\pi : E \to K$ is a principal fibration with the classifying map $p : K \to L$. Moreover, $\pi_1 \circ \pi_2 = \pi \circ g$ and the fibration $\beta = g \circ \beta_2 : BSO \to E$ is a 10-equivalence. Hence, we can consider the following situation.

$$\begin{array}{ccccc} BSO & \xrightarrow{\beta = \text{10-equiv}} & E & & \\ \downarrow \alpha & & \downarrow \pi & & \\ K & = & K & \xrightarrow{p} & L \end{array}$$

(i) Let $f : X \to K$ be a mapping given uniquely up to homotopy by the conditions $f^*(\iota_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1) = a$, $f^*(1 \otimes \iota_4 \otimes 1 \otimes 1 \otimes 1) = b$, $f^*(1 \otimes 1 \otimes \iota_8 \otimes 1 \otimes 1) = c$, $f^*(1 \otimes 1 \otimes 1 \otimes \iota_4 \otimes 1) = x$, $f^*(1 \otimes 1 \otimes 1 \otimes 1 \otimes \iota_8) = y$. Over X there is an oriented stable vector bundle ξ with $w_2(\xi) = a$, $w_4(\xi) = b$, $w_8(\xi) = c$, $p_1(\xi) = x$, $p_2(\xi) = y$ if and only if $f : X \to K$ can be lifted in the fibration α. This is possible if and only if $(k \circ f)^* = 0$, which gives the condition in (i).

(ii) Since E is a homotopy fibre of the mapping $p : K \to L$, the Puppe sequence

$$\longrightarrow \Omega K \xrightarrow{\Omega p} \Omega L \xrightarrow{q} E \xrightarrow{\pi} K \xrightarrow{p} L$$

yields the exact sequence

$$\longrightarrow [X, \Omega K] \xrightarrow{(\Omega p)_*} [X, \Omega L] \xrightarrow{q_*} [X, E] \xrightarrow{\pi_*} [X, K] \xrightarrow{p_*} [X, L]$$

Moreover, β being a 10-equivalence, $\beta_* : [X, BSO] \to [X, E]$ is a bijection for every CW-complex X of dimension ≤ 9. This means that $\gamma = \pi_* \circ \beta_* : [X, BSO] \to [X, K]$ is injective if and only if $(\Omega p)_* : [X, \Omega K] \to [X, \Omega L]$ is surjective. Hence we need to prove that the mappings $(\Omega k_1)_*$, $(\Omega k_2)_*$, $(\Omega k_3)_*$ and $(\Omega v)_*$ are surjective if and only if the conditions (A), (B), (C) and (D) are satisfied.

The computation can be carried out in the same way as in [ČV] using diagrams of the following form

$$\begin{array}{ccc} \Omega K & \xrightarrow{\Omega k_1} & K(\mathbf{Z}_4, 3) \\ \downarrow & & \downarrow \\ PK & \xrightarrow{Pk_1} & PK(\mathbf{Z}_4, 4) \\ \downarrow & & \downarrow \\ K & \xrightarrow{k_1} & K(\mathbf{Z}_4, 4) \end{array}$$

and suspensions.

$$\begin{aligned} (\Omega k_1)^*(\iota_3) = &(\Omega k_1)^*(\sigma \iota_4) = \sigma(k_1^*(\iota_4)) = 1 \otimes 1 \otimes 1 \otimes \rho_4 \iota_3 \otimes 1 \\ &- i_* \iota_1^3 \otimes 1 \otimes 1 \otimes 1 \otimes 1 - 1 \otimes i_* \iota_3 \otimes 1 \otimes 1 \otimes 1. \end{aligned}$$

Hence $(\Omega k_1)_* : [X, \Omega K] \to [X, K(\mathbf{Z}_4, 3)] : (a, b, c, x, y) \mapsto \rho_4 x - i_* a^3 - i^* b$. That is why $(\Omega k_1)_*$ is surjective if and only if

$$H^3(X; \mathbf{Z}_4) = \rho_4 H^3(X; \mathbf{Z}) + i_* H^3(X; \mathbf{Z}_2),$$

which is equivalent to Condition (A).

Now consider the mapping k_2.

$$(\Omega k_2)^*(\iota_7) = (\Omega k_2)^*(\sigma\iota_8) = \sigma k_2^*(\iota_8) = 1\otimes 1\otimes 1\otimes 1\otimes \rho_4\iota_7 \\ - 1\otimes \sigma\mathfrak{P}\iota_4\otimes 1\otimes 1\otimes 1 - 1\otimes 1\otimes i_*\iota_7\otimes 1\otimes 1 \\ - i_*\sigma(\iota_2\otimes Sq^2\iota_4)\otimes 1\otimes 1\otimes 1 - i_*\sigma(\iota_2^2\otimes \iota_4)\otimes 1\otimes 1\otimes 1.$$

We have $\sigma\iota_4^2 = \sigma Sq^4\iota_4 = Sq^4\sigma\iota_4 = Sq^4\iota_3 = 0$. Then $\rho_2\sigma\mathfrak{P}\iota_4 = \sigma\rho_2\mathfrak{P}\iota_4 = \sigma\iota_4^2 = 0$ and $\sigma\mathfrak{P}\iota_4 \in i_*H^7(K(\mathbf{Z}_2,4);\mathbf{Z}_2)$. It yields that $(\Omega k_2)_*$ is surjective if and only if

$$H^7(X;\mathbf{Z}_4) = \rho_4 H^7(X;\mathbf{Z}) + i_*H^7(X;\mathbf{Z}_2),$$

which is equivalent to Condition (B).

Next we compute $(\Omega k_3)^*$.

$$(\Omega k_3)^*(\iota_7(\mathbf{Z}_3)) = (\Omega k_3)^*(\sigma\iota_8) = \sigma k_3^*(\iota_8) \\ = 1\otimes 1\otimes 1\otimes (\sigma P_3^1\rho_3\iota_4\otimes 1 + \rho_3\sigma\iota_4^2\otimes 1 + 1\otimes\rho_3\sigma\iota_8) \\ = 1\otimes 1\otimes 1\otimes (P_3^1\rho_3\iota_3\otimes 1 + \rho_3 a\otimes 1 + 1\otimes \rho_3\iota_7).$$

Hence $(\Omega k_3)_*$ is surjective if and only if

$$H^7(X;\mathbf{Z}_3) = \rho_3 H^7(X;\mathbf{Z}) + P_3^1\rho_3 H^3(X;\mathbf{Z}),$$

which is the condition (C).

It remains to compute $(\Omega v)^*$.

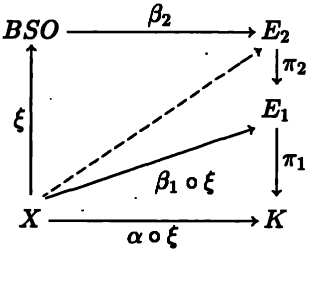
$$(\Omega v)^*(\iota_9) = (\Omega v)^*(\sigma\iota_{10}) = \sigma v^*(\iota_{10}) = 1\otimes 1\otimes \sigma Sq^2\iota_8\otimes 1\otimes 1 \\ + 1\otimes \sigma Sq^4Sq^2\iota_4\otimes 1\otimes 1\otimes 1 + 1\otimes\sigma(\iota_4 Sq^2\iota_4)\otimes 1\otimes 1\otimes 1 \\ + \sigma(Sq^1\iota_2\otimes Sq^3\iota_4)\otimes 1\otimes 1\otimes 1 + \sigma(\iota_2^2\otimes Sq^2\iota_4)\otimes 1\otimes 1\otimes 1 \\ = 1\otimes 1\otimes Sq^2\iota_7\otimes 1\otimes 1 + 1\otimes Sq^4Sq^2\iota_3\otimes 1\otimes 1\otimes 1.$$

The suspensions of products are equal to zero, which can be shown using the Serre spectral sequence. Consequently, $(\Omega v)_*$ is surjective if and only if

$$H^9(X;\mathbf{Z}_2) = Sq^2H^7(X;\mathbf{Z}_2) + Sq^4Sq^2H^3(X;\mathbf{Z}_2),$$

which is the condition (D). This completes the proof.

Proof of Corollary. Let ξ be an oriented stable vector bundle over a CW-complex X. We consider it as a mapping $\xi : X \to BSO$. Its Stiefel-Whitney and Pontrjagin classes are determined by the mapping $\alpha\circ\xi \in [X,K]$. Since $\beta_{2*} : [X,BSO] \to [X,E_2]$ is a bijection, the number of stable bundles with the same characteristic classes as ξ is given by the number of homotopy different liftings of $\alpha\circ\xi$ in the fibration $\pi_2\circ\pi_1 : E_2 \to K$.



Under Conditions (A), (B) and (C) there is just one lifting in the fibration $\pi_1 : E_1 \to K$. (The proof repeats the arguments given in the part (ii) of the previous proof.) This lifting is $\beta_1 \circ \xi$. All liftings of this mapping in the fibration $\Omega K(\mathbb{Z}_2, 10) \cong K(\mathbb{Z}_2, 9) \xrightarrow{j_2} E_2 \xrightarrow{\pi_2} E_1$ are $m(z, \beta_1 \circ \xi)$ where $z \in [X, K(\mathbb{Z}_2, 9)] \cong H^9(X; \mathbb{Z}_2)$. This shows that there are at most card $H^9(X; \mathbb{Z}_2)$ liftings.

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