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SOME EXAMPLES OF HOMOGENEOUS EINSTEIN MANIFOLDS

E. D. Rodionov

A Riemannian manifold (M, ρ) is called Einsteinian iff its Ricci tensor satisfies $Ric(\rho) = C \cdot \rho$ for some constant C .

The compact simply connected homogeneous spaces admitting homogeneous Riemannian metric of positive sectional curvature have been classified by M. Berger [E], N. Wallach [W] and B. Bergery [Be]. There are compact rank one symmetric spaces (CROSS) and also $SU(3)/i_{k,l}(S^4)$, $Sp(2)/SU(2)$, $SU(5)/Sp(2) \times T^1$, $SU(3)/T_{max}$, $Sp(3)/Sp(1) \times Sp(1) \times Sp(1)$, $F_4/Spin(8)$. The classification of all homogeneous Einstein metrics on the CROSS have been done by W. Ziller [Z]. The case of $SU(3)/i_{k,l}(S^4)$ was considered by M. Wang [Wa]. The homogeneous space $Sp(2)/SU(2)$ is isotropy irreducible and therefore it admits, up to a homothety, only one homogeneous Einstein metric.

The main purpose of the present paper is to obtain the classification of all invariant Einstein metrics on the remaining homogeneous spaces: $SU(3)/T_{max}$, $Sp(3)/Sp(1) \times Sp(1) \times Sp(1)$, $F_4/Spin(8)$, $SU(5)/Sp(2) \times T^1$.

Theorem. The classification mentioned above is given, up to a homothety, by the following table:

Space	The isotropy representation	Characteristic numbers of the metric	Sectional curvature
$SU(3)/T_{max}$	$\rho = \rho_1 \oplus \rho_2 \oplus \rho_3$	(1, 1, 1)	positive
		(2, 1, 1)	oscillate
		(1, 2, 1)	oscillate
		(1, 1, 2)	oscillate

$Sp(3)/Sp(1) \times Sp(1) \times Sp(1)$	$p = p_1 \oplus p_2 \oplus p_3$	$(1, 1, 1)$ $(3, 1, 1)$ $(1, 3, 1)$ $(1, 1, 3)$	positive oscillate oscillate oscillate
$F_4/Spin(8)$	$p = p_1 \oplus p_2 \oplus p_3$	$(1, 1, 1)$ $(7, 2, 2)$ $(2, 7, 2)$ $(2, 2, 7)$	positive oscillate oscillate oscillate
$SU(5)/Sp(2) \times T^1$	$p = p_1 \oplus p_2$	$((20-4\sqrt{7})/9, 1)$ $((20+4\sqrt{7})/9, 1)$	positive oscillate

Sketch of the proof for the cases $SU(3)/T_{max}, Sp(3)/Sp(1) \times Sp(1) \times Sp(1), F_4/Spin(8)$. Let χ be the isotropy representation of G/H and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ is the reductive decomposition of G/H with respect of the minus Killing's form of \mathfrak{g} . Then $\mathfrak{p} = p_1 \oplus p_2 \oplus p_3$, where χ acts irreducibly on each p_i ($i=1,2,3$) and $p_i \neq p_j$ for $i \neq j$. Hence a G -invariant metric on G/H is of the form: $(\cdot, \cdot) = c_1 B|_{p_1} + c_2 B|_{p_2} + c_3 B|_{p_3}$, where $c_1, c_2, c_3 \in \mathbb{R}^+$ and $B(X, Y) = -\text{tr} ad X \circ ad Y$ for all $X, Y \in \mathfrak{g}$. We suppose, without the loss of generality, that $(c_1, c_2, c_3) = (1, t, s)$. Then non-trivial computations using the formula for the sectional curvature on homogeneous Riemannian spaces show that (\cdot, \cdot) is Einstein iff

$$\begin{cases} 3st - t^2 - 3s + 1 = 0 \\ (t-s)(t+s-3) = 0 \end{cases} \quad \text{for } SU(3)/T_{max}$$

$$\begin{cases} 4st - t^2 - 4s + 1 = 0 \\ (t-s)(t+s-4) = 0 \end{cases} \quad \text{for } Sp(3)/Sp(1) \times Sp(1) \times Sp(1)$$

$$\begin{cases} 9st - 9s - 2t^2 + 2 = 0 \\ (t-s)(2t+2s-9) = 0 \end{cases} \quad \text{for } F_4/Spin(8)$$

We obtain immediately the result of our theorem by solving these systems of quadratic equations. The case of $SU(5)/Sp(2) \times T^1$ is

considered analogously.

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