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REPORT ON K–THEORY FOR CONVENIENT ALGEBRAS

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0. Introduction

Studying topological K–theory one is led to K–theory of Banach algebras by replacing bundles by the spaces of their sections. In fact the simplest proofs of some basic results in topological K–theory, in particular of Bott periodicity, heavily use Banach algebra techniques. In several applications of topological K–theory, notably to index theorems, the objects one is interested in are not general compact spaces and topological vector bundles but compact smooth manifolds and smooth vector bundles. So in fact passing to the algebraic side one should work with algebras of smooth functions which are only Fréchet algebras. This becomes more important if one is looking for non commutative analogs of algebras of smooth functions. On the other side there are quite simple topological algebras like the real or complex group algebra of an infinite discrete group which are not even Fréchet and it would be nice to have some sort of topological K–theory for such algebras.

This paper is a short report on the first step of a project, the aim of which is to generalize techniques and results of topological K–theory from Banach algebras to a much more general class of algebras which contains all complete locally convex algebras. A detailed exposition will appear elsewhere.

There arises one serious problem at the beginning of such a generalization: One of the most important techniques of K–theory for Banach algebras is to relate the K–groups of an algebra A to the stable topology of the topological groups $GL_n(A)$ of A–module automorphisms of A^n. For a Banach algebra A the group $GL_n(A)$ is the group of invertible elements in the Banach algebra of all $n \times n$ matrices with entries from A and thus it is open in this space and a topological group. This situation completely changes if one passes to more general algebras, since in this case invertible elements do not form open sets in general, the composition maps in spaces of continuous linear maps need not being jointly continuous and the inversion also is not continuous in general.

This difficulty can be overcome by forgetting about topology and passing to smooth structures in the sense of [**Frölicher**, 1980, 1981]. It then turns out that for any convenient algebra there is a natural smooth structure on the set of invertible elements such that the multiplication and inversion maps are smooth.

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1. Smooth spaces and classifying spaces of smooth groups

1.1. Definition. A *smooth space* is a set X together with a set of curves $\mathcal{C}_X \subset X^{\mathbb{R}}$ and a set of functions $\mathcal{F}_X \subset \mathbb{R}^X$ such that
(1) For any $c \in \mathcal{C}_X$ and any $f \in \mathcal{F}_X$ we have $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$.
(2) The curves and functions determine each other in the following sense: If $c \in X^{\mathbb{R}}$ is such that $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for any $f \in \mathcal{F}_X$ then $c \in \mathcal{C}_X$, and if $f \in \mathbb{R}^X$ is such that $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for any $c \in \mathcal{C}_X$ then $f \in \mathcal{F}_X$.

A map $f : X \to Y$ between smooth spaces is called smooth iff it satisfies one of the following equivalent conditions:
(1) $f \circ c \in \mathcal{C}_Y$ for all $c \in \mathcal{C}_X$
(2) $\varphi \circ f \in \mathcal{F}_X$ for all $\varphi \in \mathcal{F}_Y$
(3) $\varphi \circ f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $\varphi \in \mathcal{F}_Y$ and all $c \in \mathcal{C}_X$

Let $\underline{C^\infty}$ denote the category of smooth spaces and smooth maps.

1.2. There is an obvious notion of the smooth structure generated by a given set of curves or real valued functions. Using this one easily concludes that the category $\underline{C^\infty}$ has initial and final structures with respect to the forgetful functor to the category of sets and thus is complete and cocomplete.

Note that this implies that several standard constructions of homotopy theory like cones, suspensions, mapping cylinders and so on which can be defined as push outs, can also be done for smooth spaces. Moreover there is a natural notion of homotopies so that it is clear what is meant by the set of homotopy classes of smooth maps between two smooth spaces.

Next it turns out that for smooth spaces Y and Z there is a natural smooth structure on the set $C^\infty(Y, Z)$ of all smooth functions, such that a map $f : X \to C^\infty(Y, Z)$, where X is an arbitrary smooth space, is smooth if and only if the canonically associated map $\hat{f} : X \times Y \to Z$ is smooth. Thus the category of smooth spaces is cartesian closed, i.e. there is a natural isomorphism (which is even a diffeomorphism) $C^\infty(X, C^\infty(Y, Z)) \cong C^\infty(X \times Y, Z)$. (For a proof see [**F–K**, 1.1.7 and 1.4.3])

1.3. Examples. (1) Any finite dimensional smooth manifold with its usual smooth curves and real valued functions is a smooth space. By a theorem of [**Boman**, 1967] the definition of smoothness given in 1.1 coincides with the usual one for functions from $\mathbb{R}^n$ to $\mathbb{R}$, and from this one easily concludes that this also holds for maps between finite dimensional smooth manifolds.
(2) Any topological vector space has a natural smooth structure, namely the one generated by its topological dual. It is shown in [**F–K**, 4.3.16] that for maps between Banach spaces smoothness in the sense of 1.1 coincides with smoothness in the usual sense.

1.4. Natural topologies on smooth spaces. On a smooth space there are two obvious natural topologies. First there is the final topology with respect to all smooth curves which is important in the theory of convenient vector spaces, and second there is the initial one with respect to all real valued smooth functions. We denote these topologies by $\tau_{\mathcal{C}}$ and $\tau_{\mathcal{F}}$, respectively. By definition a smooth function between smooth spaces is continuous if one puts on both spaces either the $\tau_{\mathcal{C}}$ or the $\tau_{\mathcal{F}}$ topology.

Definition. (1) A *base space* is a smooth space for which the two natural topologies coincide and are compact.
(2) A smooth space is called smoothly paracompact iff any $\tau_{\mathcal{F}}$–open covering has a subordinate smooth partition of unity.

1.5. Definition. (1) A smooth group is a group with a smooth structure such that the multiplication and inversion maps are smooth.
(2) For a smooth group G and a smooth space X we define a smooth principal G–bundle over X as usual in differential geometry as a locally trivial (with respect to the $\tau_{\mathcal{F}}$ topology) fiber bundle with fiber G and smooth transition functions acting as left translations.

1.6. Let G be a smooth group. We want to construct a classifying space for G as a smooth variant of Milnor's construction. So we consider the set of all sequences $(t_i, g_i)_{i\in\mathbb{N}}$, where $t_i \in [0,1]$ and $g_i \in G$ such that only finitely many t_i are nonzero and $\sum_{i\in\mathbb{N}} t_i = 1$. On this set we define an equivalence relation by $(t_i, g_i) \sim (t_i', g_i')$ if and only if $t_i = t_i'$ for all i and $g_i = g_i'$ for those i for which t_i is nonzero. Let EG denote the set of equivalence classes.

Let $c : \mathbb{R} \to EG$ be a curve. Then $c(t) = (c_i(t), \tilde{c}_i(t))$ where the c_i are curves into $[0,1]$ and the $\tilde{c}_i$ are curves into G. On EG we put the smooth structure generated by the set of all curves $c : \mathbb{R} \to EG$ such that for any i the curve c_i is a smooth curve into $[0,1]$ and the restriction $\tilde{c}_i \upharpoonright c_i^{-1}((0,1]) : c_i^{-1}((0,1]) \to G$ is smooth. Now it turns out that the smooth curves for this structure are exactly those which have this property.

Next we define a right action of G on EG by $(t_i, g_i)\cdot g := (t_i, g_i \cdot g)$. Then this action is immediately seen to be smooth and free and we define BG to be the set of orbits with the final smooth structure with respect to the natural mapping $p : EG \to BG$. Then one easily shows that $p : EG \to BG$ is a smooth principal G–bundle.

Repeating the classical proofs for Milnor's construction with several changes one proves the following

1.7. THEOREM. *For any smoothly paracompact space X there is a bijection between the set of all isomorphism classes of smooth principal G–bundles over X and the set of smooth homotopy classes $[X, BG]$.*

1.8. Remark. It turns out that the constructions of associated bundles and frame bundles can also be carried out in the smooth category. Thus the space BG is also classifying for smooth fiber bundles with fixed fiber, fixed structure group and fixed action of the structure group. In particular this can be applied to the group $\mathrm{Diff}(X)$ of all smooth functions from a smooth space X to itself which have a smooth inverse (it is shown in [F–K, 1.4.8] that this is a smooth group), to conclude that the smooth space $B\,\mathrm{Diff}(X)$ is classifying for smooth fiber bundles with fiber X and without structure group.

2. CONVENIENT VECTOR SPACES, ALGEBRAS AND MODULES

2.1. Definition. (1) A *smooth vector space* is a real vector space E provided with a smooth structure such that the addition $E \times E \to E$ and the scalar multiplication $\mathbb{R} \times E \to E$ are smooth maps.
(2) A smooth vector space is called *convenient* iff its smooth structure is generated by

some set of point separating linear functionals and for any smooth curve $c : \mathbb{R} \to E$ there is a smooth curve $\dot{c} : \mathbb{R} \to E$ such that for all smooth linear functionals $\lambda : E \to \mathbb{R}$ we have $\frac{\partial}{\partial t}\big|_0 (\lambda \circ c)(t) = \lambda(\dot{c}(0))$.
(3) For convenient vector spaces E and F we write $L(E, F)$ for the vector space of all smooth linear maps from E to F and we write E' for $L(E, \mathbb{R})$.

2.2. On a convenient vector space E there is a canonical locally convex topology, namely the finest one for which E' becomes the topological dual of E. (c.f. [**F–K**, 2.1.9]). On the other hand on any locally convex vector space we can consider the smooth structure generated by the topological dual. It turns out that the resulting smooth vector space is convenient if and only if the locally convex vector spaces is Hausdorff and satisfies the following completeness condition (called c^∞–completeness or Mackey completeness): Every sequence (x_n) such that there are positive reals $t_{m,n}$ with $\lim_{m,n\to\infty} t_{m,n} = \infty$ such that the set of all $t_{m,n}(x_m - x_n)$ is bounded in E (such a sequence is called a Mackey–Cauchy sequence) converges (weakly).

So we can view any separated c^∞–complete locally convex vector space as a convenient vector space. It turns out that a linear map between two such spaces is smooth if and only if it is bounded. In fact one shows that the constructions described above establish an equivalence between the category of convenient vector spaces and smooth linear maps and the category of separated c^∞–complete locally convex spaces and bounded linear maps, and we will always identify these two categories. Note that two locally convex vector spaces can be isomorphic as convenient vector spaces without being isomorphic as locally convex spaces.

2.3. As by definition a convenient vector space is a smooth space it also carries the canonical topologies defined in 1.4 and in particular the τ_C–topology which is also called c^∞–topology or Mackey–closure topology. (We will use the notation c^∞–closed for closed in this topology etc.)

In general the c^∞–topology on a convenient vector space is not a vector space topology since the addition is only partially continuous. It turns out that the (bornological) locally convex topology constructed in 2.2 is the finest locally convex topology which is coarser than the c^∞–topology. A condition which ensures that the two topologies coincide is that the locally convex topology is metrizable.

2.4. We list without proofs some facts about the category $\underline{Con}$ of convenient vector spaces and smooth linear maps. Proofs of these results can be found in [**F–K**].
(1) The category $\underline{Con}$ is complete and cocomplete.
(2) c^∞–closed linear subspaces of convenient vector spaces are again convenient.
(3) If X is a smooth space and E is a convenient vector space then the space $C^\infty(X, E)$ is a convenient vector space.
(4) For convenient vector spaces E and F the space $L(E, F)$ is a c^∞–closed linear subspace of $C^\infty(E, F)$ and thus convenient.
(5) Multilinear maps between convenient vector spaces are smooth if and only if they are bounded.

2.5. Definition. A *convenient algebra* is a convenient vector space A together with a bilinear bounded map $\mu : A \times A \to A$ such that A is an associative algebra with multiplication μ. We will always assume that the algebra A has a unit and that all homomorphisms preserve the units.

For a convenient algebra A we denote by A^{op} the opposite algebra to A, which obviously is also a convenient algebra.

2.6. Examples. (1) Let E be a convenient vector space. Then the composition map $\circ : L(E,E) \times L(E,E) \to L(E,E)$ is smooth by cartesian closedness of the category of smooth spaces. Thus $(L(E,E), \circ)$ is a convenient algebra.
(2) Let X be a smooth space, A a convenient algebra and consider the space $C^\infty(X,A)$ which is a convenient vector space by 2.4(3). Using cartesian closedness one easily shows that the point wise multiplication in $C^\infty(X,A)$ is smooth and thus $C^\infty(X,A)$ with the point wise operations is a convenient algebra. In particular this applies to $C^\infty(X,\mathbb{R})$.

2.7. To establish the correspondence between convenient algebras and smooth groups one easily proves the following result:

PROPOSITION. *Let A be a convenient algebra. By A^* we denote the set all invertible elements of A. Let $i : A^* \to A$ be the inclusion and let $\nu : A^* \to A$ be defined by $\nu(a) := a^{-1}$. Then A^* with the initial smooth structure with respect to the maps i and ν is a smooth group.*

2.8. Convenient modules. Let A be a convenient algebra. A *convenient right module* over A is a convenient vector space M together with a bounded homomorphism of algebras $\rho_M : A^{op} \to L(M,M)$. If M and N are convenient right A–modules then a module homomorphism $f : M \to N$ is a bounded linear map such that for all $a \in A$ we have $\rho_N(a) \circ f = f \circ \rho_M(a)$. By $\mathrm{Hom}^A(M,N)$ we denote the space of all module homomorphisms. One easily verifies that this is a c^∞–closed linear subspace of $L(M,N)$ and thus a convenient vector space.

Concerning the categorical properties of modules one has the following

THEOREM. *For any convenient algebra A the category of convenient right A–modules and bounded module homomorphisms is a complete and cocomplete additive category.*

2.9. Definition. Let A be a convenient algebra. A *finitely generated projective* right A–module is a convenient right A–module P for which there exists a convenient right A–module Q such that for some n we have $P \oplus Q \cong A^n$, the n–fold direct sum of copies of A. By $\mathcal{P}(A)$ we denote the category of finitely generated projective right A–modules and bounded module homomorphisms. Using theorem 2.8 one proves:

PROPOSITION. *For any convenient algebra the category $\mathcal{P}(A)$ of finitely generated projective right A–modules is a pseudo–abelian category. (c.f. [*Ka, *I.6.7])*

2.10. On the category of convenient vector spaces there is a tensor product $\tilde{\otimes}$ which has the universal property for bounded bilinear maps (c.f. [F–K, 3.8.4]). This tensor product can be used to define for a convenient algebra A a tensor product over A which associates to a convenient right A–module M and a convenient left A–module N a convenient vector space $M\tilde{\otimes}_A N$, which has the universal property that any bilinear bounded map $f : M \times N \to E$ into an arbitrary convenient vector space such that $f(\rho(a)(m), n) = f(m, \lambda(a)(n))$ induces a unique bounded linear map $\tilde{f} : M\tilde{\otimes}_A N \to E$.

(Here ρ denotes the right action of A on M and λ denotes the left action of A on N.) This tensor product has the property that if N is also a right module over a convenient algebra B such that the actions of A and B commute then there is a natural right B-module structure on $M\tilde{\otimes}_A N$.

Now let $\varphi : A \to B$ be a bounded homomorphism of convenient algebras. Then via φ we get a left A-module structure on B and so for a right A-module M we can form the convenient vector space $M\tilde{\otimes}_A B$ which is then a right B module. Using this construction one proves:

PROPOSITION. *Any bounded algebra homomorphism $\varphi : A \to B$ between convenient algebras induces an additive functor $\mathcal{P}(\varphi) : \mathcal{P}(A) \to \mathcal{P}(B)$.*

3. K-THEORY FOR CONVENIENT ALGEBRAS

3.1. Definition. Let A be a convenient algebra, X a base space (c.f. 1.4). An *A-bundle over X* is a locally trivial smooth fiber bundle over X such that any fiber is a finitely generated projective right A-module and such that the transition functions are module homomorphisms. We allow the isomorphism type of the fibers to be different over different connected components of X.

A morphism between two A-bundles over X is a fiber respecting smooth map such that the restriction to each fiber is a module homomorphism.

By $\mathcal{E}_A(X)$ we denote the category of all A-bundles over X and their morphisms.

Using the fibered product in the category of smooth spaces as the definition of the direct sum of two A-bundles over X one shows:

3.2. PROPOSITION. *For any convenient algebra A and any base space X the category $\mathcal{E}_A(X)$ is additive.*

3.3. Our next task is to derive an analog of the theorem of Serre and Swan, i.e. to establish a correspondence between A-bundles over X and modules over the convenient algebra $C^\infty(X, A)$ via sections of the bundles. First one shows that for a locally trivial vector bundle over a smooth space with fiber a convenient vector space, the space of sections is in a natural way a convenient vector space. (In fact this result holds for a more general class of 'vector bundles', see [**F–K**, 4.6.15]). Next one verifies that for an A-bundle over X the point wise action of the convenient algebra $C^\infty(X, A)$ on the space of sections of the bundle defines a convenient right $C^\infty(X, A)$-module structure. To show that these modules are finitely generated and projective one needs the following

LEMMA. *Let A be a convenient algebra, X a base space, $\pi : E \to X$ an A-bundle over X. Then there is a natural number n and a homomorphism of A-bundles $p : X \times A^n \to X \times A^n$ such that $p \circ p = p$ and $E \cong Ker(p)$ as an A-bundle over X.*

In the case of a Banach algebra A one can prove more: In this case for any morphism on an A-bundle which is a projection, the kernel is again locally trivial and thus an A-bundle. In particular this implies that any A-bundle is a direct summand in a trivial one. But the proofs of these results use heavily the fact that the invertible elements of a Banach algebra form an open subset. Since already for quite simple convenient algebras the sets of invertible elements are not even c^∞-open (c.f. [**F–K**, 5.3.6]), I do not think that these results remain true for convenient algebras, although I do not know explicit counter examples.

3.4. It is a general theorem that to any additive category one can associate a pseudo–abelian category which has a certain universal property and is uniquely determined up to equivalence (c.f. [**Ka**, I.6.10]). Now we can formulate the analog of the Serre–Swan theorem as follows:

THEOREM. *The pseudo–abelian category associated to the additive category* $\mathcal{E}_A(X)$ *is equivalent to* $\mathcal{P}(C^\infty(X, A))$.

The results on Banach algebras mentioned at the end of 3.3 immediately imply that in this case the category $\mathcal{E}_A(X)$ itself is pseudo–abelian and thus one gets:

COROLLARY. *If* A *is a Banach algebra and* X *is a base space then the categories* $\mathcal{E}_A(X)$ *and* $\mathcal{P}(C^\infty(X, A))$ *are equivalent.*

3.5. If X and Y are base spaces, $f : X \to Y$ is a smooth map and E is an A–bundle over Y then one easily verifies that the pullback (in the category of smooth spaces) f^*E is an A–bundle over X. Moreover the pullback commutes with direct sums and thus f induces an additive functor $\mathcal{E}_A(f) : \mathcal{E}_A(Y) \to \mathcal{E}_A(X)$.

On the other hand suppose that $\varphi : A \to B$ is a bounded algebra homomorphism between convenient algebras. Then one shows that for any finitely generated projective right A–module P the functor $\mathcal{P}(\varphi)$ constructed in 2.10 induces a smooth map $B(\mathrm{Aut}(P)) \to B(\mathrm{Aut}(\mathcal{P}(\varphi)(P)))$ between the classifying spaces of the smooth groups of module automorphisms. Now let E be an A–bundle over a base space X. Then over each connected component of X the restriction of the bundle is classified by a smooth map (in fact a homotopy class) into $B(\mathrm{Aut}(P))$ for some finitely generated projective right A–module P. Composing the classifying maps with the maps constructed above one gets a smooth B–bundle over X and one verifies that via this construction φ induces an additive functor $\mathcal{E}_\varphi(X) : \mathcal{E}_A(X) \to \mathcal{E}_B(X)$.

3.6. Definition. (1) Let $\mathcal{C}$ be an additive category. We define the Grothendieck group $K(\mathcal{C})$ of $\mathcal{C}$ to be the universal abelian group associated to the commutative monoid of all isomorphism classes of objects of $\mathcal{C}$.
(2) For a convenient algebra A we put $K_0(A) := K(\mathcal{P}(A))$.
(3) For a convenient algebra A and a base space X we define $K_A(X) := K(\mathcal{E}_A(X))$.
Using the results of 2.10 and 3.5 one immediately concludes that $A \mapsto K_0(A)$ is a covariant functor and that $(A, X) \mapsto K_A(X)$ is a functor which is covariant in A and contravariant in X.

It is also possible to define higher K–groups via suspensions but we will not consider these groups in this paper.

3.7. Our final task is to give a homotopy theoretic interpretation of the group $K_A(X)$. For an A–bundle over X we get a locally constant function $X \to K_0(A)$ by assigning to each point the isomorphism class of the fiber over this point. By the universal property of the Grothendieck group this induces a group homomorphism $K_A(X) \to H^0(X, K_0(A))$, where $H^0(X, K_0(A))$ denotes the group of locally constant functions from X to $K_0(A)$, and we define $K'_A(X)$ to be the kernel of this homomorphism. One easily shows that there is a natural isomorphism $K_A(X) \cong K'_A(X) \oplus H^0(X, K_0(A))$.

3.8. For any $n \in \mathbb{N}$ let $\Phi_n^A(X)$ be the set of isomorphism classes of A–bundles over X with fiber A^n. Let $GL_n(A)$ be the smooth group of all isomorphisms of right A–modules $A^n \to A^n$, i.e. $GL_n(A) = \mathrm{Aut}(A^n)$. Then by 1.8 there is a bijection between $\Phi_n^A(X)$ and $[X, BGL_n(A)]$, the set of homotopy classes of smooth maps from X to the classifying space of the smooth group$GL_n(A)$. Adding trivial bundles with fiber A we get maps $\Phi_n^A \to \Phi_{n+1}^A$ and we denote by $\Phi^A(X)$ the direct limit of the so obtained inductive system of sets. Then it turns out that the direct sum of A–bundles induces the structure of a commutative monoid on $\Phi^A(X)$.

Now the map which sends an A–bundle with fiber A^n to the difference of its class in $K_A(X)$ and the class of the trivial bundle $X \times A^n$ in $K_A(X)$ induces a homomorphism of monoids $\Phi^A(X) \to K'_A(X)$ and one shows that this homomorphism is injective. Moreover one proves that any element of $K'_A(X)$ can be written as a difference of two elements from the image of this homomorphism and thus one gets:

LEMMA. *The group $K'_A(X)$ is isomorphic to the Grothendieck group of $\Phi^A(X)$ and the homomorphism constructed above is equivalent to the canonical homomorphism to the Grothendieck group.*

Note that since the homomorphism to the Grothendieck group is injective in this case, the passage to the Grothendieck group just means that for each element that does not already have an inverse one adjoins an inverse.

3.9. The above lemma admits a homotopy theoretic interpretation. We already know that $\Phi_n^A(X) \cong [X, BGL_n(A)]$. Now $f \mapsto f \oplus Id_A$ induces a smooth homomorphism $GL_n(A) \to GL_{n+1}(A)$ and thus a smooth map between the classifying spaces which in turn induces a map $[X, BGL_n(A)] \to [X, BGL_{n+1}(A)]$, that clearly corresponds to the map $\Phi_n^A(X) \to \Phi_{n+1}^A(X)$ constructed above. Let $[X, BGL(A)]$ denote the direct limit of the so obtained inductive system. Then there is a bijection between $\Phi^A(X)$ and $[X, BGL(A)]$.

Consider the group $K_0(A)$ as a smooth space with discrete structure. Then the maps constructed above induce maps

$$[X, K_0(A) \times BGL_n(A)] \to [X, K_0(A) \times BGL_{n+1}(A)]$$

and we define $[X, K_0(A) \times BGL(A)]$ to be the direct limit of the so obtained inductive system. Then clearly this set is again a commutative monoid and using that $H^0(X, K_0(A)) = [X, K_0(A)]$ one easily shows that one gets a monoid homomorphism

$$\varphi : [X, K_0(A) \times BGL(A)] \to H^0(X, K_0(A)) \times K'_A(X) \cong K_A(X)$$

and one easily proves:

THEOREM. *The map $\varphi : [X, K(A) \times BGL(A)] \to K_A(X)$ is injective and can be identified with the natural homomorphism to the Grothendieck group.*

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