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ORDERED SYMMETRIC SPACES

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In recent years homogeneous conal manifolds became increasingly important in various fields of mathematics. It is well known that Lorentzian manifolds play a central role in the theory of general relativity because they serve as models for space times ([HE73], [Se76]). They are also fundamental in the study of hyperbolic partial differential equations ([CB67]).

The recent applications mainly focus on homogeneous conal manifolds which are not necessarily Lorentzian. These homogeneous spaces occur in such contexts as representation theory ([Ol82]), harmonic analysis ([Fa90]), system theory (Kupka in [HLP90]), Wiener-Hopf operators on symmetric spaces ([HiNe91]), and they are very closely related to Lie semigroups ([HHL89], [La89]).

Before we can formulate the main problems we have to give some definitions.

Definition 1. A *conal manifold* M is a smooth manifold endowed with a *field of cones* θ, i. e. for each $x \in M$ the set $\theta(x)$ is a closed convex cone in the tangent space $T_x(M)$ such that $\theta(x)$ is *pointed*, i. e. it contains no non-trivial vector spaces. A *homogeneous conal manifold* M is a conal manifold for which there exists a finite dimensional Lie group G acting transitively on M such that the mappings $\mu_g : M \to M, x \mapsto g.x$ preserve the cone field, i. e.

$$d\mu_g(x)\theta(x) = \theta(g.x) \qquad \forall g \in G, x \in M.$$

If, in addition, M is a symmetric space, and the cone field θ is invariant under all transvections, M is called a *conal symmetric space*. We say that a curve $\gamma : I \to M$, where I is a real interval, is *absolutely continuous*, if for every chart $\varphi : U \to \mathbf{R}^n$ the components of the mapping $\varphi \circ \gamma : \gamma^{-1}(U) \to \mathbf{R}^n$ are absolutely continuous functions. It is clear that the derivative of an absolutely continuous curve γ exists almost everywhere on I. Such an absolutely continuous curve is said to be *conal* if

$$\gamma'(t) \in \theta(\gamma(t))$$

whenever the derivative exists. We define the *conal order* $\prec$ on M by setting $x \prec y$ if there exists a conal curve $\gamma : [t_0, t_1] \to M$ such that $\gamma(t_0) = x$ and $\gamma(t_1) = y$. Note that this is a transitive and reflexive relation. So we will also have to consider the relation

$$x \preceq y :\Longleftrightarrow (x,y) \in \overline{\{(x,y) \in M \times M : x \prec y\}}.$$

We write $\uparrow x := \{y \in M : x \preceq y\}$ and $\downarrow x := \{y \in M : x \preceq y\}$. The relation $\preceq$ has in general the disadvantage that it is not transitive. Nevertheless, we have the following lemma ([La89, p.299]).

Lemma 2. *For every homogeneous conal manifold M the relation $\preceq$ is transitive and defines an order on M. Moreover*

$$\uparrow x = \overline{\{y \in M : x \prec y\}} \qquad \textit{for all} \quad x \in M.$$

∎

More information on general conal manifolds can be found in [La89] and [Ne91b]. In the following M will always denote a homogeneous conal manifold.

Definition 3. M is said to be *globally orderable* if $\preceq$ is a partial order, i. e. if

$$x \preceq y \preceq x \Longrightarrow x = y.$$

M is said to be *globally hyperbolic* if for $x, y \in M$ the *order interval*

$$[x,y] := \{z \in M : x \preceq z \preceq y\}$$

is compact. ∎

The notion of global hyperbolicity is essential in the theory of hyperbolic differential equations (cf. [CB67]). If, in addition, M is a conal symmetric space, the compactness of the order intervals is necessary in order to define the algebra of Volterra kernels on such spaces (cf. [Fa90]).

Let us fix a base-point x_0 in the homogeneous space M. Then $M \cong G/H$, where $H = \{g \in G : g.x_0 = x_0\}$. It is clear that the cone field θ is completely determined by the cone $\theta(x_0) \subseteq T_{x_0}(M)$ and that it is H-invariant, i. e.

$$d\mu_h(x_0)\theta(x_0) = \theta(x_0) \qquad \text{for all} \quad h \in H.$$

It is also easy to see that every pointed H-invariant cone $C \subseteq T_{x_0}(M)$ defines a cone field θ_C on M by

$$\theta_C(g.x_0) := d\mu_g(x_0)C.$$

So the main problem is to decide for a given H-invariant cone C in the vector space $T_{x_0}(M)$ whether M is globally orderable (globally hyperbolic) with respect to θ_C or not. The main tools to handle these problems are the conal functions.

Definition 4. A smooth function $f \in C^\infty(M)$ is called a *strictly conal function* if

$$\langle df(x), v\rangle > 0 \qquad \text{for all} \quad x \in M, v \in \theta(x) \setminus \{0\}.$$

Now we have the following characterizations of global orderability and global hyperbolicity:

Theorem 5. *A conal homogeneous space M is globally orderable if and only i there exists a strictly conal function.*

Proof. [Ne90, Theorem I.6] ∎

Theorem 6. *Suppose that M is a conal homogeneous space M such that* $\operatorname{int} \uparrow x_0 \neq \emptyset$. *Then M is globally hyperbolic if and only if there exists a strictly conal function f and a complete Riemannian metric g on M such that*

$$\langle df(x), v\rangle \geq \|v\| := \sqrt{g_x(v,v)} \quad \text{for all} \quad x \in \uparrow x_0, v \in \theta(x).$$

Proof. [Ne90, Theorem I.32] ∎

The condition in Theorem 6 expresses that the function f increases faster than arclength along all conal curves starting in the base point x_0.

To see how these theorems apply, we specialize further to the case of conal symmetric spaces. We set $C := \theta(x_0)$ and write

$$\operatorname{Exp} : T_{x_0}(M) \to M$$

for the exponential function in x_0 which is defined by the affine connection on M.

Conal symmetric spaces have some interesting properties which are false for general conal homogeneous spaces. The following one is an essential example. One should also note that the same statement holds for Lorentzian manifolds ([HE73]).

Theorem 7. ([DN91]) *Let M be a globally orderable conal symmetric space. Then there exists a neighborhood U of x_0 such that*

$$\uparrow x_0 \cap U \subseteq \operatorname{Exp}(C).$$

■

Theorem 8. ([DN91]) *If M is a globally hyperbolic conal symmetric space, then* $\operatorname{Exp}(C)$ *is closed.* ■

It seems to be very likely that global hyperbolicity even implies that $\uparrow x_0 = \operatorname{Exp}(C)$ but we can't prove this yet. It is true for Lorentzian manifolds ([HE73]) and it works if one imposes some regularity condition on the exponential function on C.

Theorem 9. ([DN91]) *If the set* $\operatorname{Exp}(C)$ *is closed in M and C contains no singular point of the exponential function then*

$$\uparrow x_0 = \operatorname{Exp}(C).$$

■

We visualize these properties with a concrete and typical example.

Example 10. Let $M := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}$, $x_0 := (1,0,0)$, and identify $T_{x_0}(M)$ with $\{(0,a,b) : a,b \in \mathbb{R}\}$. We consider the cone $C := \{(0,a,b) : |a| \leq b\} \subseteq T_{x_0}(M)$. Then M is a symmetric space and the group $G = \mathrm{SO}(2,1)_0$ operates transitively. Let θ_C denote the G-invariant cone field on M defined by C. Then

$$\uparrow x_0 = \operatorname{Exp}(C) = \{(x,y,z) \in M : z \geq 0, x \geq 1\}.$$

This is the upper part of the region cut out from M by the geometric tangent plane through x_0. A direct calculation shows that the function defined by $f(x,y,z) := z$ is a strictly conal function. From this one directly deduces that M is globally hyperbolic. ■

This simple example fits in a larger class of symmetric spaces.

Definition 11. Let $M = G/H$ be a semisimple symmetric space, $\mathbf{g} := \mathbf{L}(G)$, and $\mathbf{h} := \mathbf{L}(H)$. Then $(\mathbf{g},\mathbf{h})$ is a semisimple symmetric pair, i. e. $\mathbf{h}$ is a subalgebra of $\mathbf{g}$ and there exists a subspace $\mathbf{q}$ of $\mathbf{g}$ such that

$$[\mathbf{h},\mathbf{q}] \subseteq \mathbf{q} \quad \text{and} \quad [\mathbf{q},\mathbf{q}] \subseteq \mathbf{h}.$$

Then there exists a compatible Cartan decomposition $\mathbf{g} = \mathbf{k} + \mathbf{p}$, i. e.

$$\mathbf{g} = \mathbf{h} \cap \mathbf{k} + \mathbf{h} \cap \mathbf{p} + \mathbf{q} \cap \mathbf{k} + \mathbf{q} \cap \mathbf{p}.$$

We say that M is *regular* if the centralizer of $Z(\mathbf{q_p}) := \{X \in \mathbf{q_p} : [X, \mathbf{q_p}] = \{0\}\}$ in $\mathbf{q}$ is not bigger than $Z(\mathbf{q_p})$ (cf. [Ola90]). ∎

In [Ne91c] we apply Theorems 6, 8 and 9 to prove the following.

Theorem 12. *Let M be an irreducible semisimple regular symmetric space and $C \subseteq \mathbf{q} \cong T_{x_0}(M)$ a pointed H-invariant cone. Then M is globally hyperbolic.* ∎

This result may also be obtained with a result of Faraut ([Fa90]) which rests on other results of Ol'shanskiĭ([Ol82]) (cf. [Hi90]). As already mentioned it is crucial for harmonic analysis on regular symmetric spaces and also in the study of Wiener-Hopf Operators.

Example 13. If G is a simple Lie group with a maximal compact subgroup K such that G/K is a bounded symmetric domain, then $M := G_{\mathbb{C}}/G$ is a regular symmetric space, where $G_{\mathbb{C}}$ denotes the complexification of G. Since the Lie algebra $\mathbf{L}(G)$ containes a pointed invariant cone W, there exists a $G_{\mathbb{C}}$ invariant cone fields θ on M with $\theta(x_0) = iW$. For each such cone field M is globally hyperbolic, and $\uparrow x_0 = \mathrm{Exp}(iW)$ (cf. [Ol82], [Fa90]). ∎

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