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NATURAL OPERATORS BETWEEN VECTOR VALUED DIFFERENTIAL FORMS

Andreas Cap

This paper is divided into three sections: The first one contains a short description of a method which can be used to determine explicitly all multilinear natural operators between vector valued differential forms and (with minor changes) between sections of several other natural vector bundles. We state the results of this determination for the linear and bilinear case.

In the second section we give a more simple description of these results and show how they can be applied to determine all linear and bilinear natural operators between ordinary differential forms.

In the last section we give definitions for algebraic structures on certain sets of natural operators.

1. Determination of multilinear natural operators

1.1. For a smooth manifold M let us consider the space of vector valued differential forms $\Omega(M;TM) = \bigoplus \Omega^p(M;TM)$, where $\Omega^p(M;TM)$ is defined to be the space of sections of the vector bundle $\Lambda^p T^*M \otimes TM$, the p–th exterior power of the cotangent bundle tensorized with the tangent bundle.

Since this is a natural vector bundle a local diffeomorphism $f : M \to N$ induces a pullback operator $f^* : \Omega^p(N;TN) \to \Omega^p(M;TM)$. A k–linear natural operator between vector valued differential forms is now defined to be a family of k–linear operators

$$A_M : \prod_{i=1}^{k} \Omega^{p_i}(M;TM) \to \Omega^r(M;TM)$$

such that the following condition is satisfied: For any two manifolds M and N, each local diffeomorphism $f : M \to N$ and all $P_i \in \Omega^{p_i}(N;TN)$ we have:

$$A_M(f^*P_1,\dots,f^*P_k) = f^*(A_N(P_1,\dots,P_k))$$

1.2. First of all a multilinear natural operator is easily seen to be local, and thus by a multilinear version of the Peetre theorem (see e.g. [C–dW–G] [Sl] or [K–M–S]) each operator A_M is a (finite order) differential operator over each compact subset. Again by naturality one easily concludes that it is of finite order everywhere and the order

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is the same in each point of each manifold. Thus we may restrict the consideration to some fixed order n.

Now an operator $A : \prod_{i=1}^{k} \Omega^{p_i}(M;TM) \to \Omega^r(M;TM)$ of order n is induced by a vector bundle homomorphism $\hat{A} : \prod_{i=1}^{k}(J^n(\Lambda^{p_i}T^*M \otimes TM)) \to \Lambda^r T^*M \otimes TM$, where J^n denotes the n-th jet prolongation, and it turns out that for a k-linear natural operator (A_M) the associated maps $\widehat{A_M}$ form a natural transformation between the functors $M \mapsto \prod_{i=1}^{k}(J^n(\Lambda^{p_i}T^*M \otimes TM))$ and $M \mapsto \Lambda^r T^*M \otimes TM$, which consists of k-linear vector bundle homomorphisms. Conversely such a natural transformation induces a k-linear natural operator.

Next it turns out that such a natural transformation is uniquely determined by the induced k-linear map between the fibers over $0 \in \mathbb{R}^m$ of the bundles, which we identify with the standard fibers.

Now $M \mapsto J^n(\Lambda^{p_i}T^*M \otimes TM)$ is a natural vector bundle, too, and thus it determines a representation of the jet group $G_m^{n+1} := inv J_0^{n+1}(\mathbb{R}^m, \mathbb{R}^m)_0$, the group of all $n+1$-jets of diffeomorphisms of $\mathbb{R}^m$, on its standard fiber, as well as on the standard fibers of all lower jet prolongations. This representation will be described in more detail later on. It turns out that a k-linear map between the standard fibers is induced by a natural transformation if and only if it is equivariant with respect to these actions of G_m^{n+1}.

1.3. Putting these considerations together we get:

THEOREM. *There is a bijective correspondence between the set of all k-linear natural operators $A_M : \prod_{i=1}^{k} \Omega^{p_i}(M;TM) \to \Omega^r(M;TM)$ of order n and the set of all G_m^{n+1}-equivariant k-linear maps $A_0 : \prod_{i=1}^{k} V_i \to V$, where V_i denotes the standard fiber of the bundle $J^n(\Lambda^{p_i}T^*M \otimes TM)$ and V denotes the standard fiber of $\Lambda^r T^*M \otimes TM$*

A rigorous proof of this theorem in a much more general version can be found in [K–M–S].

The correspondence between natural operators and equivariant maps is indeed quite simple: If one considers the elements of V_i as coordinate expressions of n-jets of vector valued differential forms at $0 \in \mathbb{R}^m$, that is as the partial derivatives up to order n at 0 of the coordinate expression of a vector valued differential form on $\mathbb{R}^m$, then A_0 is just the coordinate expression at 0 of the operator A as a differential operator.

1.4. The first step to apply the above theorem is to determine the possible orders of multilinear natural operators. Using the methods of [Mi] one easily proves:

PROPOSITION. *If $A_M : \prod_{i=1}^{k} \Omega^{p_i}(M;TM) \to \Omega^r(M;TM)$ is a k-linear natural operator, then A_M is a differential operator homogeneous of total order $r - \sum_i p_i + k - 1$.*

In particular there is no nonzero such operator for $r \le \sum p_i - k$.

Now the standard fiber of the bundle $J^n(\Lambda^{p_i}T^*M \otimes TM)$ is the vector space $\prod_{j=0}^{n}(\Lambda^{p_i}\mathbb{R}^{m*} \otimes \mathbb{R}^m \otimes S^j\mathbb{R}^{m*})$, where S^j denotes the j-th symmetric power, and thus the associated map to a k-linear natural operator of order n is:

$$A_0 : \prod_{i=1}^{k}(\prod_{j=0}^{n}(\Lambda^{p_i}\mathbb{R}^{m*} \otimes \mathbb{R}^m \otimes S^j\mathbb{R}^{m*})) \to \Lambda^r\mathbb{R}^{m*} \otimes \mathbb{R}^m$$

From homogeneity of A it follows that not all factors in this large product must be considered, but A_0 splits into a sum $A_0 = \sum_{j_1+\dots+j_k=n} A_{j_1\dots j_k}$, where

$$A_{j_1\dots j_k} : \prod_{i=1}^{k}(\Lambda^{p_i}\mathbf{R}^{m*}\otimes\mathbf{R}^m\otimes S^{j_i}\mathbf{R}^{m*})) \to \Lambda^r\mathbf{R}^{m*}\otimes\mathbf{R}^m$$

is a k-linear map.

1.5. Let us now consider the action of the group G_m^{n+1} on the standard fibers. It is defined as follows: For an element of G_m^{n+1} pick a representative, i.e. a diffeomorphism of $\mathbf{R}^m$ with this $n+1$-jet at 0. This diffeomorphism induces vector bundle automorphisms on all jet prolongations of the bundle $\Lambda^{p_i}T^*\mathbf{R}^m\otimes T\mathbf{R}^m$, which map the fibers over 0 to themselves. Up to the n-th jet prolongation these maps are independent of the choice of the representative, since there the induced automorphisms depend at most on $n+1$-jets, and so define the actions of the element of G_m^{n+1}.

This description of the representation shows two important facts:

(1) The induced representation of the subgroup $GL(m,\mathbf{R})$ is just the usual one.
(2) Considering the elements of the standard fibers as partial derivatives at 0 of the coordinate functions of vector valued differential forms on $\mathbf{R}^m$, equivariancy under the actions of G_m^{n+1} is equivalent to equivariancy under the usual transformation laws for partial derivatives.

1.6. To actually determine equivariant maps there is a method developed by I. Kolář ([Ko]). The idea of this method is to start with $GL(m,\mathbf{R})$–equivariant maps and check which linear combinations of them are G_m^{n+1}–equivariant, too.

To determine all $GL(m,\mathbf{R})$–equivariant maps consider the following diagram:

$$\begin{array}{ccc}
\bigotimes_{i=1}^k(\Lambda^{p_i}\mathbf{R}^{m*}\otimes\mathbf{R}^m\otimes S^{j_i}\mathbf{R}^{m*}) & \xrightarrow{A_{j_1\dots j_k}} & \Lambda^r\mathbf{R}^{m*}\otimes\mathbf{R}^m \\
{\scriptstyle \bigotimes_{i=1}^k(Alt_{p_i}\otimes Id\otimes Symm_{j_i})}\uparrow & & \uparrow{\scriptstyle Alt_r\otimes Id} \\
\bigotimes_{i=1}^k(\bigotimes^{p_i}\mathbf{R}^{m*}\otimes\mathbf{R}^m\otimes\bigotimes^{j_i}\mathbf{R}^{m*}) & \xrightarrow{\varphi_{j_1\dots j_k}} & \bigotimes^r\mathbf{R}^{m*}\otimes\mathbf{R}^m
\end{array}$$

Here Alt and $Symm$ denote the alternator and the symmetrizer, i.e. the canonical projections from tensor powers to exterior and symmetric powers, respectively.

Since the alternator and the symmetrizer are $GL(m,\mathbf{R})$–equivariant maps one easily sees that each $GL(m,\mathbf{R})$–equivariant map $A_{j_1\dots j_k}$ is given by applying a $GL(m,\mathbf{R})$–equivariant map $\varphi_{j_1\dots j_k}$ and taking the alternator of the result.

But the classical theory of invariant tensors gives an explicit description of a generating system for all $GL(m,\mathbf{R})$–equivariant maps $\varphi_{j_1\dots j_k}$, and thus a generating system for all $GL(m,\mathbf{R})$–equivariant maps $A_{j_1\dots j_k}$ is given by alternating these generators.

In particular from this description one sees, that for order $n>2k$ there are no nonzero $GL(m,\mathbf{R})$–equivariant maps and thus no nonzero natural operators.

Now one proceeds as follows: Take as an ansatz for A_0 a linear combination of all $GL(m,\mathbf{R})$–equivariant generators for all maps $A_{j_1\dots j_k}$. This can be viewed as the coordinate expression of a differential operator on $\mathbf{R}^m$. On this coordinate expression calculate the action of a general invertible $n+1$-jet. Equivariancy under this action is then equivalent to a large system of linear equations. The solution of these equations is a rather technical problem and is carried out in the linear and bilinear case in [Ca].

1.7. To state the results for these cases we need some operators: The Frölicher–Nijenhuis bracket: $[\ ,\ ] : \Omega^p(M;TM) \times \Omega^q(M;TM) \to \Omega^{p+q}(M;TM)$ (c.f. [**Mi**])
The insertion operator: $i : \Omega^p(M;TM) \times \Omega^q(M;TM) \to \Omega^{p+q-1}(M;TM)$ defined by $i(\varphi \otimes X)(\psi \otimes Y) = \varphi \wedge i(X)(\psi) \otimes Y$ for vector fields X and Y and differential forms φ and ψ and bilinearly extended. ($i(X)$ denotes the usual insertion operator) Note that this definition makes also sense with an arbitrary vector bundle valued differential form in the second position.
The contraction $C : \Omega^p(M;TM) \to \Omega^{p-1}(M)$, defined by $C(\varphi \otimes X) = i(X)(\varphi)$, linearly extended.
The symmetric contraction $S : \Omega^p(M;TM) \times \Omega^q(M;TM) \to \Omega^{p+q-2}(M)$ defined by $S(\varphi \otimes X, \psi \otimes Y) = i(Y)(\varphi) \wedge i(X)(\psi)$, bilinearly extended.

All these operators are natural under local diffeomorphisms.

Moreover we use that $\Omega(M;TM)$ is a graded module over the graded commutative algebra $\Omega(M)$ under the action $\varphi \wedge (\psi \otimes X) = (\varphi \wedge \psi) \otimes X$, linearly extended.

By I we denote the identity Id_{TM}, viewed as an element of $\Omega^1(M;TM)$, and by d we denote the exterior derivative of differential forms.

1.8. In the case of all linear natural operators $\Omega^p(M;TM) \to \Omega^r(M;TM)$ by (1.4) and (1.6) there can be nonzero ones only for $r = p, p+1$ or $p+2$. Generating systems for these cases are given by ($m := dim(M)$):

r	generating system	basis if
p	$Id, C(P) \wedge \mathsf{I}$	$p \geq 1, m > p$
$p+1$	$dC(P) \wedge \mathsf{I}$	$p \geq 1, m > p$
$p+2$	0	

1.9. For bilinear natural operators $\Omega^p(M;TM) \times \Omega^q(M;TM) \to \Omega^r(M;TM)$ we have to consider the five cases $r = p+q-1, \dots, p+q+3$. Generating systems for these cases are:

r	generating system	basis if
$p+q-1$	$i(P)(Q), i(Q)(P), C(P) \wedge Q, C(Q) \wedge P, i(P)(C(Q)) \wedge \mathsf{I}$,	$p, q \geq 2$
	$i(Q)(C(P)) \wedge \mathsf{I}, C(P) \wedge C(Q) \wedge \mathsf{I}, S(P,Q) \wedge \mathsf{I}$	$m > p+q$
$p+q$	$[P,Q], dC(P) \wedge Q, dC(Q) \wedge P, dC(P) \wedge C(Q) \wedge \mathsf{I}$,	$p, q \geq 2$
	$C(P) \wedge dC(Q) \wedge \mathsf{I}, i(P)(dC(Q)) \wedge \mathsf{I}, i(Q)(dC(P)) \wedge \mathsf{I}$,	$m > p+q$
	$d(i(P)(C(Q))) \wedge \mathsf{I}, d(i(Q)(C(P))) \wedge \mathsf{I}, dS(P,Q) \wedge \mathsf{I}$	
$p+q+1$	$dC(P) \wedge dC(Q) \wedge \mathsf{I}, d(i(P)(dC(Q))) \wedge \mathsf{I}$,	$p, q \geq 1$
	$d(i(Q)(dC(P))) \wedge \mathsf{I}$	$m > p+q$
$p+q+2$	0	
$p+q+3$	0	

1.10. Remark. Several parts of this result were already known. In particular it implies uniqueness (up to constants) of the Lie bracket, which was proved by D. Krupka and V. Mikolášová ([**Kr-M**]), S. van Strien ([**vS**]) and in a stronger (infinitesimal) sense by M. de Wilde and P. Lecomte ([**dW-L**]). The general result for $r = p+q$ and $dim(M) \geq p+q+2$ was proved by I. Kolář and P. Michor ([**K-M**]).

2. RELATIONS BETWEEN THE OPERATORS AND APPLICATIONS

2.1. The first aim of this section is to give a simpler description of the generating system for the bilinear operators.

PROPOSITION. *For $\omega \in \Omega^k(M)$, $P \in \Omega^p(M;TM)$, $Q \in \Omega^q(M;TM)$ and $m = dim(M)$ we have:*

(1) $C(\omega \wedge \mathsf{I}) = (-1)^k(m-k)\omega$
(2) $i(\omega \wedge \mathsf{I})(P) = p\omega \wedge P$
(3) $i(P)(\omega \wedge \mathsf{I}) = i(P)(\omega) \wedge \mathsf{I} + (-1)^{k(p+1)}\omega \wedge P$
(4) $C(i(P)(Q)) = S(P,Q) + (-1)^{p+1}i(P)(C(Q))$

PROOF: Since all operators are local and linear or bilinear we may assume that we are working on $\mathbb{R}^m$ and that $P = \varphi \otimes X$ and $Q = \psi \otimes Y$ with $\varphi \in \Omega^p(\mathbb{R}^m)$, $\psi \in \Omega^q(\mathbb{R}^m)$ and $X, Y \in \mathcal{X}(\mathbb{R}^m)$. Then $\mathsf{I} = \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i}$ and a short computation leads to the results. ∎

2.2 Remark. If we define an operator $\delta : \Omega^p(M;TM) \to \Omega^{p+1}(M;TM)$ by $\delta(P) = (-1)^{p-1}dC(P) \wedge \mathsf{I}$, then by proposition (2.1) we have $\delta^2 = \delta \circ \delta = 0$, and thus this operator gives rise to a cohomology on the space of vector valued differential forms. Moreover it can be shown, that δ is a derivation with respect to the Frölicher–Nijenhuis bracket, i.e. $\delta([P,Q]) = [\delta(P),Q] + (-1)^p[P,\delta(Q)]$, and thus there is an induced bracket on the cohomology space. This cohomology is investigated in [**M-Sch**], and it is shown there, that it contains all but the top–degree de Rham cohomology of M.

2.3 COROLLARY. *Each bilinear natural operator between vector valued differential forms can be written as a linear combination of the Frölicher–Nijenhuis bracket and of compositions of linear natural operators and insertion operators.*

PROOF: It suffices to show that we can write all operators from the list (1.9) except the Frölicher–Nijenhuis bracket as linear combinations of compositions of linear natural operators and insertion operators. We only show this for the algebraic ones (those for which $r = p + q - 1$):

From proposition (2.1) we get:

$$i(C(P) \wedge \mathsf{I})(Q) = qC(P) \wedge Q$$
$$i(C(P) \wedge \mathsf{I})(C(Q) \wedge \mathsf{I}) = qC(P) \wedge C(Q) \wedge \mathsf{I}$$
$$i(P)(C(Q) \wedge \mathsf{I}) = i(P)(C(Q)) \wedge \mathsf{I} + (-1)^{(p+1)(q-1)}C(Q) \wedge P$$
$$C(i(P)(Q)) \wedge \mathsf{I} = S(P,Q) \wedge \mathsf{I} + (-1)^{p+1}i(P)(C(Q)) \wedge \mathsf{I}$$

and thus the result holds for the algebraic operators. The other cases are similar. ∎

2.4. As an application of the results in (1.8) and (1.9) we now determine all linear and bilinear operators between ordinary differential forms, which are natural under local diffeomorphisms and do not involve top degree forms. To avoid numerical factors we define an operator $\bar{C} : \Omega^p(M;TM) \to \Omega^{p-1}(M)$ by $\bar{C}(P) := \frac{(-1)^{p-1}}{m-p+1}C(P)$. Then we clearly have $\bar{C}(\varphi \wedge \mathsf{I}) = \varphi$ for all $\varphi \in \bigoplus_{k=1}^{m-1} \Omega^k(M)$.

Let us first discuss the linear case: Here we reprove a result of R. S. Palais ([**Pa**]). In [**Ko**] it is shown that for $k > 0$ d is unique even without assuming linearity.

THEOREM. *If $A : \Omega^k(M) \to \Omega^\ell(M)$ is a nonzero linear operator, which is natural under local diffeomorphisms and $k, \ell < m := dim(M)$, then either $\ell = k$ and A is a scalar multiple of the identity, or $\ell = k+1$ and A is a scalar multiple of the exterior derivative d.*

PROOF: If A is such an operator, then $P \mapsto A(\bar{C}(P)) \wedge \mathsf{I}$ is a linear natural operator $\Omega^{k+1}(M;TM) \to \Omega^{\ell+1}(M;TM)$.

Now if $\ell \neq k, k+1$, then this operator must be identically zero, and applying $\bar{C}$ we see that $A(\bar{C}(P)) = 0$. Now putting $P := \varphi \wedge \mathsf{I}$ we get $A(\varphi) = 0$ for all $\varphi \in \Omega^k(M)$.

Next for $\ell = k+1$, $A(\bar{C}(P)) \wedge \mathsf{I}$ must be a scalar multiple of $d\bar{C}(P) \wedge \mathsf{I}$ and so we may conclude as before, that $A(\varphi)$ must be a scalar multiple of $d\varphi$.

Finally if $\ell = k$, then $A(\bar{C}(P)) \wedge \mathsf{I}$ is a linear combination of the identity and $\bar{C}(P) \wedge \mathsf{I}$. Now putting $P = \varphi \wedge \mathsf{I}$ we see that $A(\varphi) \wedge \mathsf{I}$ must be a scalar multiple of $\varphi \wedge \mathsf{I}$ and applying $\bar{C}$ the result follows. ∎

2.5. Before discussing the bilinear case we need one more result:

LEMMA. *For $\varphi \in \Omega^k(M)$, $\psi \in \Omega^\ell(M)$ and $m := dim(M)$ we have:*

(1) $S(\varphi \wedge \mathsf{I}, \psi \wedge \mathsf{I}) = (-1)^{k+\ell+1}(k\ell + k + \ell - m)\varphi \wedge \psi$

(2) $[\varphi \wedge \mathsf{I}, \psi \wedge \mathsf{I}] = (-1)^{k+1}\ell d\varphi \wedge \psi \wedge \mathsf{I} - k\varphi \wedge d\psi \wedge \mathsf{I}$

PROOF:

(1): By proposition (2.1) we have $S(P,Q) = C(i(P)(Q)) + (-1)^p i(P)(C(Q))$ and a short computation using (2.1) again leads to the result.

(2) is proved by a short computation using results of [**Mi**]. ∎

2.6. Now we can prove the theorem on bilinear natural operators between ordinary differential forms:

THEOREM. *Suppose that $A : \Omega^k(M) \times \Omega^\ell(M) \to \Omega^r(M)$ is a nonzero bilinear operator, which is natural under local diffeomorphisms, and that $k, \ell, r < m := dim(M)$. Then either $r = k + \ell$ and A is a scalar multiple of the wedge product, or $r = k+\ell+1$ and $A(\varphi, \psi)$ is a linear combination of $d\varphi \wedge \psi$ and $\varphi \wedge d\psi$, or $r = k+\ell+2$ and $A(\varphi, \psi)$ is a scalar multiple of $d\varphi \wedge d\psi$.*

PROOF: Using the operator $(P,Q) \mapsto A(\bar{C}(P), \bar{C}(Q)) \wedge \mathsf{I}$, which is a bilinear natural operator: $\Omega^{k+1}(M;TM) \times \Omega^{\ell+1}(M;TM) \to \Omega^{r+1}(M;TM)$ we argue as in the linear case.

By proposition (2.1) and lemma (2.5) all operators in the list (1.9) reduce for $P = \varphi \wedge \mathsf{I}$ and $Q = \psi \wedge \mathsf{I}$ to scalar multiples of $\varphi \wedge \psi \wedge \mathsf{I}$, to linear combinations of $d\varphi \wedge \psi \wedge \mathsf{I}$ and $\varphi \wedge d\psi \wedge \mathsf{I}$ or to scalar multiples of $d\varphi \wedge d\psi \wedge \mathsf{I}$. Thus the result follows as in the linear case. ∎

3. ALGEBRAIC STRUCTURES ON SETS OF MULTILINEAR NATURAL OPERATORS BETWEEN VECTOR VALUED DIFFERENTIAL FORMS

Following [**Sch**] and [**L–M–Sch**] we define in this section first a structure of a bigraded Lie algebra on the vector space of all multilinear natural operators between vector valued differential forms and then another such structure on the space of all graded alternating multilinear natural operators.

3.1 Definition.

(1) Let $\mathcal{M}^k_\ell$ be the space of all $k+1$-linear natural operators between vector valued differential forms, which are homogeneous of degree ℓ, i.e. which map $\prod_{i=1}^{k+1} \Omega^{p_i}(M;TM)$ to $\Omega^r(M;TM)$ with $r = \sum p_i + \ell$.
Then $\mathcal{M} = \bigoplus_{(k,\ell)\in\mathbb{Z}^2}$ is the space of all multilinear natural operators between vector valued differential forms.

(2) For $\varphi \in \mathcal{M}^k_\ell$ and $\psi \in \mathcal{M}^{k'}_{\ell'}$ define $j(\varphi)(\psi) \in \mathcal{M}^{k+k'}_{\ell+\ell'}$ by:

$$j(\varphi)(\psi)(P_0,\dots,P_{k+k'}) :=$$
$$= \sum_{i=0}^{k'-1} (-1)^{\ell\ell'+ki+\ell(p_0+\dots+p_{i-1})}\psi(P_0,\dots,P_{i-1},\varphi(P_i,\dots,P_{i+k}),\dots)$$

for $P_i \in \Omega^{p_i}(M;TM)$.

(3) For φ and ψ as above define:

$$[\varphi,\psi]^\Delta := j(\varphi)(\psi) - (-1)^{kk'+\ell\ell'} j(\psi)(\varphi)$$

3.2 THEOREM. $(\mathcal{M},[\ ,\ ]^\Delta)$ *is a bigraded Lie algebra. This means that*

(1) $[\ ,\ ]^\Delta : \mathcal{M}\times\mathcal{M} \to \mathcal{M}$ *is a bilinear map and* $[\mathcal{M}^k_\ell, \mathcal{M}^{k'}_{\ell'}]^\Delta \subseteq \mathcal{M}^{k+k'}_{\ell+\ell'}$

(2) *For* $\varphi \in \mathcal{M}^k_\ell$ *and* $\psi \in \mathcal{M}^{k'}_{\ell'}$ *we have:*
$[\varphi,\psi]^\Delta = -(-1)^{kk'+\ell\ell'}[\psi,\varphi]^\Delta$

(3) *For* $\varphi \in \mathcal{M}^k_\ell$, $\psi \in \mathcal{M}^{k'}_{\ell'}$ *and* $\omega \in \mathcal{M}$ *we have:*
$[\varphi,[\psi,\omega]^\Delta]^\Delta = [[\varphi,\psi]^\Delta,\omega]^\Delta + (-1)^{kk'+\ell\ell'}[\psi,[\varphi,\omega]^\Delta]^\Delta$

PROOF: The first two properties are trivial, while the proof of the last one requires a longer computation which is carried out in [Sch]. ∎

3.3. To treat the alternating case we first need the definition of the graded alternator: For $i = 1,\dots,k$ let $P_i \in \Omega^{p_i}(M;TM)$ and $\bar{p} := (p_1,\dots,p_k) \in \mathbb{Z}^k$. For a permutation $\sigma \in \mathcal{S}^k$ of k elements let $\sigma(\bar{p})$ be a permutation of $\sum p_i$ elements which acts as follows: divide the $\sum p_i$ elements into k disjoint blocks of length p_i and permute these blocks according to σ. Then define $s(\sigma,\bar{p}) := sign(\sigma)\cdot sign(\sigma(\bar{p}))$, where *sign* denotes the sign of a permutation.

Now we define the graded alternator $\alpha : \mathcal{M}^{k-1}_\ell \to \mathcal{M}^{k-1}_\ell$ by:

$$\alpha\varphi(P_1,\dots,P_k) := \tfrac{1}{k!}\sum_{\sigma\in\mathcal{S}^k} s(\sigma,\bar{p})\varphi(P_{\sigma 1},\dots,P_{\sigma k})$$

3.4 Definition.

(1) Let $\mathcal{A}^k_\ell := \alpha(\mathcal{M}^k_\ell) \subseteq \mathcal{M}^k_\ell$ and $\mathcal{A} := \bigoplus_{(k,\ell)\in\mathbb{Z}^2}\mathcal{A}^k_\ell$. Since it turns out that the graded alternator is idempotent $\mathcal{A}$ is stable under α.

(2) For $\varphi \in \mathcal{A}^k_\ell$ and $\psi \in \mathcal{A}^{k'}_{\ell'}$ we define $i(\varphi)(\psi) \in \mathcal{A}^{k+k'}_{\ell+\ell'}$ by:

$$i(\varphi)(\psi) := \tfrac{(k+k'+1)!}{(k+1)!(k'+1)!}\alpha(j(\varphi)(\psi))$$

(3) For φ and ψ as above we define $[\varphi,\psi]^\wedge \in \mathcal{A}^{k+k'}_{\ell+\ell'}$ by:

$$[\varphi,\psi]^\wedge := i(\varphi)(\psi) - (-1)^{kk'+\ell\ell'} i(\psi)(\varphi) = \\ = \tfrac{(k+k'+1)!}{(k+1)!(k'+1)!}\alpha([\varphi,\psi]^\Delta)$$

3.5. THEOREM. *$(\mathcal{A},[\ ,\ ]^\wedge)$ is a bigraded Lie algebra.*

PROOF: This is a computation using the properties of $[\ ,\ ]^\Delta$ which is carried out in [Sch]. ∎

Thus multilinear natural operators between vector valued differential forms lead to two natural examples of rather "small" bigraded Lie algebras.

3.6. Remark. The two brackets introduced here have the nice property that they detect certain algebraic structures. Short computations show the following results:

(1) Suppose that A is an element of $\mathcal{M}^1_0$. Then A is a natural bilinear "multiplication" on $\Omega(M;TM)$ homogeneous of weight zero. It turns out that A defines an associative multiplication if and only if $[A,A]^\Delta = 0$.

(2) Suppose that A is an element of $\mathcal{A}^1_0$. Then A is a bilinear natural graded alternating map on $\Omega(M;TM)$ homogeneous of weight zero. It turns out that A defines the structure of a graded Lie algebra, i.e. satisfies the graded Jacobi identity if and only if $[A,A]^\wedge = 0$.

This gives a probability to determine all natural associative algebra structures as well as all natural graded Lie algebra structures on $\Omega(M;TM)$.

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