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ON BOUNDARY VALUE PROBLEM OF NEUMANN TYPE FOR HYPERCOMPLEX FUNCTION WITH VALUES IN A CLIFFORD ALGEBRA

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1. Introduction

Let $\mathbf{R}_{o,n-1}$ be the universal algebra constructed by means of an orthonormal basis $\{e_2, e_3, \ldots, e_n\}$ of $\mathbf{R}^{o,n-1}$ and $e_\phi = e_1$ be the identity of $\mathbf{R}_{o,n-1}$. In the universal Clifford algebra $\mathbf{R}_{o,n-1}$ the multiplicative identity

$$e_i e_j + e_j e_i = -2\delta_{ij} \qquad i,j=2,3,\ldots,n,$$

holds. Let $\overline{\partial} = e_1 \frac{\partial}{\partial x_1} + \sum_{j=2}^{n} e_j \frac{\partial}{\partial x_j}$ be a generalized Cauchy-Riemann operator and Ω be an open set of $\mathbf{R}^n$. Then an $\mathbf{R}_{o,n-1}$ -valued $C^1(\Omega)$-function $f(x)= \sum_A f_A(x) e_A$ is called left monogenic function if $\overline{\partial} f=0$ in Ω , here $e_A = e_{\alpha_2\alpha_3\cdots\alpha_h} = e_{\alpha_2} e_{\alpha_3} \cdots e_{\alpha_h}$ are bases of $\mathbf{R}_{o,n-1}$ and $A=\{\alpha_2, \alpha_3, \ldots, \alpha_h\} \subseteq \{2,3,\ldots,n\}$ with $2 \leq \alpha_2 < \cdots < \alpha_h \leq n$.

Up to now a great number of the function theory for left monogenic function has been investigated [1][2][6]. For the use of Clifford algebra in several branches of Mathematics and Physics we refer to [2]. Recently several boundary value problem for monogenic functions have been studied (see [8][9][10][11]). In this paper we study a boundary value problem of Neumann type for hypercomplex function with values in a Clifford algebra which is a solution of a nonhomogeneous equation $\overline{\partial} w=f$ in the

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unit ball of $\mathbf{R}^n$. To this end we firstly construct a Neumann's function for Laplacian over the unit ball in $\mathbf{R}^n (n \geq 3)$. Then, making use of this Neumann's function, we solve the above boundary value problem.

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2. Neumann's function

In this section we construct an explicit representation formula for Neumann's function for Laplacian over the unit ball in $\mathbf{R}^n (n \geq 3)$ by means of the fundamental solution for Laplacian and Gegenbauer polynomials. Then, making use of this Neumann's function, the solution of the Neumann problem for the Poisson equation is represented.

In the sequel, K_n denotes a unit ball in $\mathbf{R}^n$, $K_n = \{x \mid x = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n, |x| < 1\}$. We seek for a real function $N(x,y)$ which satisfies

$$\Delta_x N(x,y) = \delta(x-y) + c, \qquad x \in K_n \tag{2.1}$$

$$\frac{\partial}{\partial n_x} N(x,y) = 0, \qquad x \in \partial K_n \tag{2.2}$$

hereby Δ_x denotes the Laplacian $\Delta_x = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $\frac{\partial}{\partial n_x}$ denotes the outer normal derivative to the boundary ∂K_n of K_n, c is a suitable constant to be determined, while y is fixed in the domain K_n. In fact this problem is not solvable for any real constant c.

Now we try to look for a solution of problem (2.1)(2.2) in the following form

$$N(x,y) = \frac{1}{\omega_n (2-n) |x-y|^{n-2}} + \frac{1}{\omega_n (2-n) |y|^{n-2} |x-y^*|^{n-2}} + g(x,y), \tag{2.3}$$

whereby ω_n stands for the area of the unit ball in $\mathbf{R}^n$, $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$, $\frac{1}{\omega_n (2-n) |x-y|^{n-2}}$ is the fundamental solution for Laplacian, namely $\Delta_x \left(\frac{1}{\omega_n (2-n) |x-y|^{n-2}} \right) = \delta(x-y)$; y^* is the symmetric point of y to the unit sphere ∂K_n, $y^*=\frac{y}{|y|^2}$. It is clear that, for $y\in K_n$, the function $\frac{1}{\omega_n(2-n)|y|^{n-2}|x-y^*|^{n-2}}$ is harmonic in K_n with respect to x.

By (2.1)(2.2), for $y\in K_n$, the function g(x,y) should satisfies

$$\Delta_x g(x,y)=c, \qquad x\in K_n \tag{2.4}$$

$$\frac{\partial}{\partial n_x}g(x,y)=-\frac{\partial}{\partial n_x}\left(\frac{1}{\omega_n(2-n)|x-y|^{n-2}}+\frac{1}{\omega_n(2-n)|y|^{n-2}|x-y^*|^{n-2}}\right), \qquad x\in\partial K_n \tag{2.5}$$

Now we suppose that the function g(x,y) has the following form

$$g(x,y)=h(x,y)+\alpha|x|^2, \tag{2.6}$$

so that the function h(x,y) satisfies, for $y\in K_n$,

$$\Delta_x h(x,y)=0, \qquad x\in K_n \tag{2.7}$$

$$\frac{\partial}{\partial n_x}h(x,y)=-\frac{\partial}{\partial n_x}\left(\frac{1}{\omega_n(2-n)|x-y|^{n-2}}+\frac{1}{\omega_n(2-n)|y|^{n-2}|x-y^*|^{n-2}}+\alpha|x|^2\right). \qquad x\in\partial K_n \tag{2.8}$$

Thus the constants c and α are related with

$$c=2n\alpha. \tag{2.9}$$

It is well known that the Neumann problem (2.7)(2.8) is solvable if and only if the constant α satisfies

$$\int_{\partial K_n}\left(\frac{\partial}{\partial n_x}\left(\frac{1}{\omega_n(2-n)|x-y|^{n-2}}+\frac{1}{\omega_n(2-n)|y|^{n-2}|x-y^*|^{n-2}}+\alpha|x|^2\right)\right)ds_x=0,$$

where ds_x stands for the areal element of ∂K_n.

Since we have

$$\int_{\partial K_n}\frac{\partial}{\partial n_x}\left(\frac{1}{\omega_n(2-n)|x-y|^{n-2}}\right)ds_x=\begin{cases}0 & y\notin\overline{K_n}\\ \frac{1}{2} & y\in\partial K_n\\ 1, & y\in K_n\end{cases}$$

and

$$\int_{\partial K_n}\frac{\partial}{\partial n_x}|x|^2 ds_x=2\omega_n,$$

so the Neumann problem (2.7)(2.8) is solvable if and only if

$$\alpha = -\frac{1}{2\omega n} \quad . \tag{2.10}$$

Thus the problem (2.1)(2.2) is solvable if and only if

$$c = -\frac{n}{\omega n} \, . \tag{2.11}$$

Now we look for the solutions of the problem (2.7)(2.8), where α is defined by (2.10). Suppose that there exists a solution for the problem (2.7)(2.8) in the following form

$$h(x,y) = \sum_{k=0}^{\infty} a_k(y)|x|^k C_k^{\frac{n-2}{2}}(\cos\theta), \tag{2.12}$$

whereby $C_k^{\frac{n-2}{2}}(t)$ are Gegenbauer polynomials, θ denotes the angle between the radial directions ox and oy, $a_k(y)$ are functions of y to be determined. It is clear that the series (2.12) is harmonic in Ω if it is uniformly convergent in Ω . By the boundary condition (2.8) now we determine functions $a_k(y)$. Notice that, for $|x|>|y|$, we have

$$\frac{1}{|x-y|^{n-2}} = \sum_{k=0}^{\infty} C_k^{\frac{n-2}{2}}(\cos\theta)|y|^k|x|^{-(n+k-2)}$$

and

$$\frac{1}{|y|^{n-2}|x-y^*|^{n-2}} = \sum_{k=0}^{\infty} C_k^{\frac{n-2}{2}}(\cos\theta)|x|^k|y|^k.$$

On the one side, by (2.8) and (2.10) we have

$$\begin{aligned}\frac{\partial}{\partial n_x} h|_{\partial K_n} &= -\frac{1}{\omega n(2-n)}\Big(\sum_{k=0}^{\infty}(-(n+k-2))C_k^{\frac{n-2}{2}}(\cos\theta)|y|^k \\ &\quad + \sum_{k=1}^{\infty} k\, C_k^{\frac{n-2}{2}}(\cos\theta)|y|^k\Big) - 2\alpha \\ &= \sum_{k=1}^{\infty}\Big(-\frac{1}{\omega n}\Big) C_k^{\frac{n-2}{2}}(\cos\theta)|y|^k.\end{aligned}$$

And on the other side, by (2.12) we have

$$\frac{\partial}{\partial n_x} h|_{\partial K_n} = \sum_{k=1}^{\infty} k\, a_k(y)\, C_k^{\frac{n-2}{2}}(\cos\theta).$$

Therefore we have

$$a_k(y) = -\frac{1}{k\,\omega n}|y|^k, \qquad k = 1,2,\cdots\cdots$$

Notice $C_o^{\frac{n-2}{2}}(\cos\theta)=1$, so we have

$$h(x,y)=\sum_{k=1}^{\infty}\frac{-1}{k\cdot\omega n}C_k^{\frac{n-2}{2}}(\cos\theta)|x|^k|y|^k+a_o(y). \tag{2.13}$$

It is easy to see that the series (2.13) is uniformly convergent in K_n with respect to x while $y\in K_n$. Now we can get the following proposition.

Proposition 2.1 Let K_n be the unit ball in $\mathbb{R}^n(n\geq3)$. Then the function $N_n(x,y)$

$$\begin{aligned}N_n(x,y)= & \frac{1}{(2-n)\omega n|x-y|^{n-2}}+\frac{1}{(2-n)\omega n|y|^{n-2}|x-y^*|^{n-2}}\\ & -\frac{1}{2\omega n}(|x|^2+|y|^2)-\frac{1}{\omega n}\sum_{k=1}^{\infty}C_k^{\frac{n-2}{2}}(\cos\theta)\frac{|x|^k|y|^k}{k}\\ & -\frac{n^2+n+2}{\omega n(4-n^2)}\end{aligned} \tag{2.14}$$

has the following properties:

i) $\Delta_x N_n(x,y)=\delta(x-y)-\frac{n}{\omega n}$, $\quad x\in K_n,\quad y\in K_n$

ii) $\frac{\partial}{\partial n_x}N_n(x,y)=0$, $\quad x\in\partial K_n,\quad y\in K_n$

iii) $N_n(x,y)=N_n(y,x)$,

iv) $\int_{K_n}N_n(x,y)\,dx=0$, $\quad y\in K_n$

whereby y^* is the symmetric point of y to the unit sphere, $y^*=\frac{y}{|y|^2}$, dx is the volume element of K_n. The function $N_n(x,y)$ is called the Neumann's function for Laplacian over the unit ball in $\mathbb{R}^n(n\geq3)$.

Indeed from above inference the properties (i)(ii)(iii) follow immediately. Now we put, for $y\in K_n$,

$$f_1(y)=\int_{K_n}\frac{1}{(2-n)\omega n|x-y|^{n-2}}\,dx,$$

$$f_2(y)=\int_{K_n}\frac{1}{(2-n)\omega n|y|^{n-2}|x-y^*|^{n-2}}\,dx,$$

$$f_3(y)=\int_{K_n}\left(-\frac{1}{\omega n}\sum_{k=1}^{\infty}C_k^{\frac{n-2}{2}}(\cos\theta)\frac{|x|^k|y|^k}{k}\right)dx.$$

Notice that the function $f_1(y)$ satisfies the Poisson equation $\Delta f_1(y)=1$,

$f_1(0)=\frac{1}{2(2-n)}$ and $f_1(y)$ is only a function of $|y|$. Then we can easily get

$$f_1(y)=\frac{1}{2(2-n)}+\frac{1}{2n}|y|^2, \qquad y\in K_n. \tag{2.15}$$

Notice also that $f_1(y)$ is continuous in the whole space $\mathbf{R}^n$, so $f_1(y)=\frac{1}{n(2-n)}$, for $|y|=1$. Moreover, $f_1(y)$ is harmonic in $\mathbf{R}^n\setminus\overline{K_n}$ and vanishes at infinity. So we can also get

$$f_1(y)=\frac{1}{n(2-n)|y|^{n-2}}, \qquad y\in\mathbf{R}^n\setminus\overline{K_n}. \tag{2.15'}$$

Therefore we have

$$\begin{aligned} f_2(y) &= \frac{1}{|y|^{n-2}}\,f_1(y^*) = \frac{1}{n(2-n)|y|^{n-2}|y^*|^{n-2}} \\ &= \frac{1}{n(2-n)}, \qquad y\in K_n \end{aligned} \tag{2.16}$$

We also have

$$-\int_{K_n}\frac{1}{2\omega n}(|x|^2+|y|^2)\,dx = -\frac{1}{2(n+2)}-\frac{1}{2n}|y|^2. \tag{2.17}$$

Next we calculate $f_3(y)$ for $y\in K_n$. Since the series $h_3(y)$

$$h_3(y) = -\frac{1}{\omega n}\sum_{k=1}^{\infty} C_k^{\frac{n-2}{2}}(\cos\theta)\,\frac{|x|^k|y|^k}{k}$$

is harmonic in K_n with respect to y while $x\in K_n$, moreover,

$$\begin{aligned} \frac{\partial}{\partial n_y}h_3(y)\Big|_{y\in\partial K_n} &= -\frac{1}{\omega n}\sum_{k=1}^{\infty} C_k^{\frac{n-2}{2}}(\cos\theta)|x|^k \\ &= -\frac{1}{\omega n}\left(\frac{1}{(1-2|x|\cos\theta+|x|^2)^{n-2/2}}-1\right). \end{aligned}$$

Then we have

$$\begin{aligned} \frac{\partial}{\partial n_y}f_3(y)\Big|_{y\in\partial K_n} &= -\frac{1}{\omega n}\int_{K_n}\left(\frac{1}{(1-2|x|\cos\theta+|x|^2)^{n-2/2}}-1\right)dx \\ &= -(2-n)f_1(1)+\frac{1}{n} = 0, \end{aligned}$$

and $f_3(y)$ is harmonic in K_n. Therefore we get

$$f_3(y) = f_3(o) = 0 , \qquad y \in K_n \qquad (2.18)$$

From (2.15)-(2.18) the property (iv) follows immediately. Thus this proposition has been proved.

In addition, it is easy to see that the Neumann's function having the properties (i)-(iv) is unique. By above Neumann's function we can get the following proposition in a classical way.

<u>Proposition 2.2</u> Let K_n be the unit ball in $\mathbf{R}^n(n\geq 3)$. Let f(x) be a continuous and differentiable function in $\overline{K}_n$. Then the Neumann problem for Poisson equation

$$\Delta u(x) = f(x) , \qquad x \in K_n \qquad (2.19)$$

$$\frac{\partial}{\partial n} u(x) = 0 , \qquad x \in \partial K_n . \qquad (2.20)$$

is solvable in the classical sense if and only if f(x) satisfies

$$\int_{K_n} f(x)\, dx = 0 \qquad (2.21)$$

and the solution may be represented in the following formula

$$u(x)=\int_{K_n} N_n(x,y)\, f(y)\, dy + c , \qquad (2.22)$$

whereby the function $N_n(x,y)$ is the Neumann's function described in the proposition 2.1 and c is an arbitrary real constant.

3. <u>Boundary value problem of Neumann type</u>

We first introduce the subalgebra $H^{(i)}(\mathbf{R}_{o,n-1})$ of $\mathbf{R}_{o,n-1}$. Fixing $i \in \{1, 2,\cdots\cdots,n\}$ and considering for i>2 the subalgebra $H^{(i)}(\mathbf{R}_{o,n-1})$ of $\mathbf{R}_{o,n-1}$ is generated by the basis elements $e_2,e_3,\cdots\cdots,e_{i-1}$, then $H^{(i)}(\mathbf{R}_{o,n-1})$ is isomorphic to the Clifford algebra $\mathbf{R}_{o,i-2}$ for $\mathbf{R}^{o,i-2}$. For i=1,2, we put $H^{(i)}(\mathbf{R}_{o,n-1})=\mathbf{R}$.

Now define for $i \in \{1,2,\cdots\cdots,n\}$ fixed the set I_i by

$$I_i = \begin{cases} \{i+1,\cdots\cdots,n\}, & \text{if } i\in\{1,2,\cdots\cdots,n-1\} \\ \phi & \text{if } i=n. \end{cases}$$

And let us introduce the projective operator $X^{(i)}$ of $\mathbf{R}_{o,n-1}$ onto its subspace $H^{(i)}(\mathbf{R}_{o,n-1})\, e_{I_i}$. Then we have for each $a=\sum_A a_A e_A \in \mathbf{R}_{o,n-1}$ that

$$X^{(i)}a=\sum_C a_C e_C$$

where C runs over all ordered subset of the form $C=D\cup I_i$ with $D\subset\{2,\cdots\cdots, i-1\}$, while for i=1

$$X^{(1)}(a)= a_{2\cdots\cdots n}e_{2\cdots\cdots n}.$$

Notice in particular that for i=n

$$X^{(n)}(\mathbf{R}_{o,n-1}) = H^{(n)}(\mathbf{R}_{o,n-1})e_1 = H^{(n)}(\mathbf{R}_{o,n-1}),$$

i.e. $X^{(n)}(\mathbf{R}_{o,n-1})$ is the clifford subalgebra of $\mathbf{R}_{o,n-1}$ generated by the basis elements $e_2,\cdots\cdots,e_{n-1}$, where

$$\mathbf{R}_{o,n-1} = X^{(n)}(\mathbf{R}_{o,n-1}) \oplus X^{(n)}(\mathbf{R}_{o,n-1})\, e_n. \tag{3.1}$$

Now consider the main involution $a\to\hat{a}$ and the conjugation $a\to\bar{a}$ on $\mathbf{R}_{o,n-1}$, i.e. if $a=\sum_A a_A e_A$, then

$$\hat{a}=\sum_A a_A\hat{e}_A, \qquad \bar{a}=\sum_A a_A\bar{e}_A ,$$

whereby for each $e_A=e_{\alpha_2}\cdots\cdots e_{\alpha_h}$,

$$\hat{e}_A=(-1)^{\#A}\, e_A \qquad \text{and} \qquad \bar{e}_A=(-1)^{\#A}\, e_{\alpha_h}\cdots\cdots e_{\alpha_2},$$

#A denoting the cardinality of A. Then for each $d\in X^{(n)}(\mathbf{R}_{o,n-1})$

$$d\cdot e_n= e_n\, \hat{d} .$$

We obtain by means of (3.1) that an arbitrary $a\in\mathbf{R}_{o,n-1}$ may be written as

$$a = b + c\, e_n = b + e_n\, \hat{c} , \qquad b,c\in X^{(n)}(\mathbf{R}_{o,n-1}).$$

Putting $\hat{c}= Y^{(n)}(a)$, we so have for all $a\in\mathbf{R}_{o,n-1}$ that

$$a = X^{(n)}(a) + e_n\, Y^{(n)}(a). \tag{3.2}$$

Notice that for n=2, we obtain that, since $X^{(2)}(\mathbf{R}_{o,1})$ is isomorphic to **R**, any $a \in \mathbf{R}_{o,1}$ admits the decomposition

$$a = b + e_2\, c, \qquad b,c \in \mathbf{R} \tag{3.3}$$

whereby $e_2{}^2 = -1$. The relation (3.3) makes us remind of the well known result that $\mathbf{R}_{o,1}$ is isomorphic (as a real algebra) to **C**. But at the same time it illustrates how the decomposition obtained in (3.2) of an arbitrary element a in $\mathbf{R}_{o,n-1}$ is a generalization of the classical representation of a complex number a in the form a=x+iy, $x,y \in \mathbf{R}$.

Now let in a classical way $\mathbf{R}^i$ $(i \geq 2)$ be identified with subspace of $\mathbf{R}^n$ given by these elements $x=(x_1,\cdots\cdots,x_n) \in \mathbf{R}^n$ for which $x_{i+1}=\cdots\cdots=x_n=0$. Then by means of the isomorphism between $\mathbf{R}^n$ and $\mathbf{R}e_1 \oplus \mathbf{R}^{o,n-1}$,

$$x=(x_1,\cdots\cdots,x_n), \qquad z_x^{(n)} = \sum_{j=1}^{n} x_j e_j,$$

we may identify $\mathbf{R}^i$ with $\mathbf{R}e_1 \oplus \mathbf{R}^{o,i-1}$. An arbitrary element of $\mathbf{R}^i (i \geq 2)$ may therefore be written as $z_x^{(i)} = x_1e_1+x_2e_2+\cdots\cdots+x_ie_i$. Notice that for $i=2,\cdots\cdots,n$,

$$|z_x^{(i)}|^2 = \overline{z_x^{(i)}}\,z_x^{(i)} = z_x^{(i)}\,\overline{z_x^{(i)}} = \sum_{j=1}^{i} x_j{}^2.$$

Now we also consider the generalized Cauchy-Riemann operator $\overline{\partial}_i$ and its conjugate operator ∂_i in $\mathbf{R}^i$, $i=2,\cdots\cdots,n$, i.e.

$$\overline{\partial}_i = \sum_{j=1}^{i} e_j \frac{\partial}{\partial x_j} = e_1 \frac{\partial}{\partial x_1} + \sum_{k=2}^{i} e_k \frac{\partial}{\partial x_k}$$

and

$$\partial_i = \overline{\overline{\partial}}_i = e_1 \frac{\partial}{\partial x_1} - \sum_{k=2}^{i} e_k \frac{\partial}{\partial x_k}\, .$$

Let $K_n^{(i)}$ and $\partial K_n^{(i)}$ be the unic ball and unit sphere in $\mathbf{R}^i$ respectively, i.e.

$$K_n^{(i)} = \{x=(x_1,x_2,\cdots\cdots,x_n): \sum_{j=1}^{i} x_j{}^2 < 1 \text{ and } \sum_{j=i+1}^{n} x_j{}^2 = 0\}$$

and

$$\partial K_n^{(i)} = \{x=(x_1,x_2,\cdots\cdots,x_n): \sum_{j=1}^{i} x_j{}^2 = 1 \text{ and } \sum_{j=i+1}^{n} x_j{}^2 = 0\}.$$

Now we consider the following integral operator T_i which are also called the Pompieu-operator and may be defined on the space of $\mathbf{R}_{o,i-1}$-valued functions g

$$(T_i g)(z_x^{(i)}) = -\frac{1}{\omega_i}\int_{K_n^{(i)}} \frac{z_x^{(i)}-z_y^{(i)}}{|z_x^{(i)}-z_y^{(i)}|^i} g(z_y^{(i)})dy_1\cdots\cdots dy_i , \quad i=2,3,\cdots\cdots,n-1, \tag{3.4}$$

whereby ω_i is the area of $\partial K_n^{(i)}$, and g is of the class $C_\alpha(0<\alpha<1)$ in $\overline{K_n^{(i)}}$. By [4] it is well known that

$$\partial_i(T_i g)=g . \tag{3.5}$$

Now let us formulate the following basic lemma.

<u>Lemma 3.1</u> The nonhomogeneous generalized Cauchy-Riemann equation

$$\overline{\partial_n}\, w=f, \qquad \text{in } K_n^{(n)} \tag{3.6}$$

is equivalent to the following equations

$$\overline{\partial_{n-1}}X^{(n)}w-\frac{\partial}{\partial x_n}\, Y^{(n)}w = X^{(n)}f, \qquad \text{in } K_n^{(n)} \tag{3.7}$$

$$\frac{\partial}{\partial x_n}X^{(n)}w+\partial_{n-1}\, Y^{(n)}w = Y^{(n)}f, \qquad \text{in } K_n^{(n)} \tag{3.8}$$

whereby $(X^{(n)}w)(x)=X^{(n)}(w(x))$ and $(Y^{(n)}w)(x)=Y^{(n)}(w(x))$.

<u>Proof</u> As $\overline{\partial_{n-1}}(e_n Y^{(n)}w)=e_n\,\partial_{n-1}(Y^{(n)}w)$, we have

$$\overline{\partial_n}\, w = (\,\overline{\partial_{n-1}} +e_n \frac{\partial}{\partial x_n}\,)(X^{(n)}w+e_n Y^{(n)}w)$$

$$=(\overline{\partial_{n-1}}\, X^{(n)}w-\frac{\partial}{\partial x_n}\, Y^{(n)}w)+e_n(\frac{\partial}{\partial x_n}\, X^{(n)}w+\partial_{n-1}Y^{(n)}w)$$

$$=X^{(n)}f+e_n Y^{(n)}f\ .$$

Thus this lemma is proved.

<u>Theorem 3.2</u> Let f(x) be a continuous differentiable $\mathbf{R}_{o,n-1}$-valued function in $\overline{K_n^{(n)}}$. Then the boundary value problem of Neumann type

$$\overline{\partial_n}\, w=f, \qquad x\in K_n^{(n)} \tag{3.9}$$

$$X^{(n)}(\frac{\partial}{\partial n}\, w)=0, \qquad x\in\partial K_n^{(n)} \tag{3.10}$$

is solvable if and only if f(x) satisfies the following integral-differential relation

$$\int_{K_n^{(n)}} (\, \partial_{n-1}(X^{(n)}f)+\frac{\partial}{\partial x_n}\, Y^{(n)}f)\, dx = 0. \tag{3.11}$$

And the solution may be represented in the following form

$$w = X^{(n)}w + e_n Y^{(n)}w,$$

$$X^{(n)}w=\int_{K_n^{(n)}} N_n(x,y)(\, \partial_{n-1}(X^{(n)}f)+\frac{\partial}{\partial y_n}\, (Y^{(n)}f))(y)\, dy + c, \tag{3.12}$$

$$Y^{(n)}w=\int_0^{x_n} (\overline{\partial_{n-1}}\, X^{(n)}w-X^{(n)}f)\, dx_n$$

$$+ T_{n-1}((-\frac{\partial}{\partial x_n}\, X^{(n)}w+ Y^{(n)}f)\,|_{x_n=0}\,) + \tilde{h}(z_x^{(n-1)}), \tag{3.13}$$

where c is an arbitrary $X^{(n)}(\mathbf{R}_{o,n-1})$-valued constant and $\tilde{h}(z_x^{(n-1)})$ is an arbitrary $X^{(n)}(\mathbf{R}_{o,n-1})$-valued function satisfying $\partial_{n-1}\tilde{h}(z_x^{(n-1)})=0$.

<u>Proof</u> If there exists a solution w(x) for the problem (3.9)(3.10) in the classical sense, then w(x) should satisfies the following problem

$$\partial_n(\overline{\partial_n}\, w)= \partial_n f, \qquad x\in K_n^{(n)} \tag{3.14}$$

$$\frac{\partial}{\partial n}\, (X^{(n)}w)=0\ . \qquad x\in \partial K_n^{(n)} \tag{3.15}$$

Then we have that $X^{(n)}w$ should satisfies the following problem

$$\Delta_n X^{(n)}w= \partial_{n-1}(X^{(n)}f)+\frac{\partial}{\partial x_n}\, (Y^{(n)}f), \quad x \in K_n^{(n)} \tag{3.16}$$

$$\frac{\partial}{\partial n}\, (X^{(n)}w)=0\ , \qquad x \in \partial K_n^{(n)}. \tag{3.17}$$

By proposition 2.2 then we get (3.11)(3.12). By lemma 3.1 and (3.7) we have

$$Y^{(n)}w=\int_0^{x_n}(\, \overline{\partial_{n-1}}\, X^{(n)}w-X^{(n)}f)\, dx_n + h(z_x^{(n-1)}),$$

where $h(z_x^{(n-1)})$ is an arbitrary $X^{(n)}(\mathbf{R}_{o,n-1})$-valued function depending only on $x_1,\cdots\cdots,x_{n-1}$. Substituting into (3.8), we get

$$\frac{\partial}{\partial x_n} X^{(n)}w + \int_0^{x_n}(\Delta_{n-1}X^{(n)}w - \partial_{n-1}(X^{(n)}f))dx_n + \partial_{n-1}h(z_x^{(n-1)})=Y^{(n)}f \; .$$

By (3.16) we have

$$\Delta_{n-1}X^{(n)}w = - \frac{\partial^2}{\partial x_n^2}X^{(n)}w + \partial_{n-1}(X^{(n)}f) + \frac{\partial}{\partial x_n} (Y^{(n)}f),$$

hence we obtain

$$\frac{\partial}{\partial x_n} X^{(n)}w \,|_{x_n=0} - Y^{(n)}f \,|_{x_n=0} + \partial_{n-1}h(z_x^{(n-1)}) = 0.$$

By (3.5) then we get

$$h(z_x^{(n-1)})= T_{n-1}((- \frac{\partial}{\partial x_n} X^{(n)}w+Y^{(n)}f) \,|_{x_n=0})+\tilde{h}(z_x^{(n-1)}),$$

whereby $\tilde{h}(z_x^{(n-1)})$ is an arbitrary $X^{(n)}(\mathbf{R}_{o,n-1})$-valued function satisfying the equation $\partial_{n-1}\tilde{h}(z_x^{(n-1)})=0$. Thus we get the formula (3.13).
Conversely if the function $w(x)$ is defined by (3.12)(3.13), then we get immediately that the functions $X^{(n)}w$ and $Y^{(n)}w$ satisfy (3.7). By (3.12) we have that the function $X^{(n)}w$ satisfies (3.16), then we get

$$\begin{aligned}
&\frac{\partial}{\partial x_n} X^{(n)}w+ \partial_{n-1}Y^{(n)}w \\
&= \frac{\partial}{\partial x_n} X^{(n)}w+\int_0^{x_n} \partial_{n-1}(\overline{\partial}_{n-1}X^{(n)}w-X^{(n)}f) \, dx_n+(- \frac{\partial}{\partial x_n} X^{(n)}w+Y^{(n)}f)|_{x_n=0} \\
&= \frac{\partial}{\partial x_n} X^{(n)}w+\int_0^{x_n} (- \frac{\partial^2}{\partial x_n^2}X^{(n)}w+ \frac{\partial}{\partial x_n}(Y^{(n)}f)) \, dx_n+(- \frac{\partial}{\partial x_n}X^{(n)}w+Y^{(n)}f)|_{x_n=0} \\
&= Y^{(n)}f \; .
\end{aligned}$$

Hence the functions $X^{(n)}w$ and $Y^{(n)}w$ also satisfy (3.8). By lemma 3.1 the function w defined by (3.12)(3.13) satisfies (3.9). Obviously (3.10) yields. Thus this theorem has been proved.

Now we introduce the integral-differential operator $\mathbf{K}_n$ defined on the space of continuous and differentiable $\mathbf{R}_{o,n-1}$-valued function $f(x)$

$$(\mathbf{K}_n f)(x)=(\mathbf{K}_n^{(1)}f)(x)+e_n(\mathbf{K}_n^{(2)}f)(x)$$

whereby

$$(\mathbf{K}_n^{(1)}f)(x)=\int_{K_n^{(n)}}N_n(x,y)\left(\partial_{n-1}(X^{(n)}f)+\frac{\partial}{\partial y_n}(Y^{(n)}f)\right)(y)\,dy,$$

$$(\mathbf{K}_n^{(2)}f)(x)=\int_0^{x_n}\left(\bar{\partial}_{n-1}(\mathbf{K}_n^{(1)}f)-X^{(n)}f\right)dx+T_{n-1}\left(\left(-\frac{\partial}{\partial x_n}(\mathbf{K}_n^{(1)}f)+Y^{(n)}f\right)\Big|_{x_n=0}\right).$$

Then we can get

Corollary 3.3 Under the assumptions of theorem 3.2 the solution of the boundary value problem (3.9)(3.10) may be represented in the following form

$$w(x)=(\mathbf{K}_n f)(x)+c+h_{n-1}(z_x^{(n-1)})e_n\ ,$$

where c is an arbitrary $X^{(n)}(\mathbf{R}_{o,n-1})$-valued constant and $h_{n-1}(z_x^{(n-1)})$ is an arbitrary $X^{(n)}(\mathbf{R}_{o,n-1})$-valued function satisfying $\bar{\partial}_{n-1}h_{n-1}(z_x^{(n-1)})=0$.

Remark Making use of the result in [11], the solution of the problem (3.9)(3.10) can be determined uniquely on condition that $(X^{(n)}w)(0)$, the functions $X^{(i)}w$ on $\partial K_n^{(i)}$ $(i=n-1,\cdots\cdots,2)$ and $(X^{(1)}w)(0)$ are also given.

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