WSGP 8

Jan Kubarski

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In: Jarolím Bureš and Vladimír Souček (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1989. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 21. pp. [235]--258.

Persistent URL: http://dml.cz/dmlcz/701444

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SOME INVARIANTS OF LIE ALGEBROIDS OF PRINCIPAL FIBRE BUNDLES

Jan Kubarski

0. INTRODUCTION

0.1. Introduced by J.Pradines [14], [15], the Lie functor for principal fibre bundles is examined in this paper. It assigns some Lie algebroid to a given principal fibre bundle (pfb for short) and plays an analogous role as the Lie functor for Lie groups.

0.2. By a (transitive) Lie algebroid (on a manifold M) we shall mean an object

$$A = (A, [\![\cdot,\cdot]\!], \gamma) \tag{0}$$

consisting of

1. a vector bundle A over M,
2. two mappings $[\![\cdot,\cdot]\!]: \mathrm{Sec}\,A \times \mathrm{Sec}\,A \longrightarrow \mathrm{Sec}\,A$ and $\gamma: A \longrightarrow TM$ such that
   (a) $(\mathrm{Sec}\,A, [\![\cdot,\cdot]\!])$ is an $\mathbb{R}$-Lie algebra,
   (b) $\gamma: A \longrightarrow TM$ is an epimorphism of vector bundles,
   (c) $[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + (\gamma\circ\xi)(f)\cdot\eta$, $\xi,\eta\in \mathrm{Sec}\,A$, $f\in C^{\infty}(M)$.

0.3. With each Lie algebroid (0) we associate a short exact sequence

$$0 \longrightarrow \boldsymbol{g}(A) \hookrightarrow A \xrightarrow{\gamma} TM \longrightarrow 0 \tag{1}$$

where $\boldsymbol{g}(A) = \mathrm{Ker}\,\gamma$, called the Atiyah sequence assigned to (0). In each fibre $\boldsymbol{g}(A)_{|x}$, a Lie algebra structure is defined by:

$$[v,w] := [\![\xi,\eta]\!](x) \quad \text{where} \quad \xi,\eta\in \mathrm{Sec}\,A,\ \xi(x)=v,\ \eta(x)=w,\quad v,w\in\boldsymbol{g}(A)_{|x}.$$

$\boldsymbol{g}(A)_{|x}$ is called the isotropy Lie algebra of (0) at x.

This paper is in final form and no version of it will be submitted for publication elsewhere

0.4. For any Lie algebroid (0) on a connected manifold M, the vector bundle $\mathfrak{g}(A)$ is a Lie algebra bundle [7], [9], [11].

0.5. A (strong) homomorphism of algebroids

$$H:(A,[\![\cdot,\cdot]\!],\gamma) \longrightarrow (A',[\![\cdot,\cdot]\!]',\gamma')$$

is a strong homomorphism $H:A \longrightarrow A'$ of vector bundles, such that

(a) $\gamma' \circ H = \gamma$,

(b) $\mathrm{Sec}\, H : \mathrm{Sec}\, A \longrightarrow \mathrm{Sec}\, A'$ is a homomorphism of Lie algebras.

H determines a homomorphism of the associated Atiyah sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{g}(A) & \hookrightarrow & A & \xrightarrow{\gamma} & TM & \longrightarrow & 0 \\ & & \big\downarrow H^{0} & & \big\downarrow H & & \| & & \\ 0 & \longrightarrow & \mathfrak{g}(A') & \hookrightarrow & A' & \longrightarrow & TM & \longrightarrow & 0 \end{array}$$

where $H^{0} = H|\mathfrak{g}(A)$.

If H is a bijection, then H^{-1} is also a homomorphism of Lie algebroids; H is then called an isomorphism of Lie algebroids.

0.6. There are three (equivalent) natural constructions of the Lie algebroid $A(P)$ for a given pfb $P = P(M,G)$ (see [1], [7], [14], [15]):

1. $A(P) = TP/G$,

2. $A(P) = A(PP^{-1})$,

3. $A(P) = W^{1}(P) \times_{G^{1}_{n}} (\mathbb{R}^{n} \times \mathfrak{g})$.

0.7. In the theory of Lie groups it is well known that two Lie groups are locally isomorphic iff their Lie algebras are isomorphic.

The question :

— What this problem looks like for pfb's ?

is answered in this paper. A suitable notion of a local homomorphism (and a local isomorphism) between pfb's is found (chapter 1).

0.8. By a local homomorphism [isomorphism] of pfb's

$$\mathcal{F}:P(M,G) \longrightarrow P'(M,G')$$

we shall mean (definition 2.1) each family

$$(2) \qquad \mathcal{F} = \{(F_t,\mu_t);\ t\in T\}$$

of partial homomorphisms [isomorphisms] $(F_t,\mu_t):P\supset D_t \longrightarrow P'$ provided some compatibility axioms are satisfied.

Every local homomorphism (2) defines an homomorphism of Lie algebroids $d\mathcal{F}:A(P)\longrightarrow A(P')$ (proposition 2.2) and conversely every homomorphism of the Lie algebroids comes from some local homomorphism of the pfb's (theorem 2.4).

0.9. Two pfb's (over the same manifold) are locally isomorphic iff their Lie algebroids are isomorphic (theorem 2.5).

From the above we see that the invariants of Lie algebroids of pfb's are the same as the invariants of local isomorphisms between pfb's.

0.10. It turns out that some invariants of pfb's are invariants of local isomorphisms so they are then de facto some notions of Lie algebroids. For example:

1. the Ad-associated Lie algebra bundle $\mathfrak{g}(P):=P\times_G\mathfrak{g}$, (see 1.10),
2. the flatness (see chapter 2),
3. the Chern-Weil homomorphism (for some local isomorphisms, chapter 3).

0.11. One of our main questions is:

— How much information about a given pfb P is carried by the Ad-associated Lie algebra bundle $\mathfrak{g}(P)$?

It turns out that sometimes none:

— If G is abelian, then $\mathfrak{g}(P)$ is trivial (see 1.10), and sometimes much, and most if G is semisimple:

— Two pfb's with semisimple structural Lie groups are locally isomorphic iff their Ad-associated Lie algebra bundles are isomorphic (Corollary 5.9).

0.12. All the differential manifolds considered in the present paper are assumed to be smooth (ie C^{∞}) and Hausdorff.

1. LIE ALGEBROID OF A PRINCIPAL FIBRE BUNDLE

In this chapter we introduce some basic definitions and facts concerning the notion of the Lie algebroid of a pfb. For more details, see [7] ÷ [9].

1.1. Take any pfb $P=P(M,G)$ with the projection $\pi:P\longrightarrow M$ and the action $R:P\times G\longrightarrow P$, and define the action

$$R^T:TP\times G\longrightarrow TP,\quad (v,a)\longmapsto (R_a)_*v,$$

R_a being the right action of a on P. Denote by

$$A(P)$$

the space TP/G of all orbits of R^T with the quotient topology. Let $[v]$ denote the orbit through v and

$$\pi^A:TP\longrightarrow A(P),\quad v\longmapsto [v],$$

the natural projection. To the end, we define the projection

$$p:A(P)\longrightarrow M,\quad [v]\longmapsto \pi z,\quad \text{if}\quad v\in T_zP.$$

For each point $x\in M$, in the fibre $p^{-1}(x)$ there exists exactly one vector space structure (over $\mathbb{R}$) such that $[v]+[w]=[v+w]$ if $\pi_P(v)=\pi_P(w)$, $\pi_P:TP\longrightarrow P$ being the projection.

$$\pi^A_{|z}:T_zP\longrightarrow A(P)_{|\pi z},$$

is then an isomorphism of vector spaces, $z\in P$.

1.2. $P(M,G)$ determines another pfb $TP(TM,TG)$ with the projection $\pi_*: TP \longrightarrow TM$ and the action $R_*: TP\times TG \longrightarrow TP$ [3]. Treating G as a closed Lie subgroup of TG, we see [4] that the structure of a Hausdorff C^∞-manifold, such that π^A is a submersion, exists in $A(P)$.

1.3. For each local trivialization $\varphi: U\times G \longrightarrow P$ of P the mapping

(3) $$\varphi^A: TU\times\mathfrak{g} \longrightarrow p^{-1}[U] \subset A(P), \quad (v,w) \longmapsto [\varphi_*(v,w)],$$

is a diffeomorphism, where $\mathfrak{g} = T_eG$.

1.4. The system

$$(A(P),p,M)$$

is a vector bundle and (3) is a (strong) isomorphism of the vector bundles (over $U\subset M$).

1.5. For a trivial pfb $P = M\times G$, we have:

$$A(M,G) = T(M\times G)_{/G} \cong TM\times\mathfrak{g}, \quad [(v,w)] \longmapsto (v,\theta^R(w)),$$

where θ^R denotes the canonical right-invariant 1-form on G.

1.6. Let $\mathrm{Sec}\,A(P)$ denote the $C^\infty(M)$-module of all C^∞ global cross-sections of $A(P)$, and $\mathfrak{X}^R(P)$ - of all C^∞ right-invariant vector fields on P. Each vector field $X\in\mathfrak{X}^R(P)$ determines a cross-section $X_o\in\mathrm{Sec}\,A(P)$ in such a way that $X_o(x) = [X(z)]$ for $z\in P_{|x}$, $x\in M$. X_o is a C^∞ cross-section. The mapping

(4) $$\mathfrak{X}^R(P) \longrightarrow \mathrm{Sec}\,A(P), \quad X \longmapsto X_o,$$

is a homomorphism of $C^\infty(M)$-modules.

1.7. For each cross-section $\eta\in\mathrm{Sec}\,A(P)$, there exists exactly one C^∞ right-invariant vector field $\eta'\in\mathfrak{X}^R(P)$ such that $[\eta'(z)] = \eta(\pi z)$. The mapping

(5) $$\mathrm{Sec}\,A(P) \longrightarrow \mathfrak{X}^R(P), \quad \eta \longmapsto \eta',$$

is an isomorphism of $C^\infty(M)$-modules, inverse to (4).

<u>1.8.</u> We define an $\mathbb{R}$-Lie algebra structure in the $\mathbb{R}$-vector space $\mathrm{Sec}\,A(P)$ by demanding that (5) be an isomorphism of $\mathbb{R}$-Lie algebras. The bracket $[\![\cdot,\cdot]\!]$ in $\mathrm{Sec}\,A(P)$ must be defined by

$$[\![\xi,\eta]\!] = ([\xi',\eta'])_0 .$$

We also take the mapping

$$\gamma : A(P) \longrightarrow TM, \quad [v] \longmapsto \pi_* v .$$

<u>1.9.</u> The object

(6) $$A(P) = (A(P), [\![\cdot,\cdot]\!], \gamma)$$

is called the <u>Lie algebroid</u> of a pfb $P(M,G)$.

Here are the fundamental properties of (6):

(a) $(\mathrm{Sec}\,A(P), [\![\cdot,\cdot]\!])$ is an $\mathbb{R}$-Lie algebra,

(b) γ is an epimorphism of vector bundles,

(c) $\mathrm{Sec}\,\gamma : \mathrm{Sec}\,A(P) \longrightarrow \mathfrak{X}(M)$, $\xi \longmapsto \gamma\circ\xi$, is a homomorphism of Lie algebras,

(d) $[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + (\gamma\circ\xi)(f)\cdot\eta$, for $f\in C^{\infty}(M)$, $\xi,\eta\in\mathrm{Sec}\,A(P)$,

(e) the vector bundle

$$\mathfrak{g}(P) := \mathrm{Ker}\,\gamma \subset A(P)$$

is a Lie algebra bundle (for definition, see for example [3,p.377]) and for an arbitrary $z\in P_{|x}$,

(7) $$\hat{z} : \mathfrak{g} \longrightarrow \mathfrak{g}(P)_{|x}, \quad v \longmapsto [A_{z*}v],$$

is an isomorphism of Lie algebras, where $A_z : G \longrightarrow P$, $a \longmapsto za$. The mapping

$$\varphi_0^A : U\times\mathfrak{g} \longrightarrow \mathfrak{g}(P)_{|U}, \quad (x,w) \longmapsto \varphi^A(\theta_x, w),$$

is a local trivialization of the Lie algebra bundle $\mathfrak{g}(P)$ (for an arbitrary local trivialization $\varphi : U\times G \longrightarrow P$), where θ_x denotes the null vector at x, and $\mathfrak{g}$ - the Lie algebra of G defined by the right-invariant vector fields on G ($\mathfrak{g} = \mathfrak{gl}(G)^0$).

By properties a $\div$ d, (6) is a Lie algebroid.

A pfb P with discrete structural Lie group has a trivial Lie algebroid, more exactly $A(P) \cong TM$.

<u>1.10.</u> $\mathfrak{g}(P)$ is canonically isomorphic to the Ad-associated Lie algebra bundle $P\times_G \mathfrak{g}$. The mapping

$$\tau : P\times_G \mathfrak{g} \longrightarrow \mathfrak{g}(P), \quad [z,v] \longmapsto \hat{z}(v),$$

is an isomorphism of Lie algebra bundles.

Thus, the <u>Ad-associated Lie algebra bundle is an invariant of the Lie algebroid of a pfb.</u>

If G is abelian, then $\mathfrak{g}(P)$ is trivial: $\mathfrak{g}(P) \cong (P\times\mathfrak{g})/_G \cong M\times\mathfrak{g}$.

<u>1.11.</u> By a (strong) <u>homomorphism</u> between pfb's $P(M,G)$, $P'(M,G')$ we shall mean a pair (F,μ) consisting of the mappings $F:P \longrightarrow P'$, $\mu:G \longrightarrow G'$ such that

(a) $\pi' \circ F = \pi$,

(b) μ is a homomorphism of Lie groups,

(c) $F(z\cdot a) = F(z)\cdot\mu(a)$.

If μ is an isomorphism then F is a diffeomorphism and (F^{-1},μ^{-1}) is also a homomorphism between pfb's. (F,μ) is then called an <u>isomorphism</u> of pfb's.

Each homomorphism $(F,\mu):P(M,G) \longrightarrow P'(M,G')$ determines a mapping $dF:A(P) \longrightarrow A(P')$, $[v] \longmapsto [F_* v]$. dF is a homomorphism of Lie algebroids.

The covariant functor $P(M,G) \longmapsto A(P)$, $(F,\mu) \longmapsto dF$, is called the <u>Lie functor</u> for pfb's.

<u>1.12.</u> As we have said in the Introduction, the Lie algebroid of a pfb P can also be defined as the Lie algebroid $A(PP^{-1})$ of the Ehresmann Lie groupoid PP^{-1}, via the construction of J.Pradines [14],[15]. We recall these constructions.

(a) Let Φ be any Lie groupoid [12]. We put $A(\Phi) = u^* T^\alpha \Phi$ where $T^\alpha\Phi = \operatorname{Ker}\alpha_*$ (α - the source, $u:M \longrightarrow \Phi$, $x \longmapsto u_x$, u_x - the unit over x). The bracket $[\![\xi,\eta]\!]$ of $\xi,\eta \in \operatorname{Sec} A(\Phi)$ is defined in such a way that the right-invariant vector field corresponding to $[\![\xi,\eta]\!]$ equals the Lie bracket of the corresponding right-invariant vector fields.

The mapping $\gamma: A(\Phi) \longrightarrow TM$ is defined by $\gamma(v) = \beta_* v$ (β - the target). The system obtained $(A(\Phi), [\![\cdot,\cdot]\!], \gamma)$ is a Lie algebroid (for details see for example [5]).

(b) The Ehresmann Lie groupoid PP^{-1} [2] is defined as follows: its space equals the space of orbits of the action $(P\times P)\times G \longrightarrow P\times P$, $((z_1,z_2),a) \longmapsto (z_1 a, z_2 a)$, the source and the target are defined by $\alpha([z_1,z_2]) = \pi z_1$, $\beta([z_1,z_2]) = \pi z_2$ ($[z_1,z_2]$ being the orbit through (z_1,z_2)), the partial multiplication by $[z_2,z_3]\cdot[z_1,z_2] = [z_1,z_3]$.

<u>1.13.</u> $\rho: A(P) \longrightarrow A(PP^{-1})$, $[v] \longmapsto \omega_{z*z} v$, $v \in T_z P$, $z \in P_{|x}$, where $\omega_z : P \longrightarrow (PP^{-1})_{|x}$, $z' \longmapsto [z,z']$, establish an isomorphism of the Lie algebroids.

<u>1.14.</u> There exists a Lie algebroid which cannot be realized as the Lie algebroid of any pfb [10].

2. THE NOTION OF A LOCAL HOMOMORPHISM BETWEEN PFB'S

In the theory of Lie groups the following theorems hold:

<u>THEOREM A.</u> If G_1 and G_2 are two Lie groups with Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively, then, for each homomorphism $h: \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ od Lie algebras, there exists a local homomorphism $H: G_1 \supset \Omega \longrightarrow G_2$ (Ω is open in G_1 and contains the unit of G_1) of Lie groups such that $dH = h$.

<u>THEOREM B.</u> Two Lie groups G_1 and G_2 are locally isomorphic iff $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic.

— What does this look like for pfb's?

First of all, we know [6],[13], that the theorems similar to the above ones hold for Lie groupoids and algebroids, as well. Thus, we have only to discover how to define a suitable notion of a local homomorphism between pfb's in order that it correspond to the notion of a local homomorphism between Lie groupoids.

Here is an answer to this problem.

<u>2.1. DEFINITION.</u> By a <u>local</u> <u>homomorphism</u> from a pfb $P(M,G)$ into a second one $P'(M,G')$ we shall mean a family

$$\mathcal{F} = \{(F_t, \mu_t);\ t\in T\}$$

such that

$$F_t : P\supset D_t \longrightarrow P,\quad D_t \text{ open in } P,$$
$$\mu_t : G\supset U_t \longrightarrow G,\quad U_t \text{ open in } G,\quad e\in U_t;$$

provided the following properties hold:

1. μ_t is a local homomorphism of Lie groups,
2. $\bigcup_t \pi[D_t] = M$,
3. $\pi'\circ F_t = \pi | D_t$ (π and π' denote the projections),
4. $F_t(z\cdot a) = F_t(z)\cdot\mu_t(a)$ for $z\in D_t$ and $a\in U_t$ such that $z\cdot a\in D_t$.
5. If $t, t'\in T$, $z\in D_t$, $a\in G$, $z\cdot a\in D_{t'}$, $a'\in G'$, $z'\in P'$ and

   $F_t(z) = z'$, $F_{t'}(z\cdot a) = z'\cdot a'$, then

   (a) $F_{t'} = R_{a'}\circ F_t\circ R_{a^{-1}}$ in some nbh of $z\cdot a$,

   (b) $\mu_{t'} = \tau_{a'^{-1}}\circ\mu_t\circ\tau_a$ in some nbh of $e\in G$ ($\tau_a(g) = aga^{-1}$, $g\in G$).

If F_t and μ_t are diffeomorphisms, then

$$\mathcal{F}^{-1} := \{(F_t^{-1}, \mu_t^{-1});\ t\in T\}$$

is a local homomorphism, and $\mathcal{F}$ is then called a <u>local isomorphism</u>.

<u>2.2. PROPOSITION</u>. Let $\mathcal{F} = \{(F_t,\mu_t);\ t\in T\} : P(M,G) \longrightarrow P'(M,G')$ be a local homomorphism between pfb's. Then

$$d\mathcal{F} : A(P) \longrightarrow A(P'),\quad [v] \longmapsto [F_{t*}v],\quad v\in T_zP,\ z\in D_t,\ t\in T,$$

is a correctly defined homomorphism of Lie algebroids.

<u>PROOF.</u> The correctness of the definition of $d\mathcal{F}$ is easy to see.

$d\mathcal{F}$ is a C^∞-homomorphism of vector bundles. Indeed, for a point $x\in M$, take an arbitrary $t\in T$ such that $x\in\pi[D_t]$. The smoothness of $d\mathcal{F}$ in some nbh of x follows from the fact that $d\mathcal{F}\circ\pi^A | \pi_P^{-1}[D_t] = \pi'^A\circ(F_t)_* | \pi_P^{-1}[D_t]$.

$\gamma'\circ d\mathcal{F} = \gamma$ is evident.

$\mathrm{Sec}(d\mathcal{F}) : \mathrm{Sec}\,A(P) \longrightarrow \mathrm{Sec}\,A(P')$ is a homomorphism of Lie algebras. Indeed, for $X\in\mathfrak{X}^R(P)$, the cross-section $d\mathcal{F}\circ X_o$ of $A(P')$ induces the right-invariant vector field $Y := (d\mathcal{F}\circ X_o)'$ on P'. The field $X|D_t$

is F_t-related to Y, which yields (by a standard calculation) that

$$(d\mathcal{F}\circ[\![\xi_1,\xi_2]\!])|\pi[D_t] = [\![d\mathcal{F}\circ\xi_1, d\mathcal{F}\circ\xi_2]\!]|\pi[D_t].$$

The free choice of $t\in T$ ends the proof. □

2.3. REMARK. (1) It is easily seen that $d\mathcal{F}$ is an isomorphism if $\mathcal{F}$ is a local isomorphism.

(2) We have $d\mathcal{F}[q(P)]\subset q(P')$ and we get the commuting diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\hat{z}} & q(P)_{|x} \\ {\scriptstyle (\mu_t)_{*e}}\downarrow & & \downarrow{\scriptstyle (d\mathcal{F})_{|x}|q(P)_{|x}} \\ \mathfrak{g}' & \xrightarrow{F_t(z)^\wedge} & q(P')_{|x} \end{array}$$

for $t\in T$, $z\in D_t$ (see (7)).

2.4. THEOREM. Let $h:A(P)\longrightarrow A(P')$ be any homomorphism of Lie algebroids. Then there exists a local homomorphism $\mathcal{F}:P(M,G)\longrightarrow P'(M,G')$ such that $d\mathcal{F}= h$.

PROOF. (For details see [8]). Take the Ehresmann Lie groupoids $\Phi:= PP^{-1}$ and $\Phi':= P'P'^{-1}$ corresponding to the pfb's $P(M,G)$ and $P'(M,G')$, respectively. Let $\tilde{h}:A(\Phi)\longrightarrow A(\Phi')$ be the homomorphism of Lie algebroids for which the diagram

$$\begin{array}{ccc} A(P) & \xrightarrow{h} & A(P') \\ \rho\downarrow & & \downarrow\rho' \\ A(\Phi) & \xrightarrow{\tilde{h}} & A(\Phi') \end{array}$$

commutes, where ρ and ρ' are natural isomorphisms described in 1.13. By theorem A, for Lie groupoids, there exists some local homomorphism $F:\Phi\supset\Omega\longrightarrow\Phi'$, Ω being open in Φ and covering all units, of Lie groupoids such that $dF=\tilde{h}$. Now, we are able to construct some local homomorphism of pfb's.

It will be the family

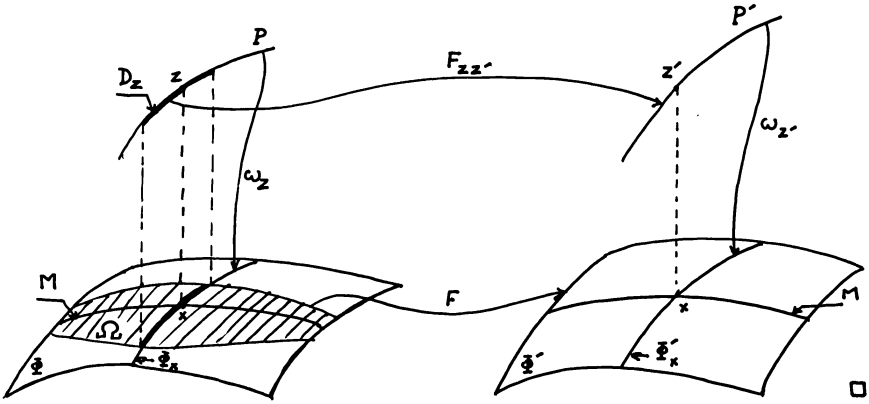
$$\mathcal{F}:=\{(F_{zz'},\mu_{zz'})\ ;\ (z,z')\in P\oplus P'\}$$

$(P\oplus P'=\{(z,z')\in P\times P';\pi z=\pi' z')\}$ where

$$F_{zz'}=\omega_{z'}^{-1}\circ F\circ\omega_z|D_z,\quad D_z=\omega_z^{-1}[\Omega\cap\Phi_{\pi z}],\quad \text{and}$$

$$\mu_{zz'} = \mu_{z'}^{-1} \circ F \circ \mu_z | U_z, \quad U_z = \mu_z^{-1}[\Omega \cap G_{\pi z}]$$

and $\omega_z : P \longrightarrow \Phi$, $z' \longmapsto [z, z']$, $G_{\pi z}$ is the isotropy Lie group at x, $\mu_z : G \longrightarrow G_{\pi z}$, $a \longmapsto [z, za]$, ($\omega_{z'}$, $\mu_{z'}$ are defined in a similar manner), see the figure:



□

2.5. THEOREM. Two pfb's (over the same manifold) are locally isomorphic iff their Lie algebroids are isomorphic. □

3. CONNECTIONS IN LIE ALGEBROIDS

3.1. DEFINITION. By a connection in Lie algebroid (0) we mean a splitting of Atiyah sequence (1), ie a mapping

$$\lambda : TM \longrightarrow A \tag{8}$$

such that $\gamma \circ \lambda = id_{TM}$, see [1], or, equivalently, a subbundle $B \subset A$ such that $A = \mathfrak{g}(A) \oplus B$.

We define its connection form $\omega^A : A \longrightarrow \mathfrak{g}(A)$ as a unique form such that $\omega^A | \mathfrak{g}(A) = id$ and $Ker\,\omega^A = Im\,\lambda$.

Let (8) be an arbitrary but fixed connection in (0) and let

$$A = A(P)$$

for some pfb $P(M,G)$. For each point $z \in P$, we define a subspace

$$H^{\lambda}_{|z} := Im[(\pi^A_{|\pi z})^{-1} \circ \lambda_{\pi z}] \subset T_z P.$$

3.2. PROPOSITION. $z \mapsto H^{\lambda}_{|z}$, $z \in P$, is a connection in $P(M,G)$. □

3.3. PROPOSITION. The correspondence

$$\lambda \mapsto H^{\lambda} \tag{9}$$

sets up a bijection between connections in (6) and in $P(M,G)$.

PROOF. Let H be any connection in $P(M,G)$. Put $B_{|x} = \pi^A_{|z}[H_{|z}]$ where $z \in P_{|x}$, $x \in M$. Evidently, $A(P)_{|x} = B_{|x} \oplus \mathfrak{g}(P)_{|x}$ and $B_{|x}$ is independent of the choice of $z \in P_{|x}$. The vector subbundle $B := \bigcup_x B_{|x}$ of $A(P)$ defines a connection

$$\lambda^H = (\gamma | B)^{-1} : TM \longrightarrow A(P).$$

The correspondence $H \mapsto \lambda^H$ is inverse to (9). □

Fix a connection H in a pfb $P(M,G)$. It determines the connection form $\omega \in \Omega^1(P;\mathfrak{g})$ and the curvature form $\Omega \in \Omega^2(P;\mathfrak{g})$. Ω is Ad-equivariant and horizontal at the same time [3], ie is a basic $\mathfrak{g}$-valued form on P. Via the classical manner (see for example [3,p.406]) the space $\Omega_B(P;\mathfrak{g})$ of all basic $\mathfrak{g}$-valued forms on $P(M,G)$ is naturally isomorphic to the space of all forms on M with values in the associated Lie algebra bundle $P\times_G\mathfrak{g}$:

$$\Omega_B(P;\mathfrak{g}) \xrightarrow{\approx} \Omega(M;P\times_G\mathfrak{g}), \quad \Theta \mapsto \tilde{\Theta},$$
$$\tilde{\Theta}(x;v_1,\dots,v_q) = [z,\Theta(z;v_1^z,\dots,v_q^z)], \quad v_i \in T_xM,$$

where $z \in P_{|x}$, while v^z denotes the horizontal lifting of v to T_zP for $v \in T_xM$.

Considering the canonical isomorphism $P\times_G\mathfrak{g} \cong \mathfrak{g}(P)$ (see 1.10), we obtain an isomorphism

$$\begin{gathered} \Omega_B(P;\mathfrak{g}) \longrightarrow \Omega(M;\mathfrak{g}(P)), \quad \Theta \mapsto \Theta_M, \\ \Theta_M(x;v_1,\dots,v_q) = \hat{z}(\Theta(z;v_1^z,\dots,v_q^z)), \quad z \in P_{|x}. \end{gathered} \tag{10}$$

Via isomorphism (10) we define the so-called <u>curvature base form</u> (or the <u>curvature tensor</u>) $\Omega_M \in \Omega^2(M;\mathfrak{g}(P))$ of H.

Now, let $\lambda : TM \longrightarrow A(P)$ be the connection in (6) corresponding to H with connection form ω^A. Of course

$$\omega^A_{|x} \circ \pi^A_{|z} = \hat{z} \circ \omega_{|z}.$$

3.4. PROPOSITION.

$$\Omega_M(X,Y) = -\omega^A([\![\lambda\circ X,\lambda\circ Y]\!]), \quad X,Y\in\mathfrak{X}(M). \tag{11}$$

PROOF. By the equality $\pi^A_{|z}(v^z) = \lambda(v)$, $v\in T_xM$, we see that, for $X\in\mathfrak{X}(M)$, the right-invariant vector field $(\lambda\circ X)'$ on P is equal to the horizontal lifting $\tilde{X}$ of X. By the classical equality $\Omega(\tilde{X},\tilde{Y}) = -\omega([\tilde{X},\tilde{Y}])$, we obtain our proposition. □

3.5. COROLLARY. $\lambda\circ[X,Y] = [\![\lambda\circ X,\lambda\circ Y]\!] + \Omega_M(X,Y)$. □

3.6. COROLLARY. The following properties are equivalent to one another:

(1) H is flat (ie $\Omega = 0$),

(2) $\Omega_M = 0$,

(3) $\mathrm{Sec}\,\lambda:\mathfrak{X}(M)\longrightarrow \mathrm{Sec}\,A(P)$ is a homomorphism of Lie algebras. □

Any connection (8) in Lie algebroid (0) is called flat iff Sec λ is a homomorphism of Lie algebras or, equivalently, if its curvature tensor Ω_M defined by (11) vanishes.

Lie algebroid (0) is called flat iff it possesses a flat cannection. A pfb P(M,G) is flat iff its Lie algebroid (6) is flat.

By theorem 2.5, we obtain

3.7. COROLLARY. If both pfb's P(M,G) and P'(M,G') are locally isomorphic and one of them is flat, then the second one is flat, too. Consequently, flatness is an invariant of local isomorphisms.

3.8. EXAMPLE. Every trivial Lie algebroid is flat. The canonical flat connection in the trivial Lie algebroid $TM\times\mathfrak{g}$ is defined by

$$\lambda: TM \longrightarrow TM\times\mathfrak{g}, \quad v\longmapsto (v,0).$$

3.9. COROLLARY. If Lie algebroid (6) of a pfb P(M,G) is trivial, then P(M,G) is flat.

4. THE CHERN-WEIL HOMOMORPHISM

The main purpose of this chapter is to calculate the Chern-Weil homomorphism of a pfb via its Lie algebroid.

We prove that the Chern-Weil homomorphisms of pfb's (over an arbitrary but fixed connected manifold M) are invariants of some local isomorphisms between them and, in the case of pfb's with connected structural Lie groups, these homomorphisms are invariants of all local isomorphisms.

Let $P(M,G)$ be any pfb with a Lie algebroid $A(P)$. Let

$$\overset{k}{\vee}\mathfrak{g}^* \quad \text{and} \quad \overset{k}{\vee}\mathfrak{g}(P)^*$$

be the k-symmetric power of the vector space $\mathfrak{g}^*$ and the vector bundle $\mathfrak{g}(P)^*$, respectively;

$$\vee\mathfrak{g}^* = \overset{k}{\oplus}(\overset{k}{\vee}\mathfrak{g}^*).$$

In the sequel any element of $\overset{k}{\vee}\mathfrak{g}^*$ (analogously of $\overset{k}{\vee}(\mathfrak{g}(P)_{|x})^*$) is treated as a symmetric k-linear homomorphism $\mathfrak{g}\times\ldots\times\mathfrak{g}\longrightarrow\mathbb{R}$ via the isomorphism

$$\overset{k}{\vee}\mathfrak{g}^* \cong \mathcal{L}^k_s(\mathfrak{g};\mathbb{R})$$

$$t_1\vee\ldots\vee t_k \longmapsto ((v_1,\ldots,v_k)\longmapsto \frac{1}{k!}\sum_{\sigma} t_{\sigma(1)}(v_1)\cdot\ldots\cdot t_{\sigma(k)}(v_k)).$$

Define the mapping

$$\circledast:(\vee\mathfrak{g}^*)_I \longrightarrow \overset{k}{\oplus}(\mathrm{Sec}\,\overset{k}{\vee}\mathfrak{g}(P)^*)$$

$$\circledast(\bar{\Gamma})_x = \overset{k}{\vee}(\hat{z}^{-1})^*(\bar{\Gamma})$$

for $\bar{\Gamma}\in(\overset{k}{\vee}\mathfrak{g}^*)_I$, where $z\in P_{|x}$, $x\in M$. From the Ad-invariance of $\bar{\Gamma}$ and the fact that $(za)^\wedge = \hat{z}\circ\mathrm{Ad}\,a$, $z\in P$, $a\in G$, we see the correctness of this definition, ie the independence of $\circledast(\bar{\Gamma})_x$ of the choice of $z\in P_{|x}$.

Denote the image $\mathrm{Im}\,\circledast^k$ ($\circledast^k:=\circledast|(\overset{k}{\vee}\mathfrak{g}^*)_I$) by $(\mathrm{Sec}\,\overset{k}{\vee}\mathfrak{g}(P)^*)_I$.

Of course,

$$\circledast^k:(\overset{k}{\vee}\mathfrak{g}^*)_I \longrightarrow (\mathrm{Sec}\,\overset{k}{\vee}\mathfrak{g}(P)^*)_I$$

is an isomorphism of vector spaces.

<u>4.1. PROPOSITION.</u> Let $\Gamma\in\mathrm{Sec}\,\overset{k}{\vee}\mathfrak{g}(P)^*$, then $\Gamma\in(\mathrm{Sec}\,\overset{k}{\vee}\mathfrak{g}(P)^*)_I$ iff, for any $z_1,z_2\in P$, we have

$$\overset{k}{\vee}(\hat{z}_1)^*(\Gamma_{\pi z_1}) = \overset{k}{\vee}(\hat{z}_2)^*(\Gamma_{\pi z_2}). \quad \square$$

<u>4.2. THEOREM.</u> The mapping

$$h^{A(P)}: \overset{k}{\bigoplus} (\mathrm{Sec} \overset{k}{\bigvee} \mathfrak{g}(P)^*)_I \longrightarrow H(M)$$

for which the diagram

$$\begin{array}{ccc} \overset{k}{\bigoplus} (\mathrm{Sec} \overset{k}{\bigvee} \mathfrak{g}(P)^*)_I & \xrightarrow{\;h^{A(P)}\;} & H(M) \\ \approx \big\uparrow \Theta & \nearrow h_P & \\ (\bigvee \mathfrak{g}^*)_I & & \end{array}$$

commutes is defined by

$$\Gamma \longmapsto [\Gamma_*(\underbrace{\Omega_M, \dots, \Omega_M}_{k\text{-times}})] \quad \text{for } \Gamma \in (\mathrm{Sec} \overset{k}{\bigvee} \mathfrak{g}(P)^*)_I$$

where Ω_M is the curvature base form of any connection in P and

$$\Gamma_*(\Omega_M, \dots, \Omega_M)(x; v_1, \dots, v_{2k})$$
$$= \frac{1}{2^k} \sum_\sigma \mathrm{sgn}\,\sigma \cdot \Gamma_x(\Omega_M(x; v_{\sigma(1)}, v_{\sigma(2)}), \dots, \Omega_M(x; v_{\sigma(2k-1)}, \dots, v_{\sigma(2k)})),$$

$v_i \in T_x M$, $x \in M$.

<u>PROOF.</u> We must only prove that

$$(12) \qquad \pi^*((\Theta^k \Gamma)_*(\Omega_M, \dots, \Omega_M)) = \Gamma_*(\Omega, \dots, \Omega)$$

where Ω_M and Ω are the curvature base form and the curvature form of the same connection in P. Both sides of (12) are horizontal forms, so, to show the theorem, we must notice the equality on the horizontal vectors only. □

Now, we describe the relationship between the Chern-Weil homomorphisms for local isomorphic pfb's.

Let $\mathcal{F} = \{(F_t, \mu_t);\ t \in T\}: P(M,G) \longrightarrow P'(M,G')$ be a local homomorphism between pfb's $P(M,G)$ and $P'(M,G')$ and let $\omega' \in \Omega^1(P', \mathfrak{g}')$ be a connection form on P' where $\mathfrak{g}' := \mathfrak{g}I(G')^\circ$.

<u>4.3. PROPOSITION.</u> There exists exactly one connection form

$$\omega \in \Omega^1(P; \mathfrak{g})$$

on P such that for each $t \in T$

$$\omega|\pi_P^{-1}[D_t] = (\mu_t)_{*e}^{-1}(F_t^*\omega').$$

PROOF. Straightforward, for details see [8]. □

The connection form ω obtained in proposition 4.3 is called *induced* by φ from ω'. ω and ω' induce some connections λ and λ' in $A(P)$ and $A(P')$, respectively, which next determine connection forms ω^A and ω'^A in them. The following diagram commutes

$$\begin{array}{ccccc} \mathfrak{g}(P) & \xleftarrow{\omega^A} & A(P) & \xleftarrow{\lambda} & TM \\ \downarrow{\scriptstyle (d\varphi)^o} & & \downarrow{\scriptstyle d\varphi} & & \| \\ \mathfrak{g}(P') & \xleftarrow{\omega'^A} & A(P') & \xleftarrow{\lambda'} & TM. \end{array}$$

4.4. PROPOSITION. The relationship between the curvature base form Ω_M and Ω'_M of ω^A and ω'^A is described by the equality

$$(d\varphi)^o_*\Omega_M = \Omega'_M.$$

PROOF. For $X, Y \in \mathfrak{X}(M)$, we have

$$\begin{aligned}(d\varphi)^o\Omega_M(X,Y) &= (d\varphi)^o(-\omega^A[\![\lambda\circ X, \lambda\circ Y]\!]) = -\omega'^A(d\varphi[\![\lambda\circ X, \lambda\circ Y]\!]) \\ &= -\omega'^A[\![d\varphi(\lambda\circ X), d\varphi(\lambda\circ Y)]\!] = -\omega'^A[\![\lambda'\circ X, \lambda'\circ Y]\!] \\ &= -\Omega'_M(X,Y). \quad \square\end{aligned}$$

4.5. PROPOSITION. If M is connected, then, for any $t, t' \in T$, there exist $a \in G$ and $a' \in G'$ such that $\mu_{t'} = \tau_{a'^{-1}} \circ \mu_t \circ \tau_a$ in some nbh of $e \in G$. □

4.6. DEFINITION. A local isomorphism φ is said to have a *property Ch-W* if for all $t \in T$

$$(13) \qquad \bigvee(\mu_t)_{*e}[(\bigvee\mathfrak{g}'^*)_I] \subset (\bigvee\mathfrak{g}^*)_I,$$

or, equivalently, if there exists $t \in T$ such that (13) holds (by proposition 4.5) provided M is connected.

4.7. EXAMPLE. φ has a property Ch-W if it satisfies one of the following properties:

(a) G is connected,

(b) there exists $t \in T$ such that μ_t can be extended to some globally defined homomorphism $G \longrightarrow G'$ (provided M is connected),

(c) there exists $t \in \Gamma$ such that for each $a \in G$, there exists $a' \in G'$ such that $\mu_{t*e} \circ \mathrm{Ad}\, a = \mathrm{Ad}\, a' \circ \mu_{t*e}$ (provided M is connected).

First, we easily show that each local isomorphism fulfilling property (c) has a property Ch-W. Now, we trivially notice that (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (c).

4.8. THEOREM. If φ has a property Ch-W, then

$$\overset{k}{\vee}(d\varphi)^{\circ *}[(\mathrm{Sec}\overset{k}{\vee}\mathfrak{g}(P')^*)_I] \subset (\mathrm{Sec}\overset{k}{\vee}\mathfrak{g}(P)^*)_I \tag{14}$$

and the following diagram commutes:

$$\begin{array}{ccccc}
(\overset{k}{\vee}\mathfrak{g}'^*)_I & \xrightarrow{\circledast'^k} & (\mathrm{Sec}\overset{k}{\vee}\mathfrak{g}(P')^*)_I & \xrightarrow{h^{A(P')}} & \\
\downarrow{\scriptstyle \overset{k}{\vee}\mu^*_{t*e}} & & \downarrow{\scriptstyle \overset{k}{\vee}(d\varphi)^\circ} & & H(M). \\
(\overset{k}{\vee}\mathfrak{g}^*)_I & \xrightarrow{\circledast^k} & (\mathrm{Sec}\overset{k}{\vee}\mathfrak{g}(P)^*)_I & \xrightarrow{h^{A(P)}} &
\end{array}$$

PROOF. To prove the commutativity of the left-hand side of the above diagram, and inclusion (14), we need to notice that

$$(\overset{k}{\vee}(d\varphi)^{\circ *} \circ \circledast'^k(\bar{\Gamma}))_x = (\circledast^k \circ \overset{k}{\vee}\mu^*_{t*e}(\bar{\Gamma}))_x ,$$

for $\bar{\Gamma} \in (\overset{k}{\vee}\mathfrak{g}'^*)_I$ and $x \in M$. To end the proof, we assert that (by 4.4)

$$\begin{aligned}
h^{A(P)} \circ \overset{k}{\vee}(d\varphi)^\circ\,(\Gamma) &= h^{A(P)}(\Gamma \circ (d\varphi^\circ \times \ldots \times d\varphi^\circ)) \\
&= [\Gamma \circ (d\varphi^\circ \times \ldots \times d\varphi^\circ)_x(\Omega_M, \ldots, \Omega_M)] \\
&= [\Gamma_*(\Omega'_M, \ldots, \Omega'_M)] \\
&= h^{A(P')}. \quad \square
\end{aligned}$$

4.9. COROLLARY. The Chern-Weil homomorphisms of pfb's are invariants of local isomorphisms having the property Ch-W. In the case of pfb's with connected structural Lie groups, the Chern-Weil homomorphisms are invariants of all local isomorphisms, so this is a notion of a Lie algebroid of such a pfb.

5. A CLASSIFICATION OF LIE ALGEBROIDS WITH SEMISIMPLE ISOTROPY LIE ALGEBRAS

Let $(A, [\![\cdot,\cdot]\!], \gamma)$ be any Lie algebroid on a manifold M with the Lie algebra bundle $\mathfrak{g}$. Let $\lambda : TM \to A$ be any connection in this Lie algebroid,

$$(15) \qquad 0 \longrightarrow \mathfrak{g} \hookrightarrow A \xrightarrow{\gamma} TM \longrightarrow 0, \quad \lambda: TM \to A$$

with the curvature base form $\Omega_M \in \Omega^2(M;\mathfrak{g})$. Corollary 3.5 states that

$$(i) \qquad [\![\lambda\circ X, \lambda\circ Y]\!] = \lambda\circ [X,Y] - \Omega_M(X,Y), \quad X,Y \in \mathfrak{X}(M).$$

λ determines a covariant derivative ∇ in $\mathfrak{g}$ by the formula

$$(ii) \qquad \nabla_X \sigma = [\![\lambda\circ X, \sigma]\!], \quad X \in \mathfrak{X}(M), \ \sigma \in \operatorname{Sec}\mathfrak{g},$$

∇ is called <u>corresponding</u> to λ. We notice that the bracket $[\![\cdot,\cdot]\!]$ in the Lie algebra $\operatorname{Sec} A$ is uniquely determined by λ, ∇, Ω_M and the Lie algebra structure in $\operatorname{Sec}\mathfrak{g}$, namely

$$(iii) \qquad [\![\lambda\circ X + \sigma, \lambda\circ Y + \eta]\!] = \lambda\circ [X,Y] - \Omega_M(X,Y) + \nabla_X \eta - \nabla_Y \sigma + [\sigma,\eta],$$

$X, Y \in \mathfrak{X}(M)$, $\sigma, \eta \in \operatorname{Sec}\mathfrak{g}$ (see also [9]).

∇ determines the so-called <u>exterior covariant derivative</u> in $\Omega(M;\mathfrak{g})$ by the classical formula.

<u>5.1. PROPOSITION.</u> The elements ∇ and Ω_M fulfil the following assertions:

(1^o) $R_{X,Y}\sigma = -[\Omega_M(X,Y), \sigma]$, $X,Y \in \mathfrak{X}(M)$, $\sigma \in \operatorname{Sec}\mathfrak{g}$, where R denotes the curvature tensor of ∇, ie

$$\nabla^2 \sigma = -[\Omega_M, \sigma], \quad \sigma \in \operatorname{Sec}\mathfrak{g}, \quad \text{(the Ricci identity)},$$

(2^o) $\nabla_X[\sigma,\eta] = [\nabla_X \sigma, \eta] + [\sigma, \nabla_X \eta]$, $X \in \mathfrak{X}(M)$, $\sigma, \eta \in \operatorname{Sec}\mathfrak{g}$, ie ∇ is a Σ-connection in $(\mathfrak{g}, \{[\cdot,\cdot]\})$, called in the sequel a <u>Σ-connection</u> in $\mathfrak{g}$, where $[\cdot,\cdot]$ denotes here a cross-section of $\mathfrak{g}^{2,1}$ such that $[\cdot,\cdot]_x$ is the Lie algebra structure of $\mathfrak{g}_{|x}$ (for the definition of Σ-connection see [3]),

(3^o) $\nabla \Omega_M = 0$ (the Bianchi identity).

<u>5.2. THEOREM.</u> (see also [8], [9]).

(a) Let a system $(\mathfrak{g}, \nabla, \Omega_M)$ be given, consisting of

(1) a Lie algebra bundle $\mathfrak{g}$ on a manifold M,

(2) a covariant derivative ∇ in $\mathfrak{g}$,

(3) a 2-form $\Omega_M \in \Omega^2(M;\mathfrak{g})$,

fulfilling conditions $(1^o) \div (3^o)$ (from proposition 5.1). Then, for arbitrary vector bundle $A \supset \mathfrak{g}$ and mappings γ, λ, such that

(*) in the diagram (15) the row is exact and $\gamma \circ \lambda = \mathrm{id}_{TM}$,

there exists in the vector space SecA exactly one Lie algebra structure $[\![\cdot,\cdot]\!]$ such that the equalities (i) and (ii) hold. The bracket $[\![\cdot,\cdot]\!]$ is defined by formula (iii).

The system $(A, [\![\cdot,\cdot]\!], \gamma)$ is a Lie algebroid with the Lie algebra bundle equal to $\mathfrak{g}$.

(b) For another vector bundle $A' \supset \mathfrak{g}$ (on M) and mappings γ', λ', fulfilling the analogous properties, there exists exactly one isomorphism $F: A' \longrightarrow A$ of Lie algebroids such that the diagram

$$\begin{array}{ccccc} \mathfrak{g} & \hookrightarrow & A' & \xrightarrow{\gamma'} & TM \\ \| & & | & & \| \\ \mathfrak{g} & \hookrightarrow & A & \xrightarrow{\gamma} & TM \end{array}$$

commutes. F is defined by the formula $F(\lambda' v + w) = \lambda v + w$, $v \in TM$, $w \in \mathfrak{g}$.

(c) If $\Omega_M = 0$, then the Lie algebroid constructed in (a) is flat.

PROOF. Straightforward calculations (see [8]). □

Let λ, $\lambda_1 : TM \longrightarrow A$ be two connections in $(A, [\![\cdot,\cdot]\!], \gamma)$. Then $c := \lambda_1 - \lambda$ has its values in the bundle $\mathfrak{g}$.

5.3. PROPOSITION. If ∇, ∇_1 are two covariant derivatives in $\mathfrak{g}$ corresponding to λ, λ_1, respectively, then $\nabla = \nabla_1$ iff $c: TM \longrightarrow \mathfrak{g}$ is a central homomorphism, ie such that $c(v)$ belongs to the centre of the Lie algebra $\mathfrak{g}_{|x}$ for $v \in T_x M$, $x \in M$. □

5.4. COROLLARY. If the isotropy Lie algebras are without the centre, then to different connections there correspond different covariant derivatives. □

Let $\mathfrak{g}$ be any bundle of semisimple Lie algebras on a manifold M.

5.5. PROPOSITION. For any Σ-connection ∇ in $\mathfrak{g}$, there exists exactly one 2-form $\Omega_M \in \Omega^2(M;\mathfrak{g})$ fulfilling condition (1^o) from proposition 5.1. Ω_M fulfils the Bianchi identity (3^o).

PROOF. It is easy to check that $R_{v,w}:\mathfrak{g}_{|x} \to \mathfrak{g}_{|x}$ for $v,w\in T_xM$ is a differentiation of the Lie algebra $\mathfrak{g}_{|x}$, R being the curvature tensor of ∇. From the assumption that $\mathfrak{g}_{|x}$ is semisimple, we have the existence and the uniqueness of an element $\Omega_M(x;v,w)\in\mathfrak{g}_{|x}$ such that

$$R_{v,w}(u) = -[\Omega_M(x;v,w),u], \quad u\in\mathfrak{g}_{|x}.$$

Of course, we have thus defined a 2-form $\Omega_M\in\Omega^2(M;\mathfrak{g})$.
By standard calculations we obtain the equalities:

$$\begin{aligned}
[\nabla\Omega_M(X,Y,Z),\sigma] &= [\nabla_X(\Omega_M(Y,Z)),\sigma] - [\nabla_Y(\Omega_M(X,Z)),\sigma] \\
&\quad + [\nabla_Z(\Omega_M(X,Y)),\sigma] - [\Omega_M([X,Y],Z),\sigma] \\
&\quad + [\Omega_M([X,Z],Y),\sigma] - [\Omega_M([Y,Z],X),\sigma] \\
&= -\nabla_X(R_{Y,Z}\sigma) + R_{Y,Z}(\nabla_X\sigma) + \nabla_Y(R_{X,Z}\sigma) \\
&\quad - R_{X,Z}(\nabla_Y\sigma) - \nabla_Z(R_{X,Y}\sigma) + R_{X,Y}(\nabla_Z\sigma) \\
&\quad + R_{[X,Y],Z}\sigma - R_{[X,Z],Y}\sigma + R_{[Y,Z],X}\sigma \\
&= 0,
\end{aligned}$$

which implies the Bianchi identity $\nabla\Omega_M = 0$.

5.6. PROPOSITION. If $\mathfrak{g}$ is the Lie algebra bundle assigned to a Lie algebroid $(A,[\![\cdot,\cdot]\!],\gamma)$, and a covariant derivative ∇ in $\mathfrak{g}$ corresponds to a connection λ, then the 2-form $\Omega_M\in\Omega^2(M;\mathfrak{g})$ defined by (1^o) is exactly the curvature base form of λ.

PROOF. We need to notice that $R_{X,Y}\sigma = -[\lambda\circ[X,Y] - [\![\lambda\circ X,\lambda\circ Y]\!],\sigma]$ knowing that $\nabla_X\sigma = [\![\lambda\circ X,\sigma]\!]$; but this is a standard calculation. □

5.7. THEOREM. For a given Lie algebra bundle $\mathfrak{g}$ whose fibres are semisimple, there exists exactly one (up to an isomorphism) Lie algebroid A for which $\mathfrak{g}(A)=\mathfrak{g}$.

PROOF. The existence: According to [3,p.380], there exists in $\mathfrak{g}$ a Σ-connection. Let A, γ, λ be elements as before (see (15) and (*) in theorem 5.2). Give any Σ-connection ∇ in $\mathfrak{g}$ and the 2-form

$\Omega_M \in \Omega^2(M;\mathfrak{g})$ fulfilling (1^o). For this homomorphism λ, we define in A some structure of a Lie algebroid $(A, [\![\cdot,\cdot]\!], \gamma)$ according to theorem 5.2 and proposition 5.5.

The uniqueness: Let $(A, [\![\cdot,\cdot]\!], \gamma)$ be any Lie algebroid for which $\mathfrak{g}(A)=\mathfrak{g}$. Let ∇^λ denote the covariant derivative in $\mathfrak{g}$ corresponding to a connection $\lambda: TM \longrightarrow A$.

LEMMA. The correspondence $\lambda \longmapsto \nabla^\lambda$ establishes a bijection between the set of all connections in $(A, [\![\cdot,\cdot]\!], \gamma)$ and the set of all Σ-connections in $\mathfrak{g}$.

PROOF. By corollary 5.4, this correspondence is an injection. Let ∇ be an arbitrary Σ-connection in $\mathfrak{g}$. Of course, $T = \nabla - \nabla_o$ is a tensor

$$T : TM \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

where ∇_o is the Σ-connection corresponding to an arbitrary but fixed connection λ_o. Besides,

$$\nabla_v \sigma = \nabla_{ov} \sigma + T(v, \sigma(x)), \quad v \in T_x M, \ \sigma \in \mathrm{Sec}\,\mathfrak{g}, \ x \in M.$$

We want to find a homomorphism $c: TM \longrightarrow \mathfrak{g}$ such that

$$\nabla_v \sigma = [\![(\lambda_o + c)v, \sigma]\!],$$

which will mean that

$$\nabla = \nabla^{\lambda_o + c}.$$

First, we notice that $T(v,\cdot): \mathfrak{g}_{|x} \longrightarrow \mathfrak{g}_{|x}$, $v \in T_x M$, is a differentiation of the Lie algebra $\mathfrak{g}_{|x}$. Because of the fact that $\mathfrak{g}_{|x}$ are semisimple, we see that the differentiation $T(v,\cdot)$ is adjoint, which means that there exists exactly one element $c(v)$ such that

$$T(v,\cdot) = [c(v), \cdot].$$

It remains to show that the mapping $c: TM \longrightarrow \mathfrak{g}$, $v \longmapsto c(v)$, is a C^∞-vector bundle homomorphism. Of course, it is a vector bundle homomorphism, so we must prove the smoothness of c only. Since $\mathfrak{g}$ is a locally trivial Lie algebra bundle, the smoothness of c is obtained locally by the following assertions:

— For a Lie algebra $\mathfrak{h}$ without the centre, a manifold N and a C^{∞}-linear representation $T: N\times\mathfrak{h}\longrightarrow\mathfrak{h}$, such that $T(v,\cdot)=[c(v),\cdot]$, $v\in N$, for some $c: N\longrightarrow\mathfrak{h}$, we have: c is C^{∞}.

This assertion is easy to show, see the diagram:

$$N\xrightarrow{c}\mathfrak{h}\rightarrowtail^{ad}\mathcal{L}(\mathfrak{h},\mathfrak{h}),\qquad \widetilde{T}: N\longrightarrow\mathcal{L}(\mathfrak{h},\mathfrak{h})$$

in which $\widetilde{T}(v)(w)=T(v,w)$.

The continuation of the proof of the theorem: Let $(A^{1},[\![\cdot,\cdot]\!]^{1},\gamma^{1})$, $(A^{2},[\![\cdot,\cdot]\!]^{2},\gamma^{2})$ be two Lie algebroids for which

$$\mathfrak{g}(A^{1})=\mathfrak{g}(A^{2})=\mathfrak{g}.$$

Take an arbitrary Σ-connection ∇ in $\mathfrak{g}$, and denote by λ_{1}, λ_{2}, the corresponding connections in A^{1}, A^{2}, respecticely (according to the lemma above). Then

$$F: A^{1}\longrightarrow A^{2},\quad \lambda_{1}v+w\longmapsto\lambda_{2}v+w,\qquad v\in TM,\ w\in\mathfrak{g},$$

is an isomorphism of Lie algebroids. Indeed

$$\begin{aligned}F([\![\lambda_{1}\circ X+\sigma,\lambda_{1}\circ Y+\eta]\!]) &= F(\lambda_{1}\circ[X,Y]-\Omega_{M}(X,Y)+\nabla_{X}\eta-\nabla_{Y}\sigma+[\sigma,\eta])\\ &=\lambda_{2}\circ[X,Y]-\Omega_{M}(X,Y)+\nabla_{X}\eta-\nabla_{Y}\sigma+[\sigma,\eta]\\ &=[\![\lambda_{2}\circ X+\sigma,\lambda_{2}\circ Y+\eta]\!].\qquad\square\end{aligned}$$

5.8. COROLLARY. Two pfb's with semisimple structural Lie groups are locally isomorphic iff their associated Lie algebra bundles are isomorphic.

5.9. QUESTION. Are two pfb's globally isomorphic provided (a) their structural Lie groups are connected, semisimple and isomorphic, (b) their Ad-associated Lie algebra bundles are isomorphic?

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INSTITUTE OF MATHEMATICS,
TECHNICAL UNIVERSITY OF ŁÓDŹ
AL. POLITECHNIKI 11
PL-90-924, ŁÓDŹ
POLAND