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THE RANK OF VECTOR FIELDS ON GRASSMANNIAN MANIFOLDS

by Ulrich Koschorke and Július Korbaš

Introduction.

It is an old and central problem in topology to decide when a given vector bundle η over a manifold M allows a nowhere vanishing section. If η is the homomorphism bundle $\eta = \mathrm{Hom}(\alpha,\beta)$ of vectorbundles α and β over M, the question can be refined considerably. Indeed, for each point x of M the fiber η_x consists of the linear maps from α_x to β_x, and it is natural to distinguish their rank (and not only whether they vanish or not). So we are lead to ask

Question. When does $\eta = \mathrm{Hom}(\alpha,\beta)$ allow a section exceeding a given minimal rank everywhere?

This question is at the base of the singularity theory (cf. [Koschorke 2]) which has numerous applications in the theory of immersions, frame fields and other vector bundle monomorphisms (and more generally, whereever morphisms of a given minimal rank are studied).

The present paper is inspired by the observations that some very classical vector bundles have a canonical interpretation as a homomorphism bundle, and so our question applies. In particular, it is wellknown that the tangent bundle of the real Grassmann manifold $G_{m,p}$ of p-planes in $\mathbb{R}^m$ has such a form

$$TG_{m,p} \cong \underset{\sim}{\mathrm{Hom}}(\gamma,\gamma^{\perp}) \quad ,$$

where $\gamma \subset \underset{\sim}{\mathbb{R}^m}$ is the canonical bundle over $G_{m,p}$, and $\gamma^{\perp}$ is its complement (for details see e.g. [Koschorke 1], p. 97). From a calculation of the Euler number it is known that $G_{m,p}$ carries a nowhere vanishing tangential vectorfield if and only if m is even and p is odd (and hence $G_{m,p}$ is an odd-dimensional manifold). Under these dimension assumptions we will actually construct a very

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concrete "linear" vectorfield, study its geometry and deduce the following

Theorem. Let $p = 2r+1$, $q = 2s+1$ and $m = p+q$. If $\binom{r+s}{r}$ or $\binom{r+s+1}{r}$ or $\binom{r+s+1}{r+1}$ is odd, then the real Grassmannian $G_{m,p}$ allows no vector field v which, when considered as section in $\operatorname{Hom}(\gamma,\gamma^{\perp})$, has rank > 1 everywhere ∎

So if we define the vectorfield rank of $G_{m,p}$ by

$$rk(G_{m,p}) = \max\{rk(v) \mid v \text{ tangential vectorfield on } G_{m,p}\}$$

where $rk(v) := \min\{\operatorname{rank}(v_x : \gamma_x \to \gamma_x^{\perp}) \mid x \in G_{m,p}\}$, we see that

$$rk(G_{m,p}) = 0 \quad \text{if } p \text{ or } m-p \text{ are even}$$

and

$$rk(G_{m,p}) = 1 \quad \text{if } p \equiv m-p \equiv 1(2) \text{ and } \binom{\frac{m-2}{2}}{\frac{p-1}{2}} \text{ or } \binom{\frac{m}{2}}{\frac{p-1}{2}} \text{ or } \binom{\frac{m}{2}}{\frac{p+1}{2}} \text{ is odd.}$$

Remark. It follows from [Koschorke 2], proposition 5.3., that $rk(G_{p+q,p}) \leq 1$ whenever the "Hankel determinant" of Stiefel-Whitney classes

$$\det(w_{p-1-i+j}(\gamma\oplus\gamma))_{1\leq i,j\leq q-1} = \det(w_{q-1-i+j}(\gamma^{\perp}\oplus\gamma^{\perp}))_{1\leq i,j\leq p-1}$$

is nontrivial. However, this criterium seems to require hard calculations which can be avoided by our explicit geometric approach. E.g. if m is even and $p=3$, or if $m \not\equiv 2(8)$ is even and $p=5$, we get from the theorem that $rk(G_{m,p}) = 1$. Already in the first case, where the Hankel determinant is simply $\bar{w}_k(\gamma)^4$ (with $k = \frac{m-4}{2}$), it takes a very involved computation to establish its nontriviality directly ∎

§ 1. The "linear" vector field v_A on $G_{m,p}$.

Throughout this paper let $p = 2r+1$, $q = 2s+1$ and $m = p+q = 2(r+s+1)$. So we can identify $\mathbb{R}^m$ with $\mathbb{C}^{r+s+1}$ and consider the composed vector bundle homomorphism

$$h : \gamma \subset G_{m,p} \times \mathbb{R}^m \xrightarrow{\text{Id} \times A} G_{m,p} \times \mathbb{R}^m \xrightarrow{\text{proj}} \gamma$$

over the real Grassmannian $G_{m,p}$ where $A : \mathbb{R}^m \to \mathbb{R}^m$ is just complex multiplication with $i = \sqrt{-1}$. Interpreted as a section of the homomorphism bundle $\underset{\sim}{\mathrm{Hom}}(\gamma,\gamma^{\perp}) \cong TG_{m,p}$, h gives rise to a tangential vectorfield v_A on $G_{m,p}$ which we are going to study now.

Given any point $g \in G_{m,p}$, note first that the kernel of $h_g : g \subset \mathbb{R}^m \xrightarrow{i} \mathbb{R}^m \xrightarrow{proj} g^{\perp}$ is obviously $g_{\mathbb{C}} := g \cap i(g)$, the largest complex subspace of the real p-plane $g \subset \mathbb{R}^m = \mathbb{C}^{r+s+1}$. The locus of minimum rank of h is

$$N = \{g_{\mathbb{C}} \oplus \ell \in G_{m,p} \mid g_{\mathbb{C}} \subset \mathbb{C}^{r+s+1} \text{ complex r-plane}, \ell \subset (g_{\mathbb{C}})^{\perp} \text{ real line}\}$$

on this flag manifold h has rank 1.

For any point $g = g_{\mathbb{C}} \oplus \ell \in N$, the resulting decomposition $\mathbb{R}^m = g_{\mathbb{C}} \oplus \ell \oplus i\ell \oplus g_{\mathbb{C}}^{\perp} = g \oplus g^{\perp}$ gives rise to a chart

$$\psi : U = \{g' \in G_{m,p} \mid g' \text{ complementary to } g^{\perp} \text{ in } \mathbb{R}^m\} \longrightarrow L_{\mathbb{R}}(g,g^{\perp})$$

defined by $\psi^{-1}(L) = \text{graph}\, L = (\mathrm{id},L)(g) \in U$ for $L \in L_{\mathbb{R}}(g,g^{\perp})$, and similarly to trivializations of $\gamma|U$ and $\gamma^{\perp}|U$. Using these to calculate the tangent behaviour of v_A at $g \in N$, one sees that the principal part of $T_g(v_A)$, which is an endomorphism of $T_g(G_{m,p}) = L_{\mathbb{R}}(g_{\mathbb{C}} \oplus \ell,\ g_{\mathbb{C}}^{\perp} \oplus (i\ell))$, is multiplication with $1-\varepsilon$ on

$$L_{\varepsilon}(g_{\mathbb{C}},\ g_{\mathbb{C}}^{\perp}) = \{L \in L_{\mathbb{R}}(g_{\mathbb{C}},\ g_{\mathbb{C}}^{\perp} \mid i \circ L = \varepsilon \cdot L \circ i\}$$

for $\varepsilon = \pm 1$.

Next consider the manifold $\mathbb{C}F(r,s,1)$ of complex flags $f \perp k \perp e \subset \mathbb{C}^{r+s+1}$ of the indicated dimensions, and denote by φ, κ and λ the corresponding vector bundles so that e.g. $\varphi \oplus \kappa \oplus \lambda = \underset{\sim}{\mathbb{C}^{r+s+1}}$. We have a natural fibration

$$\begin{aligned} \pi : N &\longrightarrow \mathbb{C}F(r,\ s,\ 1) \\ g_{\mathbb{C}} \oplus \ell &\longrightarrow (g_{\mathbb{C}},\ g_{\mathbb{C}}^{\perp},\ (\ell \oplus i\ell)) \end{aligned}$$

with fiber real projective line P^1. Choose a generic section σ of the complex vector bundle $\underset{\sim}{\mathrm{Hom}}_{\mathbb{C}}(\pi^*(\varphi),\pi^*(\kappa))$ over N and let $S \subset N$ denote its zero manifold. Extending σ to (a tubular neighbourhood of N in) $G_{m,p}$ and adding it to the "linear" section v_A of $\underset{\sim}{\mathrm{Hom}}(\gamma,\gamma^{\perp}) \cong TG_{m,p}$ studied above, we obtain a nondegenerate 1-morphism $u : \gamma \longrightarrow \gamma^{\perp}$ over all of $G_{m,p}$ (as we had seen above, v_A alone is not nondegenerate); the singularity of u, or in other words, the locus $\{g \in G_{m,p} \mid \text{rank}\, u_g = 1\}$, is precisely S.

Thus we can use the results of [Koschorke 2], in particular proposition 5.3 and fact 9.7, to compute the dual class $\mathcal{D}(S) \in H^*(G_{m,p}; \mathbb{Z}_2)$ as follows

$$\begin{aligned}\mathcal{D}(S) &= \det(w_{q-1-i+j}(\gamma^{\perp} \oplus \gamma^{\perp}))_{1\le i,\ j\le p-1} \\ &= \det(w_{p-1-i+j}(\gamma \oplus \gamma))_{1\le i,\ j\le q-1} .\end{aligned}$$

If $rk(G_{m,p}) > 1$, i.e. if there exists a 2-morphism $\tilde{u} : \gamma \to \gamma^{\perp}$ (with empty locus of rank 1 points!), then S must be zero bordant in $\mathfrak{N}_*(G_{m,p})$.

In particular, we have the following logical implications:

$$\begin{aligned} rk(G_{m,p}) > 1 &\xrightarrow{(1)} \mathcal{D}(S) \text{ vanishes in } H^*(G_{m,p}; \mathbb{Z}_2) \\ &\xleftrightarrow{(2)} [S] = 0 \quad \text{in } H_*(G_{m,p}; \mathbb{Z}_2) \\ &\xleftrightarrow{(3)} [S] = 0 \quad \text{in } H_*(N; \mathbb{Z}_2) \\ &\xleftrightarrow{(4)} w_{2rs}(\underset{\sim}{Hom}_{\mathbb{C}}(\pi^*(\varphi), \pi^*(\kappa))) = 0 \text{ in } H^*(N; \mathbb{Z}_2) \\ &\xleftrightarrow{(5)} w_{2rs}(\underset{\sim}{Hom}_{\mathbb{C}}(\varphi, \kappa)) = 0 \text{ in } H^*(\mathbb{C}F(r,s,1); \mathbb{Z}_2). \end{aligned}$$

Here conclusions (1) and (2) follow from [Koschorke 2], fact 9.7; to see (3), note that the inclusion $N \subset G_{m,p}$ restricts γ to $\gamma_{\mathbb{C}} \oplus \lambda$ and hence induces an epimorphism in $\mathbb{Z}_2$-cohomology; (4) follows from the duality of the two classes under comparison; finally (5) follows from the theorem of Leray-Hirsch which shows that π^* is injective on $H^{2*}(-; \mathbb{Z}_2)$.

Now, if the top Stiefel-Whitney class of $\underset{\sim}{Hom}_{\mathbb{C}}(\varphi, \kappa)$ vanishes over $\mathbb{C}F(r,s,1)$, so it is also trivial when restricted to the fiber $\mathbb{C}G_{r+s,r}$ or when multiplied with the Euler class of $\underset{\sim}{Hom}_{\mathbb{C}}(\varphi, \lambda)$ of of $\underset{\sim}{Hom}_{\mathbb{C}}(\lambda, \kappa)$. After relating $\underset{\sim}{Hom}_{\mathbb{C}}(\varphi, \kappa) \mid \mathbb{C}G_{r+s,r}$, $\underset{\sim}{Hom}_{\mathbb{C}}(\varphi, \varphi^{\perp})$ and $\underset{\sim}{Hom}_{\mathbb{C}}(\kappa^{\perp}, \kappa)$ to the tangent bundles of the complex Grassmannians $\mathbb{C}G_{r+s,r}$, $\mathbb{C}G_{r+s+1,r}$ and $\mathbb{C}G_{r+s+1,s}$ and after applying the theorem of Leray and Hirsch again repeatedly, we conclude in particular that the Euler numbers of these Grassmannians must be even if the Hankel determinant $\mathcal{D}(S)$ is to vanish. Since by counting Schubert cells (see [Milnor-Stasheff], §6) we always get

$$\chi(\mathbb{C}G_{m,p}) = \binom{m}{p} ,$$

this concludes the proof of the theorem stated in our introduction.

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