Bela Kis

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CONNECTION THEORY AND PARAMETER TRANSFORMATIONS

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1.1 Introduction

The classical connection theory on fiber bundles has two symmetries: a fiber symmetry and a hidden one. The fiber symmetry is related to the fiber group of the fiber bundle. Identifying the connection with its parallel translation the usual connection commutes with the action of the fiber group. This part of the connection theory is well described and has several generalizations: it includes the theory of linear and homogeneous connections as well as the theory of the principal connections [3],[1].

The hidden symmetry is related to the notion of parameter transformations of (smooth) curves. Considering the parallel translation as an operation which maps the curves of the base manifold to curves of the total space, we can see that in the classical connection theory this operation also commutes with the parameter transformations of curves. However, geometrically it is not obvious that this commutation condition must be satisfied to get a meaningful connection theory. Our goal in the present paper is to give a short description of a connection theory in which a generalized version will be used of the above condition.

In paragraph 1.2 we give a short list of the notions and notations used in this paper. The next three paragraphs of part two are devoted to build up the basic notions of our theory. These notions are worked out only for the most simple cases and we do not touch the question of the fiber symmetries, however fiber symmetry can be easily transplanted into this theory; we use vector bundles from technical reasons but the connection theory will not be linear in the "classical" way.

The third part is dealing with the first order case and contains the classical connection theory and the theory of the (over)generalized connections as special cases of our general notion.

In the last part we sketch an application from the Finsler geometry: the global description of Rund's δ-derivation [4].

1.2 Basic notions and notations

In this paragraph we collect some basic notions and notations used in the followings .

First of all we use the term "*smooth*" as the synonym of the phrase "C^∞*-differentiable*".

The triad $\xi=(E,\pi,B)$ is a **topological fiber bundle** if E , B are topological spaces and $\pi: E \longrightarrow B$ is a continuous surjection. We call spaces E and B the *total space* and the *base space* of ξ , and the map $\pi: E \longrightarrow B$ its projection. The *fibers* are the sets $\xi_p \doteq \pi^{-1}(p)$ ($p \in B$).

A topological fiber bundle $\xi=(A,\pi,B)$ is a *smooth* fiber bundle if E and B are smooth manifolds and π is a smooth submersion. ξ is *locally trivial* if for every $p \in B$ there exists an open neighborhood U_p and a continuous (smooth) map $\varphi_p: \pi^{-1}(U_p) \longrightarrow U_p \times F$ (F is a fixed space) for which the diagram

$$\begin{tikzcd}
\pi^{-1}(U_p) \arrow[rr, "\varphi_p"] \arrow[dr, "\pi"'] & & U_p \times F \arrow[dl, "pr^1"] \\
& U_p &
\end{tikzcd}$$

commutes. The total space of a locally trivial smooth fibered bundle is called *fibered manifold.* A locally trivial continuous resp. smooth bundle ξ is a continuous resp. smooth *vector bundle* if its fibers are vector spaces and the maps

$$\varphi_p^{-1} \circ \varphi_q : \pi^{-1}(U_p \cap U_q) \longrightarrow \pi^{-1}(U_p \cap U_q)$$

are linear on the fibers for every $p, q \in B$, $U_p \cap U_q \neq \phi$.

If ξ is a bundle, we will denote its total space, base space and projection by $\mathrm{tl}\,\xi$, $\mathrm{bs}\,\xi$ and $\mathrm{pr}\,\xi$ in order. If M is a manifold, its *tangent bundle* is denoted by τM . We also use the following notations: $TM := \mathrm{tl}\,\tau M$, $\mathcal{B}_M := \mathrm{pr}\,\tau M$.

We will say that the pair (α, β) is a *bundle* or *fiber preserving* map between continuous resp. smooth fiber bundles ξ and η if the diagram

$$\begin{tikzcd}
\mathrm{tl}\,\xi \arrow[r, "\alpha"] \arrow[d, "\mathrm{pr}\,\xi"'] & \mathrm{tl}\,\eta \arrow[d, "\mathrm{pr}\,\eta"] \\
\mathrm{bs}\,\xi \arrow[r, "\beta"'] & \mathrm{bs}\,\eta
\end{tikzcd}$$

is commutative and the maps α , β are continuous resp. smooth. When the base manifolds are equal and the map β is the identity we say that the bundle map (α, β) is a *strong bundle* map. If (α, β) is a bundle map between vector bundles and the restrictions of the map α to the fibers are linear we say that (α, β) is a *vector bundle map*. We also use the *'strong vector bundle map'* terminology.

The map α completely determines the map β , so we will use the simpler notation α instead of the more deductive notation (α, β).

Let $\psi : M \longrightarrow \mathrm{bs}\,\xi$ be a smooth map, where ξ is a bundle. We construct the *pull-back bundle* $\psi^!\xi$ of ξ by ψ in terms of the commutative diagram

$$\begin{tikzcd}
\mathrm{tl}\,\psi^!\xi \arrow[r, "\mathrm{ad}_\xi\psi"] \arrow[d, "\mathrm{pr}(\psi^!\xi)"'] & \mathrm{tl}\,\xi \arrow[d, "\mathrm{pr}\,\xi"] \\
M \arrow[r, "\psi"'] & \mathrm{bs}\,\xi
\end{tikzcd}$$

which is the so called *pull-back square* associated to ψ and ξ. The bundle $\psi^!\xi$ is not canonically determined: its usual representative is the bundle $(\bigcup_{p\in bs\,\xi}\psi^{-1}(p)\times\xi_p,\ pr^1, M)$. However, the vector bundle map $ad_\xi\psi$ is canonical relative to the bundle $\psi^!\xi$, i.e. $\psi^!\xi$ determines $ad_\xi\psi$ canonically.

If α is a bundle map between bundles η and ξ where the base space of η is M, then we can construct a unique strong bundle map $\psi^!\alpha : \eta \longrightarrow \psi^!\xi$ for which the diagram

$$\begin{tikzcd}
tl\,\eta \arrow[rr, "\alpha"] \arrow[dr, "\psi^!\alpha"] \arrow[dd, "pr\,\eta"'] & & tl\,\xi \arrow[dd, "pr\,\xi"] \\
& tl\,\psi^!\xi \arrow[ur, "ad_\xi\psi"'] \arrow[dd] & \\
M \arrow[rr] \arrow[dr, "id_M"'] & & bs\,\xi \\
& M \arrow[ur, "\psi"'] &
\end{tikzcd}$$

is commutative. We call this map the *pull-back factorization of* α via ψ.

The pull-back functor preserves the following properties: continuity, smoothness and local triviality. It maps vector bundles into vector bundles again.

If ξ is a vector bundle its vertical subbundle is denoted by $V\xi$. This subbundle of $\tau\,tl\,\xi$ is cannonically isomorphic with the pull-back bundle $(pr\,\xi)^!\xi$.

2.1 Germs of Curves

Let $\mathcal{T} = \{(a,b)\,|\,a<0<b;\ a,b\in M\}$ be the canonical system of the neighborhoods of $0\in\mathbb{R}$. If $p\in M$ then let $\Gamma(M)=\bigcup_{p\in M}\Gamma_p(M)$, where

$$\Gamma_p(M) \doteq \{\gamma : I \longrightarrow M \,|\, I\in\mathcal{T},\quad \gamma \text{ is a smooth 1-1 map},\ \gamma(0)=p\}$$

If $\gamma : (a,b) \longrightarrow M$ is an element of $\Gamma_p(M)$, then for any $s\in(a,b)$ let us denote by $\mathfrak{M}_s\gamma$ the map

$$(\mathfrak{M}_s\gamma)(t) \doteq \gamma(t+s),$$

whose domain is the interval $(a-s, b-s)$ and which is an element of $\Gamma_{\gamma(s)}M$. Two elements γ_1, γ_2 of $\Gamma_p(M)$ are *called germ* equivalent if there exists such an element I of $\mathcal{I}$ for which

$$I \subset \operatorname{dom}\gamma_1 \cap \operatorname{dom}\gamma_2$$

and

$$\gamma_1|_I = \gamma_2|_I .$$

This relation is an equivalence relation on $\Gamma_p(M)$. The factor set of $\Gamma_p(M)$ by it is denoted by $G_p(M)$ and its elements are the so called curve germs. If $\gamma \in \Gamma_p(M)$ then its equivalence class is denoted by γ_p. The union $G(M)=\bigcup_{p\in M} G(M)$ of all equivalence classes is called *the curve sheaf* of M. We can define a topology on $G(M)$ by the system of neighborhoods $\mathcal{U}_g$ of any $g \in G(M)$. If $g\in G_p(M)$ ($p\in M$) then the elements of $\mathcal{U}_g$ are the sets

$$\mathcal{U}(g,\gamma,I) = \{(\Pi_s\gamma)_{\gamma(s)} \mid s\in I\}$$

where $g=(\gamma)_p, I=\operatorname{dom}\gamma$.

The *canonical projection* $\pi : G(M) \longrightarrow M$ is defined by $\pi(g)=p$ iff $g\in G_p(M)$. This map is continuous. The triad $(G(M), M, \pi)$ is a (non-locally trivial) topological fiber bundle. (We denote this bundle by $G(M)$, too.)

If $\varphi : M \longrightarrow N$ is a continuous map between manifolds then the map

$$\Gamma(\varphi) : \Gamma(M) \longrightarrow \Gamma(N)$$

$$\gamma \longrightarrow \varphi\circ\gamma$$

induces a continuous fiber preserving map $G(\varphi)$ between $G(M)$ and $G(N)$.

Two elements γ_1, γ_2 of $\Gamma_p(M)$ are called *r*th order equivalent if the derivatives of the map $f\circ\gamma_1 - f\circ\gamma_2$ vanish up to *r*th order for any smooth function f defined in an open neighborhood of p. The *r*th order equivalence is an equivalence relation on $\Gamma_p(M)$; if $\gamma\in\Gamma_p(M)$ then its *r*th order equivalence class is called its *r-jet* at p and denoted by $j^r_p(\gamma)$. The set of all r-jets at p is denoted by $\dot{\mathcal{J}}^r_p(M)$; the union $\dot{\mathcal{J}}^r(M)$ of these sets is naturally a fibered manifold with the canonical projection $\pi^r : \dot{\mathcal{J}}^r(M) \longrightarrow \dot{\mathcal{J}}^r(N)$ which is defined by the relation $\tilde{\pi}^r(z) = p$ iff $z \in \dot{\mathfrak{J}}^r_p(M)$ $(p \in M)$ (see [2]).

There exists a natural map j^r between $G(M)$ and $\dot{\mathfrak{J}}^r(M)$ for which the diagram

$$\begin{tikzcd}
G(M) \arrow[r, "j^r"] \arrow[d, "\tilde{\pi}"'] & \dot{\mathfrak{J}}^r(M) \arrow[d, "\pi^r"] \\
M \arrow[r, "id_M"] & M
\end{tikzcd}$$

commutes and which is defined by the relation

$$j^r((\gamma)_p) \doteq j^r_p(\gamma)$$

This map is the so called *rth jet expansion*.

If $\varphi : M \longrightarrow N$ is a smooth map then there exists a map $j^r(\varphi)$ between manifolds $\dot{\mathfrak{J}}^r(M)$ and $\dot{\mathfrak{J}}^r(N)$ for which the diagram

$$\begin{tikzcd}
\dot{\mathfrak{J}}^r(M) \arrow[rrr, "j^r(\gamma)"] \arrow[dr, "\pi^r"] \arrow[dd] & & & \dot{\mathfrak{J}}^r(N) \arrow[dl, "\pi^r"'] \arrow[dd] \\
& M \arrow[r, "\varphi"] & N & \\
G(M) \arrow[ur, "\pi"] \arrow[rrr, "G(\varphi)"] & & & G(N) \arrow[ul, "\pi"']
\end{tikzcd}$$

is commutative. The definition of this map is:

$$j^r(\varphi)(j^r_p(\gamma)) \doteq j^r_{\varphi(p)}(\gamma \circ \varphi)$$

2.2 Subsets of G(M) and Parameter Transforms on M

Let $\mathbb{R} = (-\infty, \infty)$ be the real line. The set $G_0(\mathbb{R})$ is a topological group with a multiplication induced by the composition of the elements of $\Gamma_0(\mathbb{R})$. This group acts continuously the and freely on $G(M)$ on the right by an action also induced by the composition of functions. If $\mu \in G_0(\mathbb{R})$ then its action on $g \in G(M)$ is denoted by $R_\mu g$.

Two elements $g_1, g_2 \in G_p(M)$ are called *graph equivalent* at p if there exists such an element μ of $G_0(\mathbb{R})$ for which $g_2 = R_\mu g_1$. This equivalence is denoted by $g_1 \sim_p g_2$; the equivalence classes – the orbits – are called the *graph germs* of M at p and the equivalence class of $g \in G_p(M)$ is denoted by $\{g\}_p$.

Definition: The family $\tilde{\mathcal{A}} = \{H_\lambda | \lambda \in \Theta\}$ of disjoint subsets of $G(M)$ are called a **refinement** of $\sim_p$ if the following condition is satisfied:

(α) *if* $g_1, g_2 \in G(M)$ *are elements of the same subset* H_λ *(*$\lambda \in \Theta$ *) then* $g_1 \sim_p g_2$ *for some* $p \in M$ *.*

In this case we say that g_1 and g_2 are **congruent**.

A refinement $\tilde{\mathcal{A}} = \{H_\lambda | \lambda \in \Theta\}$ of $\sim_p$ is called a **parametrization structure** on M if there exists a subgroup $^*G$ of $G_0(\mathbb{R})$ which satisfies the following condition:

(β) *if* $g_1, g_2 \in G(M)$ *are elements of the same subset* H_λ *(*$\lambda \in \Theta$*) then there exists a* $\mu \in {}^*G$ *for which* $g_2 = R_\mu g_1$ *.*

This element μ is uniquely determined and we call it the **parameter transformation** associated to the pair (g_1, g_2). The group $^*G$ is not unique; any subgroup of $G_0(\mathbb{R})$ which contains it satisfies condition (β) . We call the smallest subgroup of $G_0(\mathbb{R})$ which satisfies condition (β) the **parameter transformation group** of $\tilde{\mathcal{A}}$.

There are two special types of the above defined notion:

(1) The parametrization structure $\tilde{\mathcal{A}} = \{H_\lambda | \lambda \in \Theta\}$ is called **full** if $G(M) = \bigcup_{\lambda \in \Theta} H_\lambda$. In this case $\tilde{\mathcal{A}}$ induces an equivalence relation $\sim$ on $G(M)$ which is finer than $\sim_p$; i.e. if $g_1 \sim g_2$ then $g_1 \sim_p g_2$. This relation is compatible with $^*G$ and in this case $\tilde{\mathcal{A}}$ is denoted by $G(M)/{}^*G$.

(2) The parametrization structure $\tilde{\mathcal{A}} = \{H_\lambda | \lambda \in \Theta\}$ is called **fat** if every $G_p(M)$ contains exactly one element of H_λ .

If $\tilde{\mathcal{A}}$ is a parametrization structure the subsets of it are also parametrization structures.

Definition: A parametrization structure $\tilde{\mathcal{F}} = \{ H_\lambda \mid \lambda \in \Theta \}$ is called **homogeneous** if the parameter transformation groups of the subsets

$$\tilde{\mathcal{F}} = \{ H_\lambda \mid \lambda \in \Theta,\ H_\lambda \subset G_p(M) \}$$

are equal.

In this paper we will use only homogeneous parametrization structures.

Examples: (i) Let $\Theta \doteq M$ and $H_\lambda = G_\lambda(M)$. The set $\tilde{\mathcal{F}} = \{H_\lambda \mid \lambda \in \Theta\}$ is a parametrization structure on M. Its parameter transformation group is the group $G_0(\mathbb{R})$. This parametrization structure is obviously full.

(ii) Two elements $g_1, g_2 \in G(M)$ have the same orientation if the parameter transformation associated to the pair (g_1, g_2) has a representative h whose first derivative is not negative on its domain. This relation is well defined and is an equivalence relation which determines a full parametrization structure $\tilde{\mathcal{F}}$ on M. Every set $G_p(M)$ contains exactly two elements of $\tilde{\mathcal{F}}$ because curves have exactly two orientations.

(iii) An element g of $G_p(M)$ ($p \in M$) is called *geometric* if it has a representative γ whose tangent is not zero in the domain of γ. If the set of geometric elements of $G_p(M)$ is denoted by $\tilde{G}_p$ then the set $\tilde{\mathcal{F}}_G = \{\tilde{G}_p \mid p \in M\}$ is a full parametrization structure on M. The parameter transformation group of $\tilde{\mathcal{F}}_G$ is called **the group of the allowed parameter transformations of** M.

2.3 General Connections

Let ξ be a fixed smooth vector bundle.

Triads $(G(b_s\xi), \pi, b_s\xi)$ and $(\mathcal{J}^r(b_s\xi), \pi^r, b_s\xi)$ are topological (smooth) vector bundles so we can construct their pull-backs via $pr\,\xi$; these bundles are denoted by $(pr\,\xi)^! G(b_s\xi)$ and $(pr\,\xi)^! \mathcal{J}^r(b_s\xi)$. We can define the action of the group $G_0(\mathbb{R})$ on the total space of $(pr\,\xi)^! G(b_s\xi)$ as the pull-back of its action on $G(b_s\xi)$; i.e. if $\mu \in G_0(\mathbb{R})$ and $g \in ((pr\,\xi)^! G(b_s\xi))_p$ then $R_\mu g \in ((pr\,\xi)^! G(b_s\xi))_p$ is determined by identity

$$(\mathrm{ad}_{G(b_s\xi)}\, pr\,\xi)(R_\mu g) = R_\mu (\mathrm{ad}_{G(b_s\xi)}\, pr\,\xi)(g).$$

Definition: The continuous fiber preserving map

$$\varphi : (pr\,\xi)^{!}\,G(bs\,\xi) \longrightarrow G(tl\,\xi)$$

is called a *(free) continuous connection* on vector bundle ξ if it satisfies the following condition:

$$G(pr\,\xi)\circ\varphi = ad_{G(bs\,\xi)}\,pr\,\xi$$

(which is the pull-back factorized version of identity

$$(pr\,\xi)^{!}\,G(pr\,\xi)\circ\varphi = id_{(pr\xi)^{!}G(bs\xi)}$$

Let $\tilde{\mathcal{F}}_{tl\xi}$ and $\tilde{\mathcal{F}}_{bs\xi}$ be parametrization structures on $tl\,\xi$ and $bs\,\xi$ respectively.

Definition: (1) Continuous connection φ is *weakly* $(\tilde{\mathcal{F}}_{tl\xi},\tilde{\mathcal{F}}_{bs\xi})$ *symmetric* if for any two elements g_1, g_2 of $(pr\xi)^{!}G(bs\,\xi)$ the fact that $(ad_{G(bs\xi)}pr\xi)(g_1)$ and $(ad_{G(bs\xi)}pr\xi)(g_2)$ are congruent implies that $\varphi(g_1)$ and $\varphi(g_2)$ are congruent.

(2) Suppose that parametrization structures $\tilde{\mathcal{F}}_{tl\xi}$ and $\tilde{\mathcal{F}}_{bs\xi}$ have common parameter transformation group $^{*}G$. A continuous connection φ is called $^{*}G$ *-symmetric* if for any $g_1, g_2 \in tl\,(pr\xi)^{!}G(bs\xi)$ the fact that $R_\mu g_1 = g_2$

for some $\mu \in G_0(\mathbb{R})$ implies that $R_\mu\ \varphi(g_1) = \varphi(g_2)$.

It is obvious that if a connection is $^{*}G$ -symmetric then it is weakly symmetric.

Smooth functions and curves can be approximated by finite segments of their Taylor-series. From the differential geometrical point of view connections which are determined by their finite approximations are very significant. The correct definition of the phrase "*finite approximation*" for connection theory is the following:

Definition: We say that the $^{*}G$ -symmetric continuous connection is *rth order determined* if there exists a continuous map

$$h : (pr\xi)^{!}\,\mathcal{J}^{r}(bs\,\xi) \longrightarrow \mathcal{J}^{r}(tl\,\xi)$$

for which

$$j^{r}_{v}\,(\varphi(v,\gamma_p)) = h\,(v, j^{r}_{p}(\gamma))$$

for every $(v,\gamma_p) \in \{v\}\times G_p(bs\,\xi) \subset tl\,(pr\xi)^{!}G(bs\xi)$ $(p = (pr\xi)(v))$ and h is smooth on the set $(ad_{G(bs\xi)}pr\xi)^{-1}(j^{r}(\bigcup_{H\subset\tilde{\mathcal{F}}_{bs\xi}} H))$

This map h is called the *horizontal map* of φ

The most often used type of *r*th order connection theory is the first order connection theory. The remaining part of the present paper is dealing with special first order connections.

3.1 First order connections

Let φ be a $^*G$ -symmetric first order determined connection on vector bundle ξ , where $^*G$ is the common parameter transformation group of parametrization structures $\tilde{\mathcal{A}}_{t\ell\xi}$ and $\tilde{\mathcal{A}}_{bs\xi}$. In this case we can identify the functor J^1 with functor d , i.e. for any manifold M the space $J^1(M)$ is the tangent space τM of M and the first jet of maps is equal to their tangent map, so we can consider the horizontal map h of φ as a fiber preserving map

$$h : (pr\xi)^! \tau bs\xi \longrightarrow \tau t\ell\xi$$

which satisfies condition

$$d\left(\varphi(v,\gamma_p)\right) = h(v, d_p\gamma)$$

for every $(v,\gamma_p) \in \{v\} \times G_p(bs\xi) \subset t\ell\,(pr\xi)^! G(bs\xi) \quad (p = (pr\xi)(v))$

Proposition: $[(pr\xi)^! d pr\xi] \circ h = id_{(pr\xi)^! \tau bs\xi}$.

Proof: If $(\tilde{\gamma})_v \doteq \varphi(v,\gamma_p)$ then $(v,\gamma_p) = id_{(pr\xi)^! G(bs\xi)}(v,\gamma_p) =$
$= ([(pr\xi)^! G(pr\xi)] \circ \varphi)(v,\gamma_p) = [(pr\xi)^! G(pr\xi)](\tilde{\gamma}_v) = (v,(pr\xi\circ\tilde{\gamma})_p)$, so $\gamma_p = (pr\xi\circ\tilde{\gamma})_p$

Now

$$\begin{aligned}[((pr\xi)^! d pr\xi)\circ h](v, d_p\gamma) &= [(pr\xi)^! d pr\xi](d_v\varphi(v,\gamma_p)) = \\ &= [(pr\xi)^! d pr\xi](d_v\tilde{\gamma}) = (v, d_p((pr\xi)\circ\tilde{\gamma})) = (v, d_p\gamma) = \\ &= id_{(pr\xi)^! \tau bs\xi}(v, d_p\gamma) \quad \text{Q.E.D.}\end{aligned}$$

Lemma: $id_{\tau t\ell\xi} - h\circ[(pr\xi)^! d pr\xi](z) \in t\ell\, V\xi$

for every $v \in t\ell\xi$, $z \in (\tau t\ell\xi)_v$.

Proof: It's known that $x \in t\ell\, V\xi$ if and only if $(d pr\xi)(x) = 0$ and this is true if and only if $((pr\xi)^! d pr\xi)(x) = 0$.

Now $((pr\xi)^! d pr\xi)(id_{\tau t\ell\xi} - h\circ((pr\xi)^! d pr\xi)) =$

$$= (pr\xi)^!(d pr\xi) - (((pr\xi)^! d pr\xi)\circ h)((pr\xi)^! d pr\xi) = 0 \quad \text{Q.E.D.}$$

Using this lemma we can define the analogues of the notions of the classical connection theory:

Definition: (1) The map $\pi_H = h\circ(pr\xi)^! d pr\xi$ is called the *horizontal projection* of φ .

(2) The map $\pi_V = id_{\tau t\ell\xi} - \pi_H$ is the *vertical projection* of φ .

(3) If $(pr\xi)^!\xi$ and $V\xi$ are identified by the map $r_\xi : V\xi \longrightarrow (pr\xi)^!\xi$

then $v = r_\xi \circ \pi_V$ is called the *vertical map* of φ.

(4) The map $D = (ad_\xi\, pr\,\xi) \circ v$ is called the *Dombrovski map* of φ. If σ is a section of ξ and v is an element of $T\ell s\,\xi$ then we call the expression $(D \circ d\sigma)(v)$ the *covariant derivative* of σ by v, and we denote it by $\nabla_v \sigma$.

Now we will study the interpretation of $^*G$ -symmetry in the first order case. Let us notice first that the tangent space $\mathcal{J}^1_0(\mathbb{R})$ can be identified with $\mathbb{R}$. This identification gives a map

$$\tau : G_0(\mathbb{R}) \longrightarrow \mathbb{R}$$

which is a semigroup homomorphism. If $^*G$ is a subgroup of $G_0(\mathbb{R})$, its image by τ is denoted usually by $^*\mathbb{R}$. The functor τ maps of the action of $\mu \in G_0(\mathbb{R})$ on $g \in G_p(M)$ into $\tau(\mu) d_p g$.

__Lemma:__ For any $(v, x) \in (pr\,\xi)^! \tau\ell s\,\xi$ and $\lambda \in {}^*\mathbb{R}$

$$h(v, \lambda x) = \lambda h(v, x)$$

__Proof:__ If $\lambda = \tau(\mu)$ ($\mu \in {}^*G$) and $x = d_p \gamma$ then

$$h(v, \lambda x) = h(v, d_p(R_\mu \gamma)) = d_v(\varphi(v, R_\mu \gamma_p)) = d_v(R_\mu \varphi(v, \gamma_p)) = \lambda h(v, x) \text{ Q.E.D.}$$

3.2 (Over)generalized Connections

In this paragraph we are dealing with a special kind of $^*G$ -symmetric connections. We give the definition in term of the map h.

__Definition:__ We say that we are given an *(over)generalized connection* on the vector bundle ξ if a continuous map

$$h : (pr\,\xi)^! \tau\ell s\,\xi \longrightarrow \tau t\ell\,\xi$$

is given for which

$$((pr\,\xi)^! d\,pr\,\xi) \circ h = id_{(pr\,\xi)^! \tau\ell s\,\xi}$$

and which is smooth on the manifold of the non-zero elements of $(pr\,\xi)^! \tau\ell s\,\xi$ and 1-homogeneous on the fibers.

We can define the vertical and horizontal projection and vertical and Dombrovski map of the (over)generalized connection similarly as in the previous paragraph. All these maps are fiber preserving and 1-homogeneous.

In the following proposition we give a description of (over)generalized connection in terms of the notions introduced in the previous paragraphs:

Let $\tilde{\mathcal{A}}_{t\ell\xi}$ and $\tilde{\mathcal{A}}_{bs\xi}$ denote the parametrization structures of $t\ell\xi$ and $bs\xi$ determined by the geometric elements of $G(bs\xi)$ and $G(t\ell\xi)$, and denote ${}^*G$ its common parameter transformation group.

__Proposition:__ There exists a 1-1 correspondence between the set of all ${}^*G$ -symmetric first order connections and the set of all (over)generalized connections on ξ .

__Proof:__ If φ is a ${}^*G$ -symmetric first order connection, then its horizontal map h is 1-homogeneous (because of $\tau({}^*G)=\mathbb{R}-\{0\}$), smooth on the set of non-zero elements of $(pr\xi)^!\,\tau bs\xi$ (which set coincides with $(ad_{\tau bs\xi}\,pr\xi)^{-1}(d(\bigcup_{H\in\tilde{\mathcal{A}}_{bs\xi}} H))$) and satisfies condition $(pr\xi)^! dpr\xi\circ h = id_{(pr\xi)^!\tau bs\xi}$, so it determines an (over)generalized connection.

On the other hand, if h is an (over)generalized connection on ξ then we can associate to any (v,γ_p) ($v\in t\ell\xi$, $p=(pr\xi)(v)$) an element $\tilde{\gamma}_v\in G_v(t\ell\xi)$ in the following way. Let us consider the first order differential equation

$$\dot{\bar{\gamma}} = h(\bar{\gamma},\dot{\gamma})$$

($\bar{\gamma}$ is an unknown curve in $t\ell\xi$).

The map h is smooth, so this equation has a local solution $\bar{\gamma}$ which satisfies the initial condition $\bar{\gamma}(0)=v$. The curve germ of γ uniquely determines the curve germ of the solution. Denote this germ by $\varphi(v,\gamma_p)$. This map is ${}^*G$ -symmetric: if $\mu\in{}^*G$ and $\mu=(g)_0$, then $\bar{\gamma}\circ g$ is the solution of the equation

$$\dot{\bar{\bar{\gamma}}} = h(\bar{\bar{\gamma}}, \tfrac{d}{dt}(\gamma\circ g))$$

Indeed,

$$\lambda\cdot\dot{\bar{\gamma}} = \tfrac{d}{dt}(\bar{\gamma}\circ g) = h(\bar{\gamma},\tfrac{d}{dt}(\gamma\circ g)) = h(\bar{\gamma},\lambda\dot{\gamma}) = \lambda\, h(\bar{\gamma},\dot{\gamma})$$

where $\lambda=\tau(\mu)$.

We can easily check that if h corresponds to φ in this way, then h is the horizontal map of φ .

3.3 Classical Connections

We show how the notion of the "classical" connection fits into the frames of the just described theory.

Classical connection theory has several settings; we will follow the way of the splitting of the vector bundle morphism; for details one can see [5].

__Definition:__ We say that a *classical connection* is given on the vector bundle ξ if we have a smooth 1-homogeneous map

$$h : (pr\,\xi)^{!}\,\tau bs\,\xi \longrightarrow \tau t\ell\,\xi$$

for which the condition $((pr\,\xi)^{!}\,d\,pr\,\xi)\circ h = id_{(pr\,\xi)^{!}\tau bs\,\xi}$ is satisfied.

Lemma: The map h is linear on the fibers.

Proof: By the Euler's theorems for homogeneous maps every smooth 1-homogeneous map is linear.

Corollary: Every classical connection is an (over)generalized connection at the same time.

If we define the horizontal and vertical projections and the vertical and Dombrovski maps as we did in the general first order case, we get back the usual objects of the classical connection theory.

Now let $\tilde{\mathcal{S}}_{t\ell\xi}$ and $\tilde{\mathcal{S}}_{bs\xi}$ be the parametrization structures determined by the graph germs of $G(t\ell\xi)$ and $G(bs\xi)$. Recall that their (common) parameter transformation group is $G_0(\mathbb{R})$.

Proposition: There is a 1-1 correspondence between the set of classical connections on ξ and the set of the $G_0(\mathbb{R})$-symmetric connections.

Proof: The proof is similar to the proof of the Proposition of the previous paragraph, the only difference is that the map h will be smooth on the whole bundle $(pr\,\xi)^{!}\,\tau bs\,\xi$, so it is linear.

4.1 Global description of Rund's δ-derivation

In this part we will describe a possible application of the (over)generalized connection theory.

First we give some motivation: The classical connection theory is historically based on the parallel translation of the Riemannian geometry; its prototype is the Levi-Civita connection.

In the Finsler geometry – which is a generalization of the Riemannian case – several connections are used but there is no such a canonical connection as the Levi-Civita connection. We will show in a Lemma that the conceptual difference between the Riemannian metrics and the Finslerian one is very similar to the difference between the theory of the classical connections and the (over)generalized one: it depends on the domain of smoothness of some maps. After that we give a reinterpretation of Rund's δ-derivation from this point of view. (We choose this derivation because it is defined on the tangent bundle of the base manifold and not in a pull-back bundle of some vector bundles as connections in the Finsler geometry usually are).

All details are ignored here; the classical description of δ-derivation can be found in [4].

Definition: (1) The pair (M, K) is called a *Finsler space* if M is a smooth manifold and $K : TM \longrightarrow \mathbb{R}$ is a continuous map which is smooth on the submanifold of the non-zero tangent vectors and satisfies the following conditions:

(A) K is homogeneous of degree 2 on the fibers of τM.

(B) The quadratic form $(\partial_{ij} K)\, z^i z^j$ (where ∂_{ij} denotes the partial derivation along the fibers by indices i, j) is positive definite.

(2) The Finsler space (M, K) is called *Riemannian space* if K is quadratic on the fibers of τM.

Lemma: The Finsler space (M, K) is a Riemannian space iff K is smooth on the whole TM.

Proof: If (M, K) is a Riemannian space then K is quadratic, so it is smooth on the whole TM.

On the other hand, if $K : TM \to \mathbb{R}$ is a smooth function then by the Euler's theorems it is quadratic, so (M, K) is a Riemannian space.

Now we can give the global definition of the δ-derivation in terms of (over)generalized connections:

Definition: On the tangent bundle τM of the Finsler space (M, K) the (over)generalized connection $h : \mathscr{C}_M^! \tau M \longrightarrow \tau M$ is called (the Rund's) δ*-derivation* if (in natural coordinates) locally it has the form

$$h(x^1, y^1, x^2) = (x^1, y^1, x^2, P(x^1, x^2)(y^1)) \qquad x^1, x^2 \in \mathbb{R}^n, \quad n = \dim M$$

where the *i*th component $P^i(x^1, x^2)(y^1)$ of $P(x^1, x^2)(y^1)$ is

$$P^i(x^1, x^2)(y^1) = P^i_l(x^1, x^2)(y^1)^l = [\gamma^i_{lj}(x^1, x^2) - c^i_{ls}(x^1, x^2)\gamma^s_{pj}(x^1, x^2)(x^2)^p](x^2)^j$$

with the notations

$$c^i_{tl} = \frac{1}{4} g^{ik} \partial_{ktl} K \; ; \quad \gamma^i_{tj} = \frac{1}{2} g^{ik} \left(\frac{\partial g_{tk}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^t} - \frac{\partial g_{tj}}{\partial x^k} \right)$$

and

$$y^1 = ((y^1)^1, \ldots, (y^1)^n); \quad x^1 = ((x^1)^1, \ldots, (x^1)^n); \quad x^2 = ((x^2)^1, \ldots, (x^2)^n),$$

where (g^{ij}) is the inverse of the matrix $(g_{ij}) = (\frac{1}{2} \partial_{ij} K)$.

It can be easily checked that h is 1-homogeneous on the fibers of $\mathcal{B}_M^! \tau M$ and the covariant derivative associated to it is the same as the δ -derivative given in [4] (p. 55, formula 3.18).

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BELA KIS
INSTITUTUM MATHEMATICUM
UNIVERSITATIS DEBRECENIENSIS
H-4010 DEBRECEN PB. 12
HUNGARY