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SINGULARITIES IN THE GEOMETRY OF AN OBSTACLE *

Stanisław Janeczko

1. Introduction.

The obstacle problem has been extensively studied in the scattering theory of Lax, Phillips and Melrose [16] . They provide the complete asymptotic properties of a scattering amplitude and detailed spectral theorem for the Laplacian on the exterior domain with respect to the strictly convex obstacle. The classical theory of diffraction formulates the similar obstacle problem [13] [14] and defining the notion of diffracted ray provides the geometrical framework for the study of the optical properties of an arbitrary optical instrument [9] [14] . It appeared (cf. [4]) that the local geometry of an obstacle determines the singularities of systems of diffracted rays (the singularities of systems of rays by reflection were exhaustively studied in [7]). Moreover the local generic models for these singularities appeared to be isomorphic to the singular orbit spaces of the complexified actions of groups generated by reflections [4] [15]. The planar obstacle problem with an inflection point is governed by the group of icosahedron. Even more general point of view on the obstacle problem was proposed recently by R. Thom [18] [19] . In his theory of interaction of so-called salient forms and pregnancies, which is the basis of the notion of preprogramm (coming from the generalization of the organizing function of DNA molecules) the main problem is to find the singularities in an appropriate obstacle problem, i.e. to determine how a propagative flux of energy can be modified in its structure under a variation of the boundary constraints.

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* This paper is in final form and no version of it will be submitted for publication elsewere.

All the mentioned above examples and theories suggest the universal meaning of the obstacle problem and evoke a desire to find an unifying approach to study of its singularities. To this end we restrict our considerations to the geometrical theory of optics and provide the necessary symplectic approach (cf. [9] [10]). Using this approach we determine the generic singularities of systems of rays in the geometric theory of diffraction and derive their analytical properties.

2. Symplectic relations in the geometrical theory of optics.

As one of the possible manifestations of the obstacle problem we can consider the optical phenomenon of diffraction, or in more general setting the theory of transformation of systems of rays by the various optical instruments [13] [14] .

The geometrical optics [13] [16] gives the general framework for describing formally what happens when a ray hits an edge or a vertex or when a ray grazes on interface or a boundary. However before Arnold [3] nobody investigated sufficiently precisely the structure of singularities of diffracted rays on smooth (not necessarily convex) surfaces (obstacles) with generic properties. First, who turned attention on geometrical theory of diffraction and introduced the notion of diffracted ray was J. Keller [13] . He derived the exact formulas in the physical analysis of the few classical diffraction problems on the particular obstacles and partially for the general smooth objects. However still without any approach to investigation of singularities appearing there. In this section we formulate the diffraction theory using the language of symplectic geometry [9] [11] and the category of symplectic relations [17] . The singularities of systems of diffracted rays are reduced to the corresponding singularities of the appropriate lagrangian submanifolds [11] .

Let $V \cong \mathbb{R}^3$ be the configurational space (with refraction coefficient $n \equiv 1$) of geometrical optics. The associated phase space (M, ω) is given by the standard symplectic reduction (cf. [10] , [20])

$$\pi_M : H^{-1}(0) \longrightarrow M,$$

where the hypersurface $H^{-1}(0)$ is described by the Hamiltonian $H: T^*V \longrightarrow \mathbb{R}$, $H(p,q) = \frac{1}{2}(|p|^2 - 1)$.

Let (M, ω), $(\tilde{M}, \tilde{\omega})$ denote the symplectic manifolds of optical rays in homogeneous media, i.e. incident rays and transformed (say diffracted) rays respectively (cf. [9] [14]). In this paper, as an optical obstacle problem we understand the problem of indication of singularities of varieties of incident and transformed rays by the corresponding optical system.

DEFINITION 2.1. (A). The <u>phase space of an optical obstacle problem</u> is the following product symplectic manifold

$$P = (M \times \tilde{M};\ \pi_2^* \tilde{\omega} - \pi_1^* \omega),$$

where $\pi_{1,2}: M \times \tilde{M} \to M, \tilde{M}$ are canonical projections.

(B). The transformation process, say reflection, refraction or diffraction of the incident rays is governed by the corresponding lagrangian subvariety of P, called an <u>obstacle variety</u>.

As an optical obstacle problem we consider now the refraction on the sample of inhomogeneous optical medium. To modelize it we assume the following refraction coefficient in $\mathbb{R}^3$:

$$(*) \qquad \hat{n}(x,y,z) = \begin{cases} 1 & \text{for } z_2 \leqslant z \leqslant z_1 \\ n(x,y,z) & \text{for } z \in W, \end{cases}$$

where $W = \{(x,y,z) \in \mathbb{R}^3;\ z_1 < z < z_2,\ z_1 < z_2\}$ and n is a smooth function in the neighbourhood of $\overline{W}$. The configurational space $\{z < z_1\}$ we call the object space and the space $\{z > z_2\}$ the image space (cf. [14]). The corresponding spaces of light rays we denote by (M, ω) and $(\tilde{M}, \tilde{\omega})$ respectively. The optical instrument between $\{z = z_1\}$ and $\{z = z_2\}$ determines the transformation of the straight lines of the object space into the straight lines of the image space. To find this transformation in our case (*) we must look at the corresponding Riemannian geometry of it. The corresponding Riemannian metric g on W; $d\sigma^2 = n^2(dx^2+dy^2+dz^2)$. For g we have a flow X_g on the cotangent bundle (T^*W, ω_W). At first we define an "energy" function $H_g: T^*W \to \mathbb{R}$ by: $H_g(p) = \frac{1}{2}\langle p,p\rangle_g$, where $\langle \cdot\, ; \cdot\rangle_g$ is the inner product on T^*W induced by g. Finally the geodesic flow X_g on T^*W is the unique vectorfield such that $\omega_W(X_g, .) = -dH_g(.)$. By the obvious relation (cf.[1]) between g and X_g on T^*W, we obtain the respective geodesics on W, in fact if $l: [a,b] \to T^*W$ is an integral curve of X_g, then $\pi_W \circ l : [a,b] \to W$ is a geodesic on (W,g). Every geodesic on (W,g) can be obtained in this way. Thus this is the main reason for the symplectic description of systems of optical rays.

By the straightforward calculation, taking the optical length parameter

$$\sigma = \int_0^s n\,ds, \quad (ds^2 = dq_1^2+dq_2^2+dq_3^2)$$

along a geodesic, we obtain in W:

$$H_g(p,q) = \frac{1}{2}(\frac{1}{n^2}|p|^2 - 1)$$

and the corresponding Hamiltonian vector field X_g;

$$X_g = \sum_i \left(\frac{1}{n^2} p_i \frac{\partial}{\partial q_i} + \frac{|p|^2}{n^3} (\frac{\partial n}{\partial q_i}) \frac{\partial}{\partial p_i} \right).$$

Thus for the geodesics on W we obtain the following equation:

$$(**) \qquad \frac{d^2 q_i}{d\sigma^2} = \frac{1}{n}(\frac{\partial n}{\partial q_i}) \sum_j (\frac{dq_j}{d\sigma})^2 - \frac{2}{n}\frac{dq_i}{d\sigma}(\sum_j \frac{\partial n}{\partial q_j} \frac{dq_j}{d\sigma}) .$$

We see that the light rays in the object space (M,ω), which are coming to the plane $\Sigma_{z_1} = \{z=z_1\}$ form the four-dimensional open subset, say $\mathcal{O}_{z_1} \subset M$, of initial conditions for the equation (**). These initial conditions propagate symplectomorphically by the flow of X_g to the plane $\Sigma_{z_2} = \{z=z_2\}$ being considered as elements of the image space $(\tilde{M},\tilde{\omega})$ covering an open subset $\mathcal{O}_{z_2}$ of it. The pairs of light rays connected one to another one in this way form the <u>canonical variety</u> of rays associated to the optical instrument. We easily obtain the following

PROPOSITION 2.2. For a sufficiently small neighbourhood V of the constant function $n: W \to 1$ (in the C^∞-Whitney topology [18]), for each element of this neighbourhood, the corresponding canonical variety of rays forms a lagrangian submanifold of P. It is a graph of a symplectomorphism $\varphi: M \to \tilde{M}$ defined, at least, on a sufficiently small open subset of M (neighbourhood of a principal ray).

REMARK 2.3. The corresponding Darboux coordinates on M (and $\tilde{M}$ respectively) adapted to the above refraction problem, say $(r_1,r_2;s_1,s_2)$ with $\omega = dr_1 \wedge ds_1 + dr_2 \wedge ds_2$, are connected to the standard phase space coordinates (p,q) of $T^*\mathbb{R}^3$ by the corresponding symplectic reduction $\pi_M: T^*\mathbb{R}^3 \supset H^{-1}(0) = \{(|p|^2-1)=0\} \to M$,

$$\pi_M(p_2,p_3;q_1,q_2,q_3) = (p_2,p_3;q_2 - \frac{q_1 p_2}{\sqrt{1-p_2^2-p_3^2}}, q_3 - \frac{q_1 p_3}{\sqrt{1-p_2^2-p_3^2}}) = (r;s),$$

where we assumed in the respective subject and image spaces $n \equiv 1$. To each point $(r_1,r_2;s_1,s_2) \in M$ is uniquely associated the corresponding ray, say

$$(q_1,q_2,q_3) = (0,s_1,s_2) + t(1, \frac{r_1}{\sqrt{1-r_1^2-r_2^2}}, \frac{r_2}{\sqrt{1-r_1^2-r_2^2}}),$$

which allows us to translate the concrete optical problems into the language of the space (M,ω) (cf. [9], [10]).

3. Equivalence and classification of the canonical varieties.

The problem to which we are coming now is to indicate analytical properties of lagrangian submanifolds $L_g \subset P$, determined by the corresponding optical metrices defined by n. It seems that this problem has no simple localization, since it is not obvious that the local properties of L_g are determined only by the local properties of the function n. However, in more general setting of a quite arbitrary optical instrument, it is interesting to investigate the generic properties of L_g assuming that all of them are realisable. This suggests us the classification problem (cf. [18]) for smooth symplectic relations (cf. [10], [9]).

Now we formulate the problem. Let $L_1, L_2 \subset P = (T^*Q \times T^*\bar{Q}, \pi_2^*\omega_{\bar{Q}} - \pi_1^*\omega_Q)$ be two symplectic relations (lagrangian submanifolds of P) from T^*Q to $T^*\bar{Q}$. By K (and H) we denote the symplectic relation in $D_1 = (T^*Q \times T^*Q;\ \pi_2^*\omega_Q - \pi_1^*\omega_Q)$ (in $D_2 = (T^*\bar{Q} \times T^*\bar{Q};\ \pi_2^*\omega_{\bar{Q}} - \pi_1^*\omega_{\bar{Q}})$) corresponding to a symplectomorphism Φ (Ψ) of T^*Q (and $T^*\bar{Q}$ respectively).

DEFINITION 3.1. Let (L_1,p_1), (L_2,p_2) be two germs of symplectic relations in P. We say that (L_1,p_1), (L_2,p_2) are equivalent iff there exist the corresponding germs of symplectic relations (K,v), (H,w) (defining for the symplectomorphisms) such that:

(i) $$L_1 = K \circ L_2 \circ H,$$

and

$$\pi_1(p_1) = \pi_2^{(Q)}(v), \quad \pi_2(p_2) = \pi_1^{(\bar{Q})}(w),$$

where $\pi_i^{(Q)}$ and $\pi_i^{(\bar{Q})}$ are the canonical projections in $T^*Q \times T^*Q$ and $T^*\bar{Q} \times T^*\bar{Q}$ respectively.

Let $F:(Q \times \bar{Q} \times \Lambda, 0) \to \mathbb{R}$ be a Morse family for symplectic relation $(L,0) \subset P$ (cf. [20]). We consider the following germ $\mathcal{F}: (Q \times \bar{Q} \times \Lambda, 0) \to \mathcal{E}_{\bar{q},\lambda}$, $\mathcal{F}(q_1,\bar{q}_1,\lambda_1) = (F(q_1,\bar{q}+\bar{q}_1,\lambda+\lambda_1),0) \in \mathcal{E}_{\bar{q},\lambda}$ where $\mathcal{E}_{\bar{q},\lambda}$ denotes the ring of germs at zero of smooth functions depending on $\bar{q}$ and λ (cf. [8]).

PROPOSITION 3.2. Let $\mathcal{F}$ be transversal in the source point 0 of the germ $(\mathcal{F},0)$ to the ideal $\mathfrak{m}^2_{\bar{q},\lambda}$ ($\mathfrak{m}_{\bar{q},\lambda}$ is the maximal ideal in $\mathcal{E}_{\bar{q},\lambda}$). Then the corresponding germ of symplectic relation (L,0) is equivalent to this one generated by the following Morse family

(ii) $$G(q,\bar{q},\lambda) = \sum_{i=1}^{\dim Q} \lambda_i(\bar{q}_i - q_i)$$

representing the identity symplectomorphism of T^*Q into $T^*\bar{Q}$.

Proof. By the straightforward calculations it is easily seen that the assumed transversality condition is sufficient for (L,0) to be the germ of symplectic relation representing a symplectomorphism $T^{*}Q \longrightarrow T^{*}\bar{Q}$ (cf. [1] , [17]). One can take K and H just represented by the inverse symplectomorphism and identity. Thus we obtain the normal form (ii).

We see (by the transversality condition) that the class of symplectic relations corresponding to symplectomorphisms is stable (cf. [8] , [18]). The further analysis of the equivalence classes for the symplectic relations can be conducted by the equivalent notion of Morse families.

Let $\varkappa : Q \times Q \times N \longrightarrow \mathbb{R}$, $\eta : \bar{Q} \times \bar{Q} \times M \longrightarrow \mathbb{R}$ be Morse families corresponding to K and H respectively. Then using the properties of composition of symplectic relations (cf. [17] , [20]), we obtain:

PROPOSITION 3.3. Morse family for the symplectic relation given by the composition (i) is following,

$$G: Q \times \bar{Q} \times W \longrightarrow \mathbb{R},$$

$$G(q,\bar{q},w) = \varkappa(q,q',\nu) + F(q',\bar{q}',\lambda) + \quad (\bar{q}',\bar{q},\mu),$$

where $w=(q',\nu,\bar{q}',\lambda,\mu)$ is the Morse family parameter.

Thus the symplectic relation $(\tilde{L},0) \subset P$ with the Morse family $\tilde{F}: Q \times \bar{Q} \times \tilde{W} \longrightarrow \mathbb{R}$ is equivalent to (L,0) iff there exists diffeomorphism R such that the following diagram commutes

$$\begin{array}{ccc} Q \times \bar{Q} \times W & \xrightarrow{R} & Q \times \bar{Q} \times W \\ & \searrow^{\varkappa} \quad \swarrow_{\varkappa} & \\ & Q \times \bar{Q} & \end{array}$$

and

$$G \circ R = \tilde{F} + q,$$

where q is a nondegenerate quadratic form of remaining Morse parameters (cf. [21]). By the above mentioned reduction of Morse parameters one can reduce G to obtain Morse family with $w \in \mathbb{R}^{\dim Q + \dim \bar{Q}}$ (cf. [20]).

Infinitesemal properties of the orbits of that equivalence relation differ from the standard one in singularity theory of mappings. Because of the product structure of P, preserved by the equivalences the problem of classification of the normal or prenormal forms seems to be quite difficult and contain the complicated functional modulus. However some generic local properties of symplectic relations implied by the concrete physical models can be established. We leave it to the forthcoming paper. Some remarks concerning of this problem can be found also in [10] .

4. Singularities in systems of diffracted rays.

Diffracted rays are produced, for example, when an incident ray hits an edge of an impenetrable screen. In this case the incident ray produces infinitely many diffracted rays (see [13]), which make the same angle with the edge as does the incident ray.

Diffraction on the point aperture in a plane screen, according to the preceeding framework is represented by the following symplectic relation (canonical variety)

$$M \times \tilde{M} \supset L = \{(r,s;\tilde{r},\tilde{s});\ s_1 = s_2 = \tilde{s}_1 = \tilde{s}_2 = 0\}.$$

Hence the diffracted rays corresponding to the point source rays N in (M,ω) are given as a symplectic image (cf. [10], [20], [9]),

$$L(N) = \{(\tilde{r},\tilde{s});\ \tilde{s}_1 = \tilde{s}_2 = 0\} \subset (\tilde{M},\tilde{\omega}).$$

It is easy to verify that the canonical variety corresponding to diffraction on the straight edge of a thin screen $\{(q_1,q_2,q_3);\ q_1 = 0,\ q_3 \geqslant 0\}$ in $\mathbb{R}^3$ is given by the following Morse family:

$$G(s_1,s_2,\tilde{s}_1,\tilde{s}_2,\lambda) = \lambda_1 s_2 + \lambda_2 \tilde{s}_2 + \lambda_3(s_1 - \tilde{s}_1).$$

It contains all cones of diffracted rays produced by incident rays in general position according to the edge (cf. [13], Fig. 12). We recall the symplectic structure on $M \times \tilde{M}$, namely,

$$\Omega = \sum_i (d\tilde{r}_i \wedge d\tilde{s}_i - dr_i \wedge ds_i), \quad (i=1,2)$$

Thus the lagrangian submanifold of diffracted rays produced by the rays normally incident on the edge (i.e. the source at infinity) is governed by

$$F(\tilde{s}_1,\tilde{s}_2,\lambda) = \lambda \tilde{s}_2.$$

Taking the point source beam of incident rays in general position $(a,b,c) \in \mathbb{R}^3$ we derive the following generating family for diffracted beam:

$$F(\tilde{s}_1,\tilde{s}_2,\lambda) = \lambda_1\lambda_6 - a\sqrt{1-\lambda_4^2-\lambda_5^2} + (\lambda_7 - b)\lambda_4 + (\lambda_6 - c)\lambda_5 +$$

$$+ \lambda_2 \tilde{s}_2 + \lambda_3(\lambda_7 - \tilde{s}_1).$$

We can generalize the problem to consider an aperture in a plane screen to be a generic smooth curve. In this case, on the basis of the beautiful paper [6] we obtain,

PROPOSITION 4.1. For the generic shape of an aperture in a plane screen. The wave-front evolution by diffraction of a normally incident ray beam in the neighbourhood of each ray going through an edge is described by the following generating family:

$$\psi(q,t;\lambda_1,\lambda_2) = \lambda_1 w_1(\lambda_2) + q_1\lambda_1^2(1 + w_2^2(\lambda_2)) + q_2\lambda_1 - q_3\lambda_1 w_2(\lambda_2) - t,$$

where $w_1(\lambda_2) = w(\lambda_2) - \lambda_2 w'(\lambda_2)$, $w_2(\lambda_2) = w'(\lambda_2)$, $w(\lambda_2) = a_2\lambda_2^2 +$

$+ a_3\lambda_2^3 + O(\lambda_2^4)$ is a smooth function defining the Taylor map corresponding to the aperture curve.

Proof. Let $f(s_1,s_2)=0$ be a defining equation for an aperture curve. Thus a generating family for the canonical variety $L\subset P$,

$G(s_1,s_2,\tilde{s}_1,\tilde{s}_2,\lambda)=\lambda_1 f(s_1,s_2)+\lambda_2 f(\tilde{s}_1,\tilde{s}_2)+\lambda_3(s_1-\tilde{s}_1)+\lambda_4(s_2-\tilde{s}_2)$. The beam N at infinity in (M,ω) is generated by

$$F(s) \equiv 0.$$

Hence $L(N)$ is generated by the family:

$$\varphi(\tilde{s}_1,\tilde{s}_2,\lambda) = \lambda f(\tilde{s}_1,\tilde{s}_2),$$

and the corresponding diffracted lagrangian variety in $T^*\mathbb{R}^3$ is generated by

$$\bar{\psi}(q;\alpha)=\lambda f(\mu_1,\mu_2)+\mu_1\lambda_1+\mu_2\lambda_2+q_2\lambda_1+q_3\lambda_2+q_1\sqrt{1-\lambda_1^2-\lambda_2^2},$$

where $\alpha=(\lambda,\lambda_1,\lambda_2,\mu_1,\mu_2)$.

On the basis of [6], taking the local representation of f, and reducing the parameters of the family $\bar{\psi}$, we obtain the desired result.

EXAMPLE 4.2. Let us take $f(\mu_1,\mu_2)=\mu_1-\mu_2^2$, then we obtain the generating family for the diffracted variety in $T^*\mathbb{R}^3$;

$$\hat{\psi}(q,\lambda_1,\lambda_2)= q_1\lambda_1^2(1+4\lambda_2^2)+q_2\lambda_2-2\lambda_1\lambda_2 q_3-\lambda_1\lambda_2^2.$$

We see that it is not stable in the standard sense [21] and not longer a differentiable submanifold of $T^*\mathbb{R}^3$. An analysis of the generic properties of these varieties in the neighbourhood of the ordinary and inflection points of the aperture boundary (cf. [6], [13]) we leave to the forthcoming paper.

Now we can adapt an introduced symplectic framework to describe the diffraction problem on a smooth closed obstacle. The problem is substantially connected to the Riemannian obstacle problem (cf. [2], [4]), i.e. determination of geodesics on a Riemannian manifold with smooth boundary. Any geodesic on such manifold is C^1 and consists generically finitely many so-called switchpoints where geodesic has an initial or end point according to lie in interior part of the manifold or on the boundary. Cauchy uniqueness for manifolds with boundary states that every boundary point (point of an obstacle) has a neighbourhood in which: if two geodesic segments with the same initial point, initial tangent vector and length do not coincide, then one of them has its right endpoint in the interior part of the manifold and is an involutive of the other (it the case of the plane it lies on an appropriate involute of the obstacle curve [13]). By an involutive of a geodesic γ is meant a geodesic which has the same initial point, initial tangent vector and length as .

Let us consider an open subset S of an obstacle surface in $\mathbb{R}^3$. By l_1 we denote the initial tangent line to the geodesic segment γ on S. Let l_2 be a tangent line to S. We say that l_2 is subordinate to l_1 with respect to an obstacle S if l_2 (or its piece in $(\mathbb{R}^3,S)$) belongs to the geodesic segment with the same initial point and the same tangent vector as γ has.

PROPOSITION 4.3. The canonical variety

$$L=\{(l,\tilde{l})\in P;\ \tilde{l} \text{ is subordinate to } l \text{ with respect to an obstacle } S\subset\mathbb{R}^3\}$$

is a lagrangian subvariety of P.

Proof. At each point p of S we can choose the geodesic polar coordinates

$$\varphi :]0,\varrho[\times\mathbb{R}\longrightarrow \mathcal{O}_p\subset S,$$
$$\varphi(r,\theta)=\exp_p(r\cos\theta\, e_1(p)+r\sin\theta\, e_2(p)),$$

where $\{e_1(p),e_2(p)\}$ is an orthonormal basis of T_pS and $\exp_p$ is the corresponding exponential map (cf.[1]). Let (l_θ,p) denotes a line tangent to the associated geodesic starting at p. We have

$$u(\theta,r,p) = ((l_\theta,p);((\tfrac{\partial}{\partial r}\exp_p(r\cos\theta\, e_1(p)+r\sin\theta\, e_2(p)))\big|_{(r,\theta)},$$
$$\exp_p(r\cos\theta\, e_1(p)+r\sin\theta\, e_2(p)))\in L\subset M\times\tilde{M},$$

and the mapping

$$\mathbb{R}^4 \ni (r,\theta,p)\xrightarrow{U} u(r,\theta,p)\in P$$

is a lagrangian immersion, i.e.

$$U^*(\tilde{\omega}\ominus\omega) = 0.$$

Let N_p denotes the lagrangian submanifold of rays in (M,ω) starting at the same point p of the object space.

PROPOSITION 4.4. For the generic position of $p\in\mathbb{R}^3$ and the generic obstacle surface S the only possible germs of systems of diffracted rays $L(N)\subset(\tilde{M},\tilde{\omega})$ are symplectomorphic to these ones generated by the following generating families (not necessary Morse families[10]):

1. Smooth case

$$F(\tilde{s}_1,\tilde{s}_2)\equiv 0.$$

2. The wing singularity

$$F(\tilde{s}_1,\tilde{s}_2,\lambda) = \frac{1}{40}\lambda^5+\frac{1}{6}\lambda^3 s_1+\frac{1}{2}\lambda s_1^2.$$

3. The open swallowtail singularity

$$F(\tilde{s}_1,\tilde{s}_2,\lambda)=\frac{1}{576}\lambda^7+\frac{1}{30}\lambda^5\tilde{s}_1+\frac{1}{24}\lambda^4\tilde{s}_2+\frac{1}{6}\lambda^3\tilde{s}_1^2+\frac{1}{2}\lambda^2\tilde{s}_1\tilde{s}_2+\frac{1}{2}\lambda\tilde{s}_2^2.$$

Proof of this proposition follows immediately on the basis of Proposition 4.2 in [10] and the general methods of [4].

Now we can describe explicitely the generic diffracted wave--front evolution in the presence of a smooth obstacle in $\mathbb{R}^3$;

COROLLARY 4.5. For the generic obstacle in Euclidean three-space the only singular models of surface-diffracted wave-fronts are given by the following phase families (cf. [12]):

1. (inflection points - geodesic has an asymptotic direction).

$$\varphi(q_1,q_2,q_3,\lambda_1,\lambda_2,\lambda_3) = -\frac{1}{40}\lambda_3^5-\frac{1}{6}\lambda_3^3\lambda_1-\frac{1}{2}\lambda_3\lambda_1^2+q_1\sqrt{1-\lambda_1^2-\lambda_2^2}+q_2\lambda_1+q_3\lambda_2.$$

2. (biasymptotic points - geodesic is tangent to a line of asymptotic points).

$$\varphi(q_1,q_2,q_3,\lambda_1,\lambda_2,\lambda_3) = -\frac{1}{576}\lambda_3^7-\frac{1}{30}\lambda_3^5\lambda_1-\frac{1}{24}\lambda_3^4\lambda_2-\frac{1}{6}\lambda_3^3\lambda_1^2-\frac{1}{2}\lambda_3^2\lambda_1\lambda_2-$$
$$-\frac{1}{2}\lambda_3\lambda_2^2+q_2\lambda_1+q_3\lambda_2+q_1\sqrt{1-\lambda_1^2-\lambda_2^2}.$$

REMARK 4.6. The wave-front evolution "1." in the above corollary ane can obtain also by the Legendre transform introduced in the singularity theory of functions on manifolds with singular boundary [15], [5]. Let $(T^*\mathbb{C}^n-\{0\}, \mathbb{C}^n, \pi_{\mathbb{C}^n}, \vartheta_{\mathbb{C}^n})$ and $(T^*\mathbb{C}^n-\{0\}, M^n, \tilde{\pi}, \vartheta)$ be two special symplectic structures on $(T^*\mathbb{C}^n, \pi_{\mathbb{C}^n})$. Here M^n denotes the manifold of hyperplanes in $\mathbb{C}^n$ and $\tilde{\pi}$: $T^*\mathbb{C}^n-\{0\} \to M^n$ is the canonical lagrangian fibration associated to it. This fibration induces a symplectomorphism $\alpha: T^*\mathbb{C}^n-\{0\} \to T^*M^n$ (and its projectivisation $\tilde{\alpha}: PT^*\mathbb{C}^n \to PT^*M^n$) such that $\tilde{\pi} = \pi_{M^n}\circ\alpha$ and $\vartheta = \alpha^*\vartheta_{M^n}$. In local map α is given in the following way:

$$\alpha(p_1,\dots,p_n;x_1,\dots,x_n)=(-x_2p_1,\dots,-x_np_1,p_1;\frac{p_2}{p_1},\dots,\frac{p_n}{p_1},x_1+x_2\frac{p_2}{p_1}+$$
$$+\dots+x_n\frac{p_n}{p_1}),$$

where the hyperplanes in $\mathbb{C}^n$ are parametrized in the following way

$$p_1x_1+p_2x_2+\dots+p_nx_n-p_1\tilde{x}_1-p_2\tilde{x}_2-\dots-p_n\tilde{x}_n = 0$$

Let $V = \{z\in\mathbb{C}^n;\ h(z) = 0\}$ be a complex hypersurface (boundary) with isolated critical zero. We consider the lagrangian variety (cf. [11])

$$\tilde{V} = T^*_{V-\{0\}}\mathbb{C}^n \subset T^*\mathbb{C}^n.$$

It is obvious that $\alpha(\tilde{V})$ is a costrained lagrangian subvariety of T^*M^n with constraint

$$\mathcal{L}(V) = (\pi_{M^n}\circ\alpha)(V)$$

called a Legendre transform of V (cf. [15]). It appears thet the singularities of the wave-front evolution in the presence of an obstacle are diffeomorphic to the appropriate Legendre transforms of the singular boundaries with isolated critical points of type A_k. Their connection to the Coxeter groups generated by reflections is settled in [10].

REMARK 4.7. It appears (cf. [12],[4]) that the singularities in the obstacle geometry can be represented in the symplectic space of binary forms derived in [12]. Using the invariant theory of binary forms the three-dimensional case, described in [4] was generalized in [12], where the analytical structure of the generalized open swallowtails was indicated and the standard reduction procedure was substantially extended. Let us recall briefly these results.

We say that two binary forms $f(x,y)$, $g(x,y)$ are apolar if their apolar covariant (see [12]) $\langle f|g\rangle$ is identically zero form. If f, g are written umbrally; $f = \langle U|[\alpha\, u]^n\rangle$, $g = \langle U|[\beta\, u]^m\rangle$, say $m \leqslant n$ then the corresponding apolar covariant $\langle f|g\rangle$ is the binary form of degree $n-m$ defined umbrally by

$$\langle f|g\rangle = \langle U|[\alpha\ \beta]^m [\alpha\, u]^{n-m}\rangle$$

Let (M^{n+1}, ω) be the unique symplectic space of binary forms (derived in [12]). The canonical subspaces in M^{n+1}, say $C^{(1)}$, $0 \leqslant 1 \leqslant \frac{n-1}{2}$, of all binary forms apolar to its 1-derivatives with respect to x are called the canonical apolar subspaces. They form the coisotropic varieties in (M^{n+1}, ω), (cf. [1]). To the space of binary forms of degree n one can associate the corresponding space of polynomials of one variable putting $y=1$. In order to have the polynomial symplectic spaces adapted to the investigations of singularities in the variational obstacle problem we associate to every symplectic space (M^{n+1}, ω) the canonically reduced symplectic space Q^{n-1} of polynomials of degree $n-1$, where leading term has constant coefficient $1/(n-1)!$. $Q^{n-1} = C_o/\sim$, where "$\sim$" is given by the coisotropic submanifold $C_o = \{ f \in M^{n+1};\ n!\,a_n = 1\}$, $(a_i)_0^n$ are coefficients of binary forms. Q^{n-1} is identified canonically with the space of derivatives $\frac{d}{dx}(f(x,1))$, $f \in M^{n+1}$ belonging to C_o, namely

$$Q^{n-1} \ni \varphi(x) = \frac{x^{2k+2}}{(2k+2)!} + q_1\frac{x^{2k+1}}{(2k+1)!} + \dots + q_{k+1}\frac{x^{k+1}}{(k+1)!} - p_{k+1}\frac{x^k}{k!} + \dots + (-1)^{k+1}p_1$$

endowed with the reduced symplectic structure

$$\omega' = \sum_{j=1}^{k+1} dp_j \wedge dq_j .$$

THEOREM (12). The apolar subspaces $C^{(l)}$, $l=1,\dots,\frac{n-1}{2}$, of (M^{n+1},ω) induce the corresponding coisotropic subspaces of (Q^{n-1},ω'), say $\tilde{C}^{(l)}$, $\tilde{C}^{(l)}=\{\varphi\in Q^{n-1};\ \tilde{P}_s^{(l)}(q,p)=0,\ s=1,\dots,l\}$, $l=1,\dots,k+1$, where

$$\tilde{P}_s^{(l)}=\frac{(-1)^k}{n!}\sum_{i=1}^{k-s+1}\binom{n-1}{i}\binom{n}{i}^{-1}q_ip_i+\frac{1}{n!^2}\sum_{i=k-s+2}^{k+1}(-1)^iq_iq_{n-s-i}$$

$$\binom{n-1}{i}i!(n-s-i)!+\frac{(-1)^k}{n!^2}\sum_{i=k+2}^{n-1}\binom{n-1}{i}i!(n-s-i)!p_{n-i}q_{n-s-i}+a_s.$$

Hence we have that the reduced symplectic space corresponding to the homogeneous system $(Q^{n-1},\omega',\tilde{C}^{(l)})$ is identified with the following space of polynomials

$$Z=\left\{\frac{x^{2k+1}}{(2k+1)!}+q_1\frac{x^{2k-1}}{(2k-1)!}+\dots+q_k\frac{x^k}{k!}-p_k\frac{x^{k-1}}{(k-1)!}+\dots+(-1)^kp_1\right\}$$

endowed with the reduced symplectic structure

$$\bar{\omega}=\sum_{i=1}^{k}dp_i\wedge dq_i.$$

The study of Hilbert's zero-forms and connected with them the corresponding spaces of polynomials with root having a prescribed multiplicity, provides the following result (see [12]):

THEOREM ([12]). Let $m\geqslant\left[\frac{n}{2}\right]$. Then the set of polynomials of Z having a root of multiplicity m, say $L^{(n)}_{m-1}$, form the isotropic varieties in $(Z,\bar{\omega})$. The maximal isotropic variety, i.e. for $m=\left[\frac{n}{2}\right]$, is a lagrangian variety symplectomorphic, in the case of $n=7$, to the system of rays on an obstacle, with the highest generic singularity, so-called open swallowtail singularity.

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