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CLIFFORD AND GRASSMANN LIKE ALGEBRAS

A.K. Kwaśniewski

This paper is in final form and no version of it will be submitted for publication elsewhere.

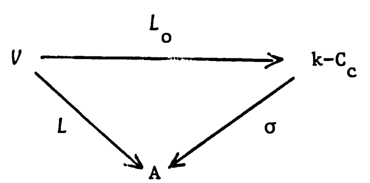
I. Introduction

The main purpose of this talk is to draw an attention to new, recently discovered applications of generalized Clifford algebras and also to various generalizations of Clifford and Grassmann algebras on their own.

There exist at least two natural generalizations of Clifford algebra concept and - correspondingly - of Grassmann one.

The first generalization is based on the observation that Clifford algebras $C_n^{(2)}$ form a very special case of an algebra extension of the group $\Gamma = Z_2 \oplus Z_2 \oplus \ldots \oplus Z_2$ (n summands). Hence, algebra extensions C_δ of any finite, abelian group might be considered as generalization of $C_n^{(2)}$ algebra with an appropriately associated Grassmann like algebras G_ρ .

The second generalization of $C_n^{(2)}$ into k-C_n is to consider the following commutative diagram with arbitrary, natural k instead of k=2 only:



(1.1)

where V is an n-dimensional vector space, k-C_n and A are associative algebras, and L, L_o are corresponding monomorphisms with the property

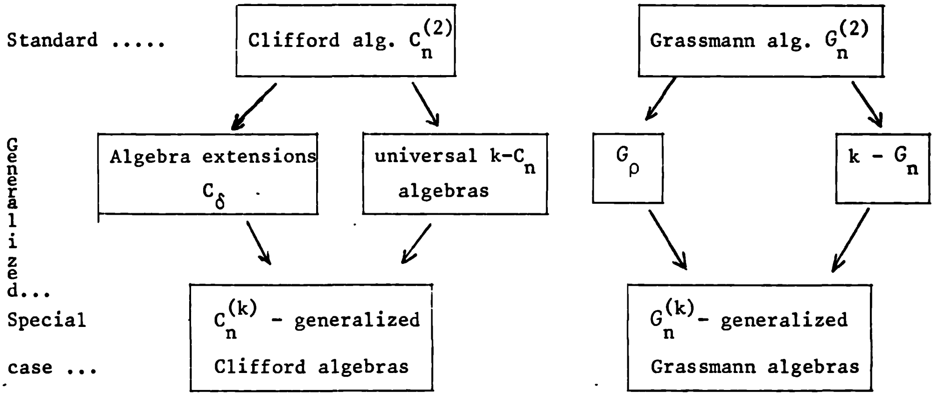
$$L_o(x)^k = Q_k(x)\,\mathbb{I}$$

$$L(x)^k = Q_k(x)\,\mathbb{I}$$

while $\sigma \in \mathrm{Hom}(k\text{-}C_n, A)$.

For k=2, Q_2 is a quadratic form, for k=3 - cubic, etc.

The generalizations of algebras to be considered in this paper are represented by the scheme:



Grassmann like algebras are associated to the corresponding algebras from the Clifford algebra part of the scheme and k-C_n, G_ρ, k-G_n algebras seem to be new.
At first we establishe basic relations between the algebras of this scheme and then discusses in detail two new applications of some of them, while listing several others applications to mathematical physics.
Our talk is organized as follows.
For the reader's convenience we deliver at the beginning a rather detailed information on C_δ algebras following [22,6].

Then we introduce all other algebras from the above scheme. The application to be considered in the first part of the talk is an explicit construction of Lie Γ graded algebras [19,6].
The second application is the subject of the separate paper which follows this one. There , under assumption of uniqueness of the existing curves, we find critical curves for all nonstandard, planar Potts models due to extensive use of $C_n^{(k)}$ algebra properties.

Before we proceed, let us briefly sketch, what is known to us about the already existing applications of generalized and C_δ Clifford like algebras.
The importance of usual Clifford algebras for mathematical physics is nowadays out of question and their use is inevitable.
Here we want to indicate that the field of applicability of their generalizations is rather vast and also contains various branches of physics.
Generalizations of Clifford algebras were introduced quite independently by the authors of [11,12,15]. In the present paper we derive, in a canonical way, an ultimate generalization of Clifford algebras so that the precedent $C_n^{(k)}$ generalized Clifford algebras serve as an epimorphic image of the ones introduced by the author.

These new k-C_n Clifford-like algebras are naturally of primary importance for those algebraic problems of physics in which universality of the arising algebra is crucial.

In parallel, we introduce the corresponding Grassmann-like algebras, which are expected to play a similar role with respect to Clifford-like algebras as the Grassmann ones with respect to the usual Clifford algebras.

First applications of generalized Clifford algebras date from late sixties [16], then followed by another ones in another branch of physics [17]. $C_n^{(k)}$ algebras were also shown to play decisive role in solving problems of general involutional transformations [18].

Quite recently a remarkable application of these algebras was found while constructing and classifying the so called ε-Lie Γ graded algebras [19,15], hence equivalently - an application to modular quantization (see [4] and ref.therein).

It is already well known that algebra extensions (including $C_n^{(k)}$ algebras) of finite groups as well as generalized Dirac groups form an excellent tool for deriving and classifying projective representations of finite groups; a subject of primary importance for quantum theory of crystals. For that applications see the review [9].

Let us also mention some other applications. Namely the generalized Pauli algebra and corresponding Dirac group were used in [13] to solve an inverse problem for harmonic vibrations of cyclic molecules not restricted to the closest neighbour approximation.
It is also to be noted that generalized Clifford and Grassmann algebras are of potential importance for generalizations of the Ising model as the Onsager formula for partition function in the case of Ising model [21] can be derived from algebraic properties of Clifford algebras only. Hence our k-C_n algebras seem to be natural tool for investigating Potts model with multisite interactions [20]. It is to be noted also that $C_2^{(k)}$ appears as typical algebra of quantum models with Z_k symmetry and its role in deriving conservations laws in quantum Potts models is decisive [1]. We close this incomplete list of applications by two remerks. Namely, the ultrageneralized complex analysis of [8] should make use of k-C_n algebras as these are the most general object linearizing the equation that defines this ultraanalyticity.

Finally it is to be noted that the notion of "generalized numbers" used recently by the authors of [14] to study Fermi-Bose symmetry with reference to Klein transformations - is a very special case of our G_ρ-Grassmann like algebras introduced in [6] and also presented here.

II. Preliminaries

The generalized Clifford algebra is a special case of algebra extension of Γ over $\mathbb{C}$ where (22.II).

Def.

An algebra C is an algebra extension of Γ over $\mathbb{C}$ iff

1) $C = \bigoplus_{\vec{\alpha}\in\Gamma} C_{\vec{\alpha}}$ (is Γ graded) 2) $\dim C_{\vec{\alpha}} = 1$, $C_{\vec{\alpha}}C_{\vec{\beta}} = C_{\vec{\alpha}+\vec{\beta}}$ $\vec{\alpha},\vec{\beta}\in\Gamma$

It is known [22.I] that there exists a bijective correspondence between isomorphic classes of algebra extensions of Γ over $\mathbb{C}$ and cohomology classes of $H^{(2)}(\Gamma,\mathbb{C}^*) \simeq$ $\simeq P_{a.s.}(\Gamma,\mathbb{C}^*)$.
$P_{a.s.}(\Gamma,\mathbb{C}^*)$ denotes the group of all antisymmetric pairings i.e. mappings $\delta:\Gamma\times\Gamma\to\mathbb{C}^*$ which are 1) bimorphisms and 2) $\delta(\vec{\alpha},\vec{\alpha})=1$, $\vec{\alpha}\in\Gamma$.
As $H^2(\Gamma,\mathbb{C}^*)\simeq Z_k\oplus \dots \oplus Z_k$ ($n(n-1)/2$ summands) we have $k^{n(n-1)/2}$ different algebra extensions C_δ ($\delta\in P_{a.s.}(\Gamma,\mathbb{C}^*)$) of Γ over $\mathbb{C}$. C_δ can be thought of as the algebra generated by generators $\gamma_1,\dots,\gamma_n$ satisfying

$$\gamma_i\gamma_j = \omega_{ij}\gamma_j\gamma_i \quad , \qquad \gamma_i^k = 1 \qquad i,j=1,\dots,n$$

where $\omega_{ij}=\delta(s_i,s_j)$ while $\{s_i\}_{i=1}^n$ are generators of Γ and $\delta\in P_{a.s.}(\Gamma,\mathbb{C})$. Because δ is an antisymmetric pairing, $\omega_{ij}=\omega^{\alpha_{ij}}$, where $\alpha_{ij}\in Z_k$, ω is primitive k-th root of unity and this $(\alpha_{ij})=(n\times n)$ matrix is antisymmetric in the sense of Z_k ring. Note that the additive group of these (α_{ij}) matrices is isomorphic to $H^2(\Gamma,\mathbb{C}^*)$.
For any $\delta\in P_{a.s.}(\Gamma,\mathbb{C}^*)$ we then have

$$\delta(\vec{\alpha},\vec{\beta}) = \omega^{\langle\vec{\alpha}|A\vec{\beta}\rangle} \quad ; \qquad \vec{\alpha},\vec{\beta}\in\Gamma$$

where $A=(\alpha_{ij})$ matrix and $\langle\vec{\alpha}|\vec{\beta}\rangle = \sum_{i=1}^n \alpha_i\beta_i$; $\alpha_i,\beta_i\in Z_k$.
The special choice of A, namely $\alpha_{ij}=1$ for $i<j$ gives $C_n^{(k)}$ generalized Clifford algebras.
These C_δ Clifford like algebras were applied to construct and classify [19,5] ε-Lie Γ graded algebras defined below of potential importance for physics.

III. GRASSMANN-LIKE ALGEBRAS G_δ

C_δ Clifford like algebras provide an example of δ Lie Γ graded commutative algebras [19,5] and this point of view enables one to introduce Grassmann like algebras along the same lines.
For completeness we start with necessary definitions.
Def.

$\varepsilon: \Gamma\times\Gamma \to \mathbb{C}^*$ is said to be a commutation factor iff
1) ε is a bimorphism 2) $\varepsilon(\vec{\alpha},\vec{\beta})\,\varepsilon(\vec{\beta},\vec{\alpha}) = 1$ $\vec{\alpha},\vec{\beta}\in\Gamma$.
∎

In the case of Γ admitting a Γ_o subgroup of index 2 one may define:

Def.

Let $\Gamma = \Gamma_o \cup \Gamma_1$, ε_o is said to be the Grassmann commutation factor if

$$\varepsilon_o : \Gamma\times\Gamma \to \mathbb{C}^* , \quad \varepsilon_o(\vec{\alpha},\vec{\beta}) = \begin{cases} -1; & \vec{\alpha},\vec{\beta}\in\Gamma_1 \\ 1; & \text{otherwise} \end{cases}$$

∎

The group of commutation factors is given by either $P_{a.s.}(\Gamma,\mathbb{C}^*)\cup\varepsilon_o P_{a.s.}(\Gamma,\mathbb{C}^*)$ or $P_{a.s.}(\Gamma,\mathbb{C}^*)$ depending on whether Γ admits Γ_o subgroup of index 2 or not.
The ε-Lie Γ graded algebra is then defined as follows:

Def.

Let L be Γ graded vector space equipped with bilinear mapping $<\ ,\ >:L\times L \to L$. Let $x_{\vec{\alpha}}$, $y_{\vec{\beta}}$, $z_{\vec{\delta}}\in L$, $\vec{\alpha},\vec{\beta},\vec{\gamma}\in\Gamma$, denote homogeneous elements. L is then said to be ε Lie Γ graded algebra iff it is Γ graded algebra under $<\ ,\ >$ multiplication and

1) $\vec{\alpha},\vec{\beta}\in\Gamma$; $<x_{\vec{\alpha}}, y_{\vec{\beta}}> = -\,\varepsilon(\vec{\alpha},\vec{\beta})\,<y_{\vec{\beta}}, x_{\vec{\alpha}}>$ (ε skew symmetric)

2) $\vec{\alpha},\vec{\beta},\vec{\gamma}\in\Gamma$ $\quad <x_{\vec{\alpha}},<y_{\vec{\beta}},z_{\vec{\gamma}}>> = <<x_{\vec{\alpha}},y_{\vec{\beta}}>,z_{\vec{\gamma}}> + \varepsilon(\vec{\alpha},\vec{\beta})<y_{\vec{\beta}},<x_{\vec{\alpha}},z_{\vec{\gamma}}>>$

(ε Jacobi identity) where ε is a commutation factor.

∎

Still one more definition is necessary.

Def.

Let U be associative Γ graded algebra. ass. U is said to be ε Lie Γ graded algebra associated to U iff

1) ass.U = U as Γ graded vector spaces and

2) $<\ ,\ >$: ass.U $\times$ ass.U $\to$ ass.U is defined via

$$<x_{\vec{\alpha}},y_{\vec{\beta}}> = x_{\vec{\alpha}}y_{\vec{\beta}} - \varepsilon(\alpha,\beta)y_{\vec{\beta}}x_{\vec{\alpha}} .$$

∎

One sees now that ass.C_δ is a commutative δ Lie Γ graded algebra, hence C_δ is an epimorphic image of the universal enveloping algebra of a commutative δ Lie Γ graded algebra [2,p.26]. This is seen in the following way.
Let V be maximally Γ graded vector space i.e. $V=\bigoplus_{\gamma\in\Gamma}V_\gamma$, $\dim V_\gamma=1$. Let $S_\delta=T/I_\delta$ be the δ symmetric algebra of V where T is the tensor algebra of V while I_δ is an ideal of T generated by the elements

$$x_{\vec{\alpha}} \otimes y_{\vec{\beta}} - y_{\vec{\beta}} \otimes x_{\vec{\alpha}}\delta(\vec{\alpha},\vec{\beta}) \ ; \qquad \vec{\alpha},\vec{\beta}\in\Gamma$$

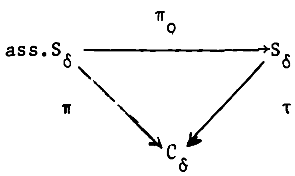
Of course the vector space V can be considered as commutative δ Lie Γ graded algebra where by definition $<x_{\vec{\alpha}},y_{\vec{\beta}}>=0$; $\vec{\alpha},\vec{\beta}\in\Gamma$. S_δ is then the universal enveloping algebra of this δ commutative δ Lie Γ graded algebra V. S_δ may be identified with the algebra of all polynomials in the basis elements $\{x_{\vec{\alpha}}\}_{\vec{\alpha}\in\Gamma}$ of V, which satisfy relations

$$x_{\vec{\alpha}}x_{\vec{\beta}} = \delta(\vec{\alpha},\vec{\beta})x_{\vec{\beta}}x_{\vec{\alpha}}$$

Of course $(x_{\vec{\alpha}})^k \in Z[S_\delta]$ as $\delta(0,\vec{\beta})=\delta(\vec{\beta},0)=1$ $\vec{\beta}\in\Gamma$. S_δ is naturally Γ graded with the grading induced by that of V

i.e. $x_{\vec{\alpha}_1},\ldots\ldots,x_{\vec{\alpha}_r} \in (S_\delta)_{\vec{\gamma}}$ iff $\sum_{i=1}^{r}\vec{\alpha}_i = \vec{\gamma}$, $\vec{\alpha}_i,\vec{\gamma}\in\Gamma$, and $S_\delta = \bigoplus_{\vec{\gamma}\in\Gamma}(S_\delta)_{\vec{\gamma}}$.

The center $Z[S_\delta]$ is not trivial as it contains the subalgebra $W[\{Z_{\vec{\alpha}}\}]$ of all polynomials in variables $\{Z_{\vec{\alpha}}\}_{\vec{\alpha}\in\Gamma}, Z_{\vec{\alpha}} \equiv (x_{\vec{\alpha}})^k$ independently of the δ chosen.
Because of the commutative diagram



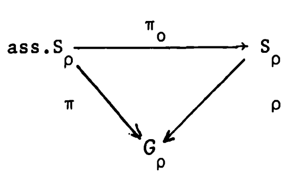
π_o - trivial isomorphism ass.$S_\delta \to$ ass.S_δ
π - epimorphism ass.$S_\delta \to$ ass. C_δ
τ - epimorphism of Γ graded associative algebras

C_δ is τ epimorphic image of S_δ , epimorphism τ being that sending $(x_{\vec{\alpha}})^k$ $\vec{\alpha}\in\Gamma$ into $\mathbb{I}\in C_\delta$.
With the Γ graded vector space V given this commutative diagram provides a definition of C_δ with τ being that above i.e.

$$C_\delta \simeq S_\delta/_{\ker\tau}$$

∎

This seemingly affected presentation of C_δ Clifford like algebras opens the way to introduce Grassmann like algebras G_ρ along the same lines. Namely: consider now $S_\rho = {}^T/_{I_\rho}$ with $\rho=\varepsilon_o\delta$; $\delta\in P_{a.s.}(\Gamma,\mathbb{C}^*)$ where Γ admits a T_o subgroup of index 2. Again we have the commutative diagram



where this time the epimorphism ρ of Γ graded algebras (S_δ onto G_ρ) sends all $(x_{\vec{\alpha}})^k$; $\vec{\alpha}\in\Gamma$ into zero. Hence

$$G_\rho \simeq S_\rho/_{\ker_\rho} .$$

∎

To end these considerations we find it interesting to give a simple example.

Example.

Let $\Gamma=Z_{2l}$; $\Gamma_o=\{2i;\ i=0,\dots,l-1\}$, $\Gamma_1=\{2i+1;\ i=0,\dots,l-1\}$. Since $H^2(Z_k,\mathbb{C}^*)\simeq\{0\}$ we have only one δ and $\delta\equiv 1$. Therefore we end up with only one Grassmann like algebra $G_{\varepsilon_o}(1)$. Note that ε_o commutation factor can be written now as

$$\varepsilon_o(\alpha,\beta) = (-1)^{\alpha\beta} \qquad ; \qquad \alpha,\beta\in Z_{2l} \ .$$

IV. THE ALGEBRAS $C^{(k)}_{\{\beta_1,\dots,\beta_n\}}$

We consider now, for completeness, the dimodule algebra construction of $C_n^{(k)}$ algebras [7]. $C^{(k)}_{\{\beta_1,\dots,\beta_n\}}$ algebra, defined below, is to be important for the forthcoming application. Consider the generalized Pauli algebra $C_2^{(k)}$. This is a special case of (central simple) generalized quaternion algebra $A_\omega(a,b)$, $a,b\in\mathbb{C}^*$; discussed in [10, §15]. i.e. $C_2^{(k)}=A_\omega(1,1)$.

In [7] it was observed that $A_\omega(a,b)=C^{(k)}(a)\#C^{(k)}(b)$ where $C^{(k)}(a)$ is the Z_k dimodule algebra generated by λ subjected to the relation $\lambda^k=a\mathbb{1}$, $a\in\mathbb{C}^*$; and # denotes the smash product of dimodule algebras [7]. Z_k group action on $C^{(k)}(a)$ is defined via $\lambda\to\omega\lambda$.

A similar generalization of $C_n^{(k)}$ is an algebra $\#_{i=1}^{n} C^{(k)}(a_i)$, $a_i\in\mathbb{C}^*$ the generators $\gamma_1,\dots,\gamma_n$ of which satisfy relations: $\gamma_i^k = \Pi a_i$ $\gamma_i\gamma_j=\omega\gamma_j\gamma_i$ $i<j$; $i,j=1,\dots,n$. With help of smash product of Z_k graded dimodule algebras one can obtain besides $C_n^{(k)}$ algebra extension of Γ, some other algebra extensions. Here they are:let $C_\beta^{(k)}$ be the k dimensional algebra over $\mathbb{C}$ with the basis $1,\lambda,\dots,\lambda^{k-1}$ where $\lambda^k=1$. Take the grade of λ to be $\beta\in Z_k$ and let β be comprime with k.

Define the Z_k action on $C_\beta^{(k)}$ via: $\kappa\in Z_k$, ${}^\kappa\lambda=\omega^\beta\lambda$.

$C_\beta^{(k)}$ becomes then a dimodule algebra. Consider now the smash product

$$C^{(k)}_{\beta_1}\ \#\ C^{(k)}_{\beta_2}\ \#\ \dots\ \#\ C_{\beta_n} \equiv C_{\{\beta_1,\dots,\beta_n\}}$$

Its generators

$$\begin{aligned}\gamma_1 &= \lambda_1 \# 1 \# \dots \# 1\\ \gamma_2 &= 1 \# \lambda_2 \# 1 \dots \# 1\\ &\vdots\\ \gamma_n &= 1 \# \dots \# 1 \# \lambda_n\end{aligned}$$

satisfy then

$$\gamma_i\gamma_j = \omega^{\beta_j}\gamma_j\gamma_i \qquad i>j$$

$\gamma_i^k = \mathbb{1}$ $\qquad i,j = 1,\dots,n$.

For any choice of $\beta_1,\dots,\beta_n \in Z_k$ (β's comprime with k) we obtain "several" algebra extensions $C^{(k)}_{\{\sigma(\beta_1),\dots,\sigma(\beta_n)\}}$; $\sigma\in S_n$. (S_n group of permutations ≡ symmetric group). All these extensions are isomorphic as algebras.

V. THE UNIVERSAL k-C_n CLIFFORD ALGEBRAS

Up to now we have presented Clifford like algebras which generalized $C_n^{(2)}$ algebras due to the observation that $C_n^{(2)}$ is an algebra extension of $\Gamma = Z_2\oplus\dots\oplus Z_2$ (n summands).
In this section we introduce new Clifford like algebras - denoted by k-C_n, which are universal in a sense of the commutative diagram (1.1) (see the Introduction).
We also introduce in this section new Grassmann like algebras.
1. k-C_n algebras are defined by the commutative diagram (1.1) up to isomorphism and it is clear that

$$k - C_n \simeq T(\nu)/I(Q_k) \tag{5.1}$$

where $T(\nu)$ is the tensor algebra of ν and $I(Q_k)$ is the ideal of T generated by the elements

$$\{\underbrace{x\otimes \dots \otimes x}_{k} - Q_k(x)\mathbb{1}\}_{x\in\nu} \quad ,$$

The mapping Q_k ("k - ubic" form) is defined via:
<u>Def.</u>

$$Q_k : \nu \to \mathbb{C} \ ;$$

1) $Q_k(\lambda\vec{x}) = \lambda^k Q_k(x)$; $\qquad \vec{x}\in\nu$; $\lambda\in\mathbb{C}$

2) the mapping $B_k: \ \underbrace{\nu\times\dots\times\nu}_{k} \to \mathbb{C}$;

$$k!B_k(\vec{x}_1,\dots,\vec{x}_k) \equiv \sum_{l=1}^{k}(-1)^{k-1} \sum_{(i_1,\dots,i_l)\in S\binom{k}{l}} Q_k(\vec{x}_{i_1} + \dots + \vec{x}_{i_l})$$

is k-linear.
$S\binom{k}{l}$ denotes the family of subsets of $\{1,\dots,k\}$ that count l elements. ∎

This generalized notion of quadratic and cubic [3,p.114] forms is achieved by the process of polarization typical for multilinear structures. Clearly we have the identity

$$k!B_k(x_1,\dots,x) \equiv Q_k(x)$$

The Clifford-like algebra k-C_n has n generators. Namely: let$\{\hat{\gamma}_i\}_{i=1}^n$ be the "k-orthonormal" basis of ν i.e.

$$B(\hat{\gamma}_{i_1},\dots,\hat{\gamma}_{i_k}) = \delta(i_1,\dots,i_k) \qquad i_1,\dots,i_k=1,\dots,n$$

where $\delta(i_1,\dots,i_k) = \begin{cases} 1 & i_1= \dots =i_k \\ 0 & \text{otherwise.} \end{cases}$

Let ρ be canonical epimorphism $\rho : T \to T_{I(Q_k)}$; then $[\rho(x)]^k=Q_k(x)\mathbb{1}$, $x\in\nu$. (In the following we shall not distinguish $x\in\nu$ from its monomorphic image $\rho(x)\in$k-C_n). It is ease to see now that

$$\frac{1}{k!}\{\hat{\gamma}_{i_1},\dots,\hat{\gamma}_{i_k}\} = \delta(i_1,\dots,i_k); \quad \hat{\gamma}\text{'s k-}C_n \tag{5.2}$$

where

$$\{x_1,\dots,x_k\} \equiv \sum_{\sigma\in S_k} x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(k)}$$

Hence k-C_n Clifford like algebra is generated by $\{\hat{\gamma}_i\}_{i=1}^n$ satisfying (5.2). Clearly all this is due to the identity

$$k!B(x_1,\dots,x_k) \equiv \{x_1,\dots,x_k\} \qquad x_1,\dots,x_k \in \nu \simeq \rho(\nu).$$

2. We present now a class of matrix representations of k-C_n Clifford like algebras. It is obvious and characteristic that i) $C_n^{(2)}$ is the faithful representation of 2-C_n ii) $C_n^{(2)}$ is the only algebra extension of $Z_2 \oplus \dots \oplus Z_2$ (n summands) out of $2^{n(n-1)/2}$ possible, that has this property. For k≥3 $C_n^{(k)}$ is also a representation of k-C_n but not faithful and neither it is the only algebra extension of $\Gamma \equiv Z_k \oplus \oplus \dots \oplus Z_k$ (n>1 summands) representing k-C_n.

Let us prove what is stated above. Consider the diagram (1.1); k≥3. $\hat{\gamma}_1,\dots,\hat{\gamma}_n$ satisfying (5.2) are $\alpha_o\equiv\rho|_\nu$ images of a k orthonormal basis in ν. Denote by $\gamma_1,\dots,\gamma_n$ the α_o image of this basis in an algebra A. If A is chosen to be an algebra extension of Γ for which $\gamma_1,\dots,\gamma_n$ satisfy (5.2) then

$$0 \neq \hat{\gamma}_1\hat{\gamma}_2^2 - \omega^{2\alpha_{12}}\hat{\gamma}_2^2\hat{\gamma}_1 \xrightarrow{\sigma} 0$$

which shows that this very algebra A is not faithful representation of k-C_n. Now we show that

Lemma V.1.

Generators $\gamma_1,\dots,\gamma_n$ of $C_n^{(k)}$ satisfy (5.2)

Proof:

The case $i_1=\dots=i_k$ in (5.2) is trivial hence we assume otherwise. Consider first say $i_1=1$, $i_l\neq 1$ $l=2,\dots,k$. Then

$$\{\gamma_1,\gamma_{i_2},\dots,\gamma_{i_k}\}\sim\sum_{l=0}^{k-1}\omega^l\sum_{\sigma\in S_{k-1}}\gamma_{\sigma(i_2)}\cdots\gamma_{\sigma(i_k)}\gamma_1=0$$

Similarly for $i_l\neq 1$ $l=3,\dots,k$

$$\{\gamma_1,\gamma_1,\gamma_{i_3},\dots,\gamma_{i_k}\}\sim\sum_{0\leq i<j\leq k-1}\omega^{i+j}\{\gamma_{i_3},\dots,\gamma_{i_k}\}\gamma_1\gamma_1=0$$

and so on. The choice of $i_1=1$ (and so on) is replaced by the choice of other smallest number out of $\{i_1,\dots,i_k\}$ in the case $1\notin\{i_1,\dots,i_k\}$. Lemma is thus proved due to the famous zero

$$\sum_{0\leq i_1<\dots<i_s\leq k-1}\omega^{i_1}\cdots\omega^{i_s}=0;\qquad s\leq k-1$$

∎

In the same manner one proves a more general

Lemma V.2.

The $\gamma_1,\dots,\gamma_n$ generators of $C^{(k)}_{\{\beta_1,\dots,\beta_n\}}$ algebra extension of Γ (see IV) satisfy (5.2). ∎

3. Analogously to 2-C_n case one can define Grassmann like (k-G_n) algebras via k-C_n Clifford like algebras.

Def.

k-G_v is the algebra generated by $1,\Theta_1,\dots,\Theta_v$ where

$$\{\Theta_{i_1},\dots,\Theta_{i_k}\}=0;\qquad i_1,\dots,i_k=1,2,\dots,v \tag{5.3}$$

∎

One immediately gets the representation (denoted by $G_v^{(k)}$) of k-G_v via $C_{2v}^{(k)}$. Namely let $\mathbb{C}\ni\kappa$, $\kappa^k=-1$, then define

$$\Theta_i\equiv\gamma_i+\kappa\gamma_{v+i};\qquad i=1,\dots,v;\qquad \gamma\text{'s - generators of } C_{2v}^{(k)}.$$

Clearly these Θ's do satisfy (5.3).

It is also obvious that $\Theta^{(l)}$'s $l=0,\dots,k-1$

$$\Theta_i^{(l)}=\gamma_i+\omega^l\kappa\gamma_{v+i}\qquad i=1,\dots,v;\qquad \gamma\text{'s - generators of } C_{2v}^{(k)},$$

represent generators of k-G_v.

It is then easy to show that

$$\left\{\Theta_{i_1}^{(l_1)},\ldots,\Theta_{i_k}^{(l_k)}\right\} = (1-\omega^{l_1+\ldots+l_k})\delta(i_1,\ldots,i_k)$$

and also we have a kind of "Z_k - Witt decomposition":

$$\gamma_i = \frac{1}{k}\sum_{l=0}^{k-1}\Theta_i^{(l)} \; ; \qquad \gamma_{v+i} = \frac{1}{k}\sum_{l=0}^{k-1}\omega^{k-l}\Theta_i^{(l)}$$

∎

Having thus introduced new types of Clifford and Grassmann like algebras we end this section with three remarks.

Remarks:

1. A natural question arises wheteher k-G_n is relevant to a kind of projective geometry as $G_n^{(2)}$ is via Plücker coordinates. This question and other similar expectations (of analogous to the case k=2 geometrical facts) are however naive as the group of linear transformations leaving invariant "k-ubic" (k>2) for $Q_k(\vec{x})=\sum_{i=1}^{n} x_i^k$ is finite - of order $k^n n!$. This linear group is given [11] by transformations

$$i=1,\ldots,n \qquad x_i \to \omega^{\alpha_i} x_{\sigma(i)} \quad , \quad \alpha_i \in Z_k \, , \qquad \sigma \in S_n \; .$$

2. The generators of k-C_n algebra, satisfying (5.2) linearize "k-ubic" form Q_k; $v\in\vec{x} \to Q(\vec{x}) = \sum_{i=1}^{n} x_i^k$ i.e.

$$\mathbb{I}\sum_{i=1}^{n} x_i^k = \Big(\sum_{i=1}^{n} x_i\hat{\gamma}_i\Big)^k \tag{5.4}$$

This property of $\{\hat{\gamma}_i\}_{i=1}^{n}$ is due to the obvious identity valid in any associative algebra A

$$k!\Big(\sum_{i=1}^{n} a_i\Big)^k \equiv \sum_{i_1,\ldots,i_k=1}^{n}\{a_{i_1},\ldots,a_{i_k}\} \; ; \qquad a_i\in A.$$

As for the linearization of (5.4), any representation of (5.2) relations will do, of course - $C_n^{(k)}$ algebra generators included.

∎

3. Due to the property (5.4), algebras k-C_n can be applied to ultrageneralized complex analysis as that of [8].

VI. EXPLICIT CONSTRUCTION OF ε-LIE Γ GRADED ALGEBRAS

In this section we apply C_δ algebras for the explicit construction of ε-Lie Γ graded algebras [19] defined in the section III. From the work of Scheurnert [19] one already learns that the theory of ε-Lie Γ graded algebras can be considerably reduced to the theory of graded Lie algebras or Lie superalgebras.
It was then noticed by the author [5] that the group of commutation factors, de-

pending on whether the grading group Γ has or has not a subgroup Γ_o of index 2, is given correspondingly by

$$H^2(\Gamma:\mathbb{C}^*) \cup \varepsilon_o H^2(\Gamma:\mathbb{C}^*) \qquad \text{or} \qquad H^2(\Gamma:\mathbb{C}^*) .$$

On the other hand, we know from section III, that the $H^2(\Gamma:\mathbb{C}^*)$ group is isomorphic to the additive group of $(\ _{ij})$ matrices - antisymmetric in the sense of Z_k ring. Therefore any commutation factor is of the form: either $\varepsilon_o\delta$ or δ where

$$\delta(s_i,s_j) = \omega^{\cdot ij} \qquad , \qquad i,j=1,\dots,n .$$

Therefore an ε Lie Γ graded algebra can be obtained: either from Lie Γ graded algebra and ε_o Lie Γ graded algebra or Lie Γ graded algebra only, according to the construction of [19,5]. (ε_o Lie algebra is called by physicists - a Lie superalgebra).

The construction now runs as follows.

Let S_δ denotes the δ-Lie commutative algebra associated to the algebra extension C_δ of the grading group.

Associated - means that multiplication in S_δ is defined via $\langle x_\alpha, y_\beta\rangle = x_\alpha y_\beta +$ $- \varepsilon(\alpha,\beta) y_\beta x_\alpha$, where x_α, y_β are homogeneous elements of C_δ .

Then L_ε - an ε-Lie Γ graded algebra, is isomorphic to $S_\delta \# L_{\varepsilon_o}$, or in the other possible case, to $S_\delta \# L$, where L is just a Lie algebra graded by the same group. The product # was defined in [19].

Definition:

$$S_\delta \# L_\varepsilon = \bigoplus_{\alpha\in\Gamma} (S_\delta)_\alpha \otimes (L_\varepsilon)_\alpha$$

where $S_\delta = \bigoplus_{\alpha\in\Gamma} (S_\delta)_\alpha$; $L_\varepsilon = \bigoplus_{\alpha\in\Gamma} (L_\varepsilon)_\alpha$

and the multiplication is defined by

$$\langle a_\alpha \otimes A_\alpha, b_\beta \otimes B_\beta\rangle =: a_\alpha b_\beta \otimes \langle A_\alpha, B_\beta\rangle$$

a_α, A_α, b_β, B_β - homogeneous elements.

■

$S_\delta \# L_\varepsilon$ is evidently an $\delta\varepsilon$ Lie Γ graded algebra.

Hence, once the algebras C_δ, $\delta\in H^2(\Gamma,\mathbb{C}^*)$ are known, we are able to construct all ε Lie Γ graded algebras in either of two classification cases i.e. containing or not a subgroup of index 2. In the following we consider only $\Gamma = Z_k \oplus \dots \oplus Z_k$ (n-summands) grading group. For this very Γ grading group $H^2(\Gamma,\mathbb{C}^*) = Z_k \oplus \dots \oplus Z_k$ (n(n-1)/2 summands) [6] and we have $k^{n(n-1)/2}$ different algebra extensions C_δ of Γ over $\mathbb{C}$.

$\mathbb{C}_\delta$ can be thought of as the algebra generated by generators $\gamma_1,\dots,\gamma_n$ satisfying

$$\gamma_i\gamma_j = \omega_{ij}\gamma_j\gamma_i \quad ; \qquad \gamma_i^k = 1, \qquad i,j=1,\dots,n$$

where $\omega_{ij} = \delta(s_i,s_j)$ while $\{s_i\}_{i=1}^n$ form the set of generators of Γ and $\delta\in H^2(\Gamma,\mathbb{C}^*)$.

Via straightforward calculation it can be then shown that for any $\delta\in H^2(\Gamma:\mathbb{C}^*)$ one has

$$\delta(\vec{\alpha},\vec{\beta}) = \omega^{\langle\vec{\alpha}|A\vec{\beta}\rangle} \quad ; \qquad \vec{\alpha},\ \vec{\beta}\in\Gamma = Z_k\oplus\dots\oplus Z_k \quad \text{(n summands)},$$

where A is a matrix $A=(\alpha_{ij})$ and $\langle\vec{\alpha}|\vec{\beta}\rangle = \sum_1^n \alpha_i\beta_i \quad \alpha_i\ ,\ \beta_j\in Z_k$, and the notation underlines the fact, that Γ is a naturally the Z-module of dimension n. All ε Lie Γ graded algebras are then given explicitly by S_δ # L and S_δ # L_{ε_o} or by S_δ # L where $\delta\in H^2(\Gamma,\mathbb{C}^*)$.

L is a corresponding Γ graded Lie algebra while L_{ε_o} is a Γ graded Lie superalgebra.

Given L (and/or L_{ε_o}) and its basis $\{X_{i,\vec{\alpha}}\}$ the basis $\{Y_{i,\vec{\alpha}}\}$ of δ Lie Γ graded algebra S_δ # L is of the form

$$Y_{i,\vec{\alpha}} = \gamma_{\vec{\alpha}}\otimes X_{i,\vec{\alpha}} \qquad \vec{\alpha}\in\Gamma$$

where $\gamma_{\vec{\alpha}}=\gamma_1^{\alpha_1}\dots\gamma_n^{\alpha_n}$; $(\alpha_1,\dots,\alpha_n)\in\Gamma = Z_k\oplus\dots\oplus Z_k$ (n summands) is the canonical basis of C_δ.

The structure constants (primed) of the δ Lie Γ graded algebra S_δ # L (and/or S_δ # L_{ε_o} ...) are expressed by those of L according to

$$C'^{\alpha+\beta}_{\alpha,\beta} = \sigma(\alpha,\beta)\ C^{\alpha+\beta}_{\alpha,\beta}$$

(with "internal" indexes i,j,k of grading subspaces - being suppressed) where the 2 - cocycle σ is related to δ via relations

$$\delta(s_i,s_j) = \begin{cases} 1 & i<j \\ \delta(s_i,s_j) & i>j \end{cases}$$

$$\delta(s_i,s_j) = \sigma(s_i,s_j)\ \sigma^{-1}(s_j,s_i)$$

and $\{s_i\}_1^n$ is the set of generators of Γ. A very special case of the commutation factor is that of δ or $\delta\varepsilon_o$ where δ is defined by $A=(\alpha_{ij})$ matrix of the form

$$\alpha_{ij} = 1 \quad , \qquad i<j \quad .$$

One easily recognizes then, that C_δ becomes in this case a $C_n^{(k)}$ generalized Clifford algebra [12].

In this very case of δ, the multiplier $\sigma(\vec{\alpha},\vec{\beta})$ occurring in the formula for structure constants, is of the form

$$\sigma(\vec{\alpha},\vec{\beta}) = \omega^{\sum_{i>j} \alpha_i\beta_j} \quad .$$

For $\Gamma=Z_k$, as $H^2(Z_k;\mathbb{C}^*)=\{0\}$, we have two at most ε-Lie algebra structures;Lie algebra and Lie superalgebra for k=2l and only Z_k graded Lie algebra for k=2l+1;$l\in N$.

An immediate application of the above explicit construction can be now made in parastatistical theories (see [4] for the references).

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