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In: Zdeněk Frolík and Vladimír Souček and Jiří Vinárek (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1985. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 9. pp. 109--116.

Persistent URL: http://dml.cz/dmlcz/701392

LIE ALGEBRAS CONNECTED WITH ASSOCIATIVE ONES

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0. Introduction.

The aim of this note is to present a purely algebraic approach to different types of infinite-dimensional Lie algebras arising in analysis and geometry, which is based on natural connections between the Lie and some associative algebras. As examples of such Lie algebras will be taken:

i) Lie algebras of vector fields on manifolds (the classes C^∞ and C^ω, i.e. real-analytic, will be considered),

ii) Lie algebras of C^∞ (C^ω) functions on symplectic manifolds with the Poisson brackets,

iii) C^*- algebras as Lie algebras.

The Lie algebra $D^\infty(M)$ ($D^\omega(M)$) of all C^∞ (C^ω) vector fields on a C^∞ (C^ω) manifold M (we assume manifolds to be finite-dimensional and paracompact) is a classical example of an algebraic object growing on topological one. On the other hand, many algebraic objects usually have topological interpretations, as in the well-known model: a compact topological space $X \longmapsto$ the Banach algebra $C(X)$ of all continuous functions on $X \longmapsto$ the structure space of $C(X)$ (which is homeomorphic to X). It allows us to interprete isomorphisms between $C(X_1)$ and $C(X_2)$ as homeomorphisms between X_1 and X_2.

A similar approach to study isomorphisms of the Lie algebras $D^\infty(M)$, due to Pursell and Shanks and applied in a number of situations by Omori (see [7] and [6]), is to investigate their ideals.

Indeed, for a compact smooth manifold M, any maximal Lie ideal of $D^\infty(M)$ is of the form $D^\infty_{(p)}(M) = \{ X \in D^\infty(M) : j^\infty_p(X) = 0 \}$, where j^∞_p denotes the infinite jet at p, for some $p \in M$, i.e. consists of all vector fields which are flat at p.

An isomorphisms of such Lie algebras induces then a bijection between the underlying manifolds, which proves to be a diffeomorphism.

It is essential in this proofs to localize by partitions of unity,

what is impossible in the analytic case. Moreover, for a connected C^{ω} manifold M we have $D^{\omega}_{(p)}(M)=0$. However, one may also consider the isotropy subalgebras $D^{\alpha}_{p}(M)=\{X\in D^{\alpha}(M):\ X(p)=0\}$ (α denotes ∞ or ω in this paper), for $p\in M$.
It is the idea due to Wojtyński [8] that isotropy subalgebras are precisely the maximal Lie subalgebras of finite codimension. To avoid partitions of unity in the considerations, arguments must be purely algebraic.

1. Lie bimodules.

The general model we propose is the following.

(1.1) Definition. Let A be an associative commutative algebra over a field of characteristic zero, and let D(A) be the Lie algebra of all derivations of A. This Lie algebra is also a left A-module in the obvious way.
If L is a Lie subalgebra of D(A) which is also an A-submodule, then the pair (A,L) is called a Lie bimodule.

A connection between the algebraic structures for a Lie bimodule (A,L) gives the formula

$$[fX,gY]=f(Xg)Y-g(Yf)X+fg[X,Y]\quad ,$$

where $f,g\in A$ and $X,Y\in L$.

(1.2) Example. Let M be a C^{α} manifold, let F be a C^{α} foliation on M, let $C^{\alpha}(M)$ be the associative algebra of all C^{α} functions on M, and let $D^{\alpha}(F)$ be the Lie algebra of all C^{α} vector fields on M tangent to the leaves of F. Then $(C^{\alpha}(M),D^{\alpha}(F))$ is a Lie bimodule.

Let (A,L) be a Lie bimodule, and let J be an ideal of A. It is easy to see that

$$L_J=\{X\in L:\ XA\subset J\}$$

is a Lie subalgebra of L, and that

$$L_{(J)}=\{X\in L_J:\ Y_1(\dots(Y_n(XA))\dots)\subset J \text{ for all } Y_1,\dots,Y_n\in L\}$$

is a Lie ideal of L. Moreover, L_J and $L_{(J)}$ are A-submodules of L and they are, in a sense, the most important types of Lie subalgebras and Lie ideals of L.
Note that the terms, like XA in the above definitions and [A,X] or AA in the sequel, always denote linear span of respective products.
We interprete the defined objects in the following example.

(1.3)Example. Let M be a C^{α} manifold, let $p \in M$, and let J be an ideal of $C^{\alpha}(M)$ of the well-known form $p^{*} = \{f \in C^{\alpha}(M): f(p) = 0\}$.
Then $(D^{\alpha}(F))_J = \{X \in D^{\alpha}(F): X(p) = 0\}$ is the isotropy subalgebra $D^{\alpha}_p(F)$ for each C^{α} foliation F on M,
and $(D^{\alpha}(M))_{(J)} = \{X \in D^{\alpha}(M): j^{\infty}_p(X) = 0\}$ is the "Shanks' and Pursell's" Lie ideal $D^{\alpha}_{(p)}(M)$.

(1.4)Theorem. (see [2] and [4]) Let (A,L) be a Lie bimodule.
If K is a Lie ideal in L, then there is an ideal I of A such that $IL \subset K$ and $K \subset L_{(J)}$ for each prime ideal J of A containing I.
In particular, if $AL=L$ and if every proper ideal of A is contained in some maximal ideal (e.g. A has unity), then every proper Lie ideal of L is contained in a maximal Lie ideal and every maximal Lie ideal of L is of the form $L_{(J)}$ for a maximal ideal J of A.

(1.5)Corollary. (Shanks, Pursell [7]) For $D^{\infty}_c(M)$ being the Lie algebra of all C^{∞} vector fields on M with compact supports, every maximal Lie ideal of $D^{\infty}_c(M)$ consists of vector fields which are flat at a given point of M.

Proof. Maximal ideals of the associative algebra $C^{\infty}_c(M)$ of all C^{∞} functions on M with compact supports are of the form p^{*} for $p \in M$ and each proper ideal of $C_c(M)$ is contained in maximal one.
Observe that $(C^{\infty}_c(M), D^{\infty}_c(M))$ is a Lie bimodule and that $C^{\infty}_c(M)D^{\infty}_c(M) = D^{\infty}_c(M)$, since every vector field with compact support equals itself multiplicated by a compactly supported smooth function.
By (1.4), maximal Lie ideals of $D^{\infty}_c(M)$ are of the form $(D^{\infty}_c(M))_{(p^{*})}$, i.e. consist of vector fields which are flat at p.

(1.6)Corollary. (see [2]) Let M be a connected compact C^{ω} manifold. Then the Lie algebra $D^{\omega}(M)$ of all C^{ω} vector fields on M is simple.

Proof. $C^{\omega}(M)$ has unity, every maximal ideal of this algebra is of the form p^{*} for $p \in M$, and $D^{\omega}_{(p)}(M) = \{0\}$.

If M is a non-compact C^{α} manifold, then there are maximal ideals of $C^{\alpha}(M)$ which are not of the form p^{*}. Nevertheless, the p^{*} ideals still have an algebraic characterization, namely, they are precisely the maximal ideals of finite codimension (see [2]).
Let us denote the set of all maximal finite-codimensional ideals of an associative commutative algebra A by $\mathfrak{M}_{ass}(A)$. We can then write

$\mathfrak{m}_{ass}(C^{\alpha}(M)) = \{p^*: p \in M\}$.

The isotropy subalgebras $D_p^{\alpha}(F)$ are of finite codimension. It leads to the following definition.

(1.7) Definition. A Lie bimodule (A,L) will be called admissible iff
i) every proper ideal I of A is contained in a prime ideal of A and $AL = L$ (e.g. A has unity) ,
ii) $\dim(L/L_J) < +\infty$ for each $J \in \mathfrak{m}_{ass}(A)$ and $LA = A$.

(1.8) Theorem. (see [4]) Let M be a C^{α} manifold. A Lie bimodule $(C^{\alpha}(M),L)$ is admissible if and only if there are vector fields $X_1,\dots,X_n \in L$ with no common zeros.

(1.9) Remark. Note that L in the above theorem really consists of vector fields, since $D(C^{\alpha}(M)) = D^{\alpha}(M)$ (see [3]) .

(1.10) Example. Let F be a C^{α} foliation on a manifold M . Then $(C^{\alpha}(M),D^{\alpha}(F))$ is an admissible Lie bimodule.

(1.11) Example. Let L be the Lie algebra of those smooth vector fields on $\mathbb{R}^2$ which are of the form $f(x,y)\partial_y$, where $f \in C^{\infty}(\mathbb{R}^2)$, on the set $\{(x,y) \in \mathbb{R}^2: y > 0\}$ - the upper half of the plane. Then $(C^{\infty}(R^2),L)$ is an admissible Lie bimodule.

(1.12) Theorem. (see [2]) Let (A,L) be an admissible Lie bimodule. Then the mapping $J \longmapsto L_J$, where $J \in \mathfrak{m}_{ass}(A)$, is a bijection between $\mathfrak{m}_{ass}(A)$ and the set of all maximal finite-codimensional Lie subalgebras of L .

(1.13) Corollary. Given a C^{α} manifold M suppose that $(C^{\alpha}(M),L)$ is an admissible Lie bimodule. Then there is a one-one correspondence between the points of M and the maximal finite-codimensional Lie subalgebras of L given by

$$M \ni p \longmapsto L_p = \{X \in L : X(p) = 0\} .$$

Having the points of M interpreted in terms of the Lie algebra L , we can prove the following generalization of the Shanks and Pursell's result.

(1.14) Theorem. (see [2], [4]) Let M_i be a C^{α} manifold, and let L_i be a Lie algebra of C^{α} vector fields on M_i such that $C^{\alpha}(M_i)L_i \subset L_i$, and that there are $X_1,\dots,X_n \in L_i$ with no common zeros, $i=1,2$. Then a mapping $s:L_1 \longrightarrow L_2$ is an isomorphisms of the Lie algebras if and only if there is a C^α diffeomorphisms $u:M_1 \longrightarrow M_2$ such that $s=u_*$,where u_* is the natural action of the diffeomorphism u on vector fields.

(1.15)Remark. If $L_i = D^\alpha(F_i)$,where F_i is a C^α foliation on M_i , $i=1,2$,then it is not hard to see that the diffeomorphism u from (1.14) has in this case to map leaves of F_1 diffeomorphically onto the leaves of F_2 .Similarly,automorphisms of the Lie algebra L from (1.11) are generated by diffeomorphisms of $\mathbb{R}^2$ which preserve the upper half of the plane and the foliation x=const on it .

2. AL-algebras and hamiltonian vector fields.

Consider now an associative algebra A with the natural Lie algebra structure given by the bracket $[X,Y] = XY - YX$ .It is easy to verify that the Lie and associative products are connected by the following formulas.

(2.1) $$[X,YZ] = [X,Y]Z + Y[X,Z] \quad .$$

(2.2) $$[X,YZ] + [Y,ZX] + [Z,XY] = 0 \quad .$$

It is interesting that the above formulas appear in the symplectic geometry as follows.

Let (M,β) be a C^α symplectic manifold,i.e. let M be a C^α manifold and let β be a C^α closed and non-degenerate 2-form on M . Since β is non-degenerate,it induces an isomorphism $w:TM \longrightarrow T^*M$ of the tangent and cotangent bundles given by $w(X) = -i(X)\beta$,where i denotes the inner product.Vector fields on M corresponding (with respect to w) to exact 1-forms on M are called hamiltonian (with respect to the symplectic structure β) vector fields on M .One can show that the set $D^\alpha(M,\beta)$ of all C^α hamiltonian vector fields is a Lie subalgebra of $D^\alpha(M)$ .This subalgebra is not a $C^\alpha(M)$-module,so the methods developed in the previous section are of no use in this case.Nevertheless,a new algebraic model can be found.

We have a natural linear mapping

$$V:C^\alpha(M) \ni f \longmapsto V_f \in D^\alpha(M,\beta)$$

defined by $V_f = w^{-1}(df)$ .One can check that $C^\alpha(M)$ with the bracket $(f,g) = V_f(g)$ (usually called the Poisson bracket) is a Lie algebra.Moreover,the mapping $V:C^\alpha(M) \longrightarrow D^\alpha(M,\beta)$ is a surjective homomorphism of the Lie algebras with the kernel Const(M) consisting of locally constant functions on M .

Since the vector field V_f is a derivation of the associative algebra $C^\alpha(M)$,we have

$$(f,gh) = V_f(gh) = V_f(g)h + gV_f(h) = (f,g)h + g(f,h)$$

for all $f,g,h \in C^\alpha(M)$,so the equality (2.1) holds true.It is easy to see that (2.1) implies in this case (2.2) ($C^\alpha(M)$ is commutative as an associative algebra).

All this can be generalized as follows.

(2.3)Definition. Let A be an associative and simultaneously a Lie algebra such that that this both structures are connected by the identities (2.1) and (2.2) .

Then A will be called an associative-Lie algebra (AL-algebra) .

For an AL-algebra A ,the set of all maximal finite-codimensional associative both-sides ideals will be denoted by $\mathfrak{M}_{ass}(A)$.

For a subspace $B \subset A$,we denote by N(B) the Lie normalizer of B , i.e. $N(B) = \{X \in A: [X,B] \subset B\}$.

The following theorem,for AL-algebras with commutative associative part,is due to Atkin [1].(For the proof see also [5].)

(2.4)Theorem. Let A be an AL-algebra such that $AA = A$,and that $0 < \dim(A/N(I)) < +\infty$ for each $I \in \mathfrak{M}_{ass}(A)$.

Then the mapping $I \longmapsto N(I)$,where $I \in \mathfrak{M}_{ass}(A)$,is a bijection between $\mathfrak{M}_{ass}(A)$ and the set of all maximal finite-codimensional Lie subalgebras of A which are not Lie ideals of A (do not contain $[A,A]$).

Since $\mathfrak{M}_{ass}(C^\alpha(M)) = \{p^*: p \in M\}$,and since $N(p^*) = \{f \in C^\alpha(M): (f,g)(p) = 0 \text{ if } g(p)= 0 ,\ g \in C^\alpha(M)\} = \{f \in C^\alpha(M): df(p)= 0\}$,we get the following corollary for a C^α symplectic manifold (M,β) .

(2.5)Corollary. Let (M,β) be a C^α symplectic manifold.Then each maximal finite-codimensional Lie subalgebra of $C^\alpha(M)$ which is not a Lie ideal (with respect to the Poisson bracket) is of the form $\{f \in C^\alpha(M): df(p)= 0\}$ for some $p \in M$.

Maximal finite-codimensional Lie subalgebras of $D^\alpha(M,\beta)$ are of the form $D_p(M,\beta) = \{X \in D^\alpha(M,\beta): X(p)= 0\}$, $p \in M$.

Having the points of M interpreted in the algebraic terms,one can prove the following theorem about isomorphisms.

(2.6)<u>Theorem</u>. Let (M_i,β_i) be a C^α symplectic manifold, $i=1,2$. Then
i) a mapping $s: D^\alpha(M_1,\beta_1)\longrightarrow D^\alpha(M_2,\beta_2)$ is an isomorphism of the Lie algebras of hamiltonian vector fields if and only if there is a C^α diffeomorphism $u:M_2\longrightarrow M_1$ and a non-vanishing function $c\in \mathrm{Const}(M_2)$ such that $\beta_2 = cu^*(\beta_1)$ and $s = u_*^{-1}$,
ii) a mapping $\hat{s}: C^\alpha(M_1)\longrightarrow C^\alpha(M_2)$ is an isomorphism of the Lie algebras of functions with the Poisson brackets if and only if there is a C^α diffeomorphism $u:M_2\longrightarrow M_1$, a non-vanishing function $c\in\mathrm{Const}(M_2)$, and a linear $K:C^\alpha(M_1)\longrightarrow \mathrm{Const}(M_2)$ containing the derived algebra $C^\alpha(M_1)^{(1)}=(C^\alpha(M_1),C^\alpha(M_1))$ in its kernel such that $\beta_2= cu^*(\beta_1)$ and $\hat{s} = cu^* + K$.

Note that a C^∞ version of i) of the above theorem (for conformally symplectic vector fields) is due to Omori [6].

(2.7)<u>Remark</u>. One can show that for a connected 2n-dimensional C^α symplectic manifold (M,β) the derived algebra $C^\alpha(M)^{(1)}$ equals $\{f\in C^\alpha(M):\ f\eta = d\gamma$ for some (2n-1)-form γ of the class $C^\alpha\}$, where $\eta=\beta^n$ is the volume form generated by β.
Hence $C^\alpha(M)^{(1)}= C^\alpha(M)$ if M is non-compact, and

$$C^\alpha(M)^{(1)} = \{ f\in C^\alpha(M):\ \textstyle\int_M f\eta = 0 \}$$

if M is compact.
Thus the linear K on $C^\alpha(M)$ with the kernel containing $C^\alpha(M)^{(1)}$ has to be trivial if M is connected and non-compact, and has to have the form $K(f) = a\int_M f\eta$, a=const, if M is connected and compact.

Finally, we give an example of a correspondence between associative and Lie ideals in C^*-algebras (which are AL-algebras with the natural associative and Lie algebra structures).

(2.8)<u>Theorem</u>. (see [5]) If A is a C^*-algebra such that the derived Lie algebra $[A,A]$ is dense in A, then

$$I\longmapsto \mathrm{ad}^{-1}(I) = \{X\in A:\ [X,A]\subset J\}$$

is a one-one correspondence between the closed maximal associative ideals I of A and the closed maximal Lie ideals of A.

(2.9)<u>Example</u>. Put $A = B(H)$ - the C^*-algebra of all bounded linear operators on an infinite-dimensional separable Hilbert space H. The only maximal closed associative ideal in A is the ideal $C(H)$ of all compact operators. Moreover, $[A,A] = A$. Thus the only maximal closed Lie ideal in A is $\mathrm{ad}^{-1}(C(H)) = C(H)\oplus\mathbb{C}\,\mathrm{Id}$.

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