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by Kunio Yoshino (*)

Abstract

Using transforms of analytic functionals with non-compact carrier, Polya's theorem concerning arithmetic entire functions is generalized to arithmetic non-entire functions.

1. Introduction

In 1920 Polya (see [7])) proved the following

Theorem. Suppose that the function f(z) satisfies the following conditions :

(1) $|f(z)|\leqq Ce^{\alpha|z|}$ $(z\in C)$

(2) $f(N)\subset Z$.

If $\alpha<\log 2$, then f(z) is a polynomial with rational coefficients.

Recently, this theorem has been generalized by several authors to the case of entire functions of several complex variables (see [2], [3], [4]).

In this paper, we investigate Polya's theorem for non-entire functions of several complex variables.

The following theorem is our main result.

Theorem 1. Let f(z) be holomorphic in $\Gamma=\{z\in C^n:\mathrm{Re}\, z_i>0,\ 1\leqq i\leqq n\}$ and satisfy the following conditions :

(1) For any $\varepsilon>0$, there exists a constant $C_\varepsilon>0$ such that

$$|f(z)|\leqq C\ e^{a(z)} \qquad (\mathrm{Re}\ z_i\geqslant\varepsilon,\ 1\leqq i\leqq n)$$

where a(z) is a convex function of homogeneous degree 1.

(2) $f(N^n)\subset Z$.

Furthermore let $L\subset C^n$ be defined by

$L=\{\zeta\in C^n:\mathrm{Re}\langle\zeta,z\rangle\leqq a(z),\ \forall z\in\Gamma\}$

and suppose that the i-th projection $L_i=pr_i(L)$ of L is contained in $\{\zeta_i\in C:|e^{\zeta_i}-1|<1\}$ for all i $(1\leqq i\leqq n)$.

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Then f(z) is a polynomial with rational coefficients.

To prove Theorem 1 we use the Fourier-Borel and Avanissian -Gay transforms of analytic functionals with unbounded carrier and in the sections 2 and 3 we define the Fourier-Borel and Avanissian - Gay transforms of such functionals. In section 4 we recall the definition of the transfinite diameter and its properties while in section 5, we give the proof of Theorem 1.

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2. The Fourier-Borel transform of analytic functionals with unbounded carrier

In this section we first recall the definition of the Fourier-Borel transform of analytic functionals with unbounded carriers and also mention the Ehrenpreis-Martineau type theorem due to J.W. DE. Roever. Let L be a closed convex set which is bounded in the imaginary direction in C^n and put

$$H_b(L:\varphi)=\{f(z)\in O(\overset{\circ}{L})\cap C(L):\sup_{z\in L}|f(z)e^{-\varphi(z)}|<+\infty\}\ ,$$

where $\varphi(z)$ is a real valued function and $O(\overset{\circ}{L})$ and $C(L)$ denote respectively the spaces of holomorphic functions defined in the interior of L, and the space of continuous functions in L.

Put

$$Q(L:K')=\lim_{\varepsilon\downarrow 0}\ \text{ind}_{\varepsilon'\downarrow 0}\ H_b(L_\varepsilon:-h_K{}'(z)-\varepsilon'|z|)$$

where L_ε stands for the ε-neighbourhood of L and $h_{K'}(z)$ is the supporting function of the compact convex subset K' of C^n.

An element of the dual space Q'(L:K') of Q(L:K') is called an analytic functional with carrier L and of type $h_{K'}(z)$. Let us recall that if L is a compact convex subset of C^n then Q'(L:K') concides with the space of analytic functionals O'(L) in the sense of A. Martineau, and that if $L=R^n$ and K'={0} then Q'(L:K') coincides with the space of Fourier-hyperfunctions studied by M.Sato and T. Kawai.

When the exponential function $\exp(\sum_{i=1}^{n}\zeta_i z_i)=\exp(\zeta z)$ belongs to Q(L:K'), the Fourier-Borel transform $\tilde{T}(z)$ of T∈Q'(L:K') is defined as follows :

$$\tilde{T}(z)=\langle T_\zeta,\exp(\zeta z)\rangle.$$

Now let Γ be an open convex cone in C^n, let $a(z)$ be a convex function on Γ of homogeneous degree one and put

$$\Omega(a;\Gamma)=\{\zeta\in C^n:\mathrm{Re}<\zeta,z>\leqq a(z).\ \forall z\in\Gamma\} .$$

Then the following generalized Ehrenpreis-Martineau type theorem is valid :

Theorem 2 (J.W. DE Roever [9])
The Fourier-Borel transform is a linear topological isomorphism from $Q'(\Omega(a:\Gamma):\{0\})$ onto $\mathrm{Exp}(\Gamma:a)$, where $\mathrm{Exp}(\Gamma:a)=\lim\ \mathrm{proj}\ H_b(\Gamma+\varepsilon(z_0);\ a(z)+\varepsilon|z|)$ and z_0 is a fixed complex vector contained in Γ with $|z_0|=1$.

Note that the space $H_b(\Gamma+\varepsilon(z_0):a(z)+\varepsilon|z|)$ may be defined in a similar as $H_b(L:\varphi)$.
We close this section by giving two examples of $\Omega(a:\Gamma)$ in the case of $n=1$.
Take

$$\Gamma=\{z\in C:\mathrm{Re}\,z>0\}.$$

Example 1. If $a(z)=\alpha|z|$ with $\alpha>0$, then $\Omega(a:\Gamma)=$
$\{\zeta\in C:|\zeta|=\alpha\}\cup\{\zeta\in C:|\mathrm{Im}\zeta|\leqq\alpha,\mathrm{Re}\zeta\leqq 0\}$ (see Figure 1).

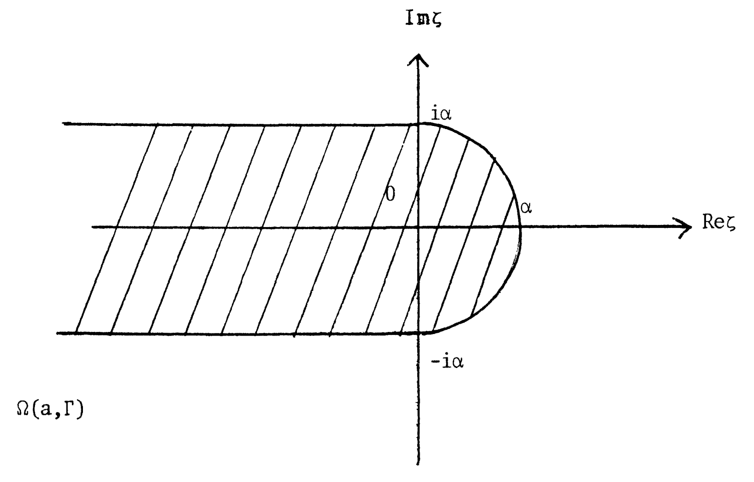


Figure 1.

Example 2 (see [8]) If $a(z)=|z|\{\cos\varphi\log(2\cos\varphi)+\varphi\sin\varphi\}$ where $z=|z|e^{i\varphi}$ $(-\pi/2<\varphi<\pi/2)$, then $\Omega(a:\Gamma)=\{\zeta\in\mathbb{C}:|e^{\zeta}-1|<1\}$. (see Figure 2)

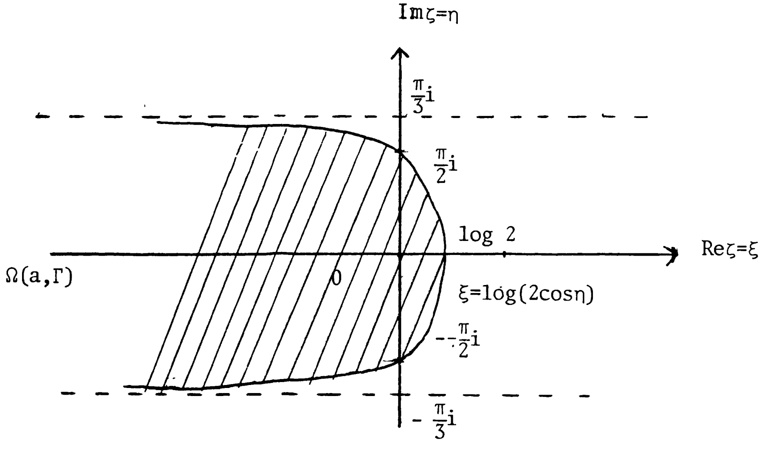


Figure 2

3. The Avanssian-Gay transform of analytic functionals with unbounded carrier

In [3], the Avanissian-Gay transform is introduced for analytic functionals with compact carrier, while in [6] and [10] it has been generalized to the case of analytic functionals with unbounded carrier. According to [3], [6] and [10]. Let us first recall the definition of the Avanissian-Gay transform of analytic functionals with unbounded carrier.

Assume that the closed convex set L is bounded in the imaginary direction and also bcunded below in the real direction. More precisely, we assume there exist $a_i\in R$ $(1\leq i\leq n)$ and compact sets K_i $(1\leq i\leq n)$ having a width less than 2π such that

$$L\subset \prod_{i=1}^{n}(a_i+R_+ +\sqrt{-1}K_i)$$

where $R_+=[0,\infty)$

Furthermore let $T\in Q'(L:\{0\})$; then the Avanissian-Gay transform $G_T(w)$ of T is given by

$$G_T(w) = \langle T_\zeta, \prod_{i=1}^{n} \frac{1}{1-w_i e^{\zeta_i}} \rangle$$

Some properties of $G_T(w)$ are listed in

Proposition 1. (see [3], [6], [10])

(1) $$G_T(w) \in O(\prod_{i=1}^{n} \{\mathbb{C} \setminus \exp(-L_i)\})$$

where L_i is i-th projection of L ($1 \leq i \leq n$)

(2) $$G_T(w) = (-1)^n \sum_{m \in N^n} T(-m) w_1^{-m_1} \ldots w_n^{-m_n} \qquad (|w_i| > e^{-a_i})$$

where $m = (m_1, \ldots, m_n) \in N^n$.

(3) Let $K_i = [k_1^{(i)}, k_2^{(i)}]$ with $k_2^{(i)} - k_1^{(i)} < 2\pi$, $1 \leq i \leq n$.

Then for all $\varepsilon > 0$ and $\varepsilon' > 0$, there exists a constant $C_{\varepsilon,\varepsilon'} > 0$, such that

$$|G_T(w)| \leq C_{\varepsilon,\varepsilon'} |w_1|^{-\varepsilon'} \ldots |w_n|^{-\varepsilon'}$$

$(\varepsilon - k_1^{(i)} \leq \arg \omega_i \leq 2\pi + \varepsilon - k_2^{(i)}; 1 \leq i \leq n)$ $\qquad 1 \leq i \leq n)$

(4)

$$\langle T, h \rangle = \left(\frac{1}{2\pi i}\right)^n \int_{\Gamma_1 x \ldots x \Gamma_n} G_T(e^{-\zeta_1}, \ldots, e^{-\zeta_n}) h(\zeta) d\zeta_1 \ldots d\zeta_n$$

for all $h \in Q(\prod_{i=1}^{n} (a_i + R_+ + \sqrt{-1} K_i) : \{0\})$, hereby $\Gamma_i = \partial(a_i + R_+ + \sqrt{-1} K_i)$

Moreover

$$T(z) = \left(\frac{1}{2\pi i}\right)^n \int_{\Gamma_1 x \ldots x \Gamma_n} G_T(e^{-\zeta_1}, \ldots e^{-\zeta_n}) e^{\zeta z} d\zeta_1 \ldots d\zeta_n$$

$$= \left(\frac{-1}{2\pi i}\right)^n \int G_T(w_1, \ldots w_n) w_1^{-z_1-1} \ldots w_n^{-z_n-1} \, dw_1 \ldots dw_n$$

$$\partial \exp(-\Gamma_1) x \ldots x \exp(-\Gamma_n)$$

4. Transfinite diameter and the Martineau-Šeinov theorem about Laurent series of several complex variables

In this section we recall the definition of the transfinite diameter of a compact set K in the complex plane and the Martineau-Šeinov theorem about Laurent series for functions of several complex variables.

Let K be a compact set in the complex plane and put

$$V_n = \max_{\substack{z_i \in K \\ z_j \in K}} \prod_{1 \leq i < j \leq n} |z_i - z_j|$$

Then it is well known that $\tau(K) = \lim_{n\to\infty} V_n^{\frac{2}{n(n-1)}}$ exists for any compact $K \subset \mathbb{C}$ and it is called the transfinite diameter of K (see [1] and [12]).

Some properties of the transfinite diameter of a compact set K are listed in

Proposition 2. Let K_i $(i=1,2)$ be compact subsets of $\mathbb{C}$.

(1) $K_1 \subseteq K_2 \Rightarrow \tau(K_1) \leqq \tau(K_2)$

(2) $\tau(K_1) \leqq \frac{1}{2\pi}(\text{length of } \partial K_1)$

Some examples of transfinite diameters are now given (see [1] and [12]).

Example 3. If $K = \{z \in \mathbb{C} : |z| = r\}$, then $\tau(K) = r$.

Example 4. If $K = \{z \in \mathbb{C} : |z| = r,\ |\arg z| \leqq \alpha\}$, then $\tau(K) = r \sin\frac{\alpha}{4}$.

Example 5. If $K = [a,b]$, $a,b \in \mathbb{R}$, then $\tau(K) = \frac{b-a}{4}$

Theorem 3. (Martineau [5] and Šeinov [11]). Suppose that G(w) is holomorphic in $\prod_{j=1}^{n} (\mathbb{C} \setminus F_j)$, where F_j is a polynomially convex compact set and $\tau(F_j) < 1$ for all j $(1 \leqq j \leqq n)$. Suppose furthermore that G(w) has the following Laurent expansion at infinity

$$G(w) = \sum_{v \in \mathbb{N}^n} \frac{a_v}{w^v} \qquad (a_v \in \mathbb{Z}).$$

Then

$$G(w) = \frac{A(w_1,\dots,w_n)}{B_1(w_1)\dots B_n(w_n)}$$

where $A(w_1,\dots, w_n) \in Z[w_1,\dots, w_n]$, $B_i(w_i) \in Z[w_i]$ and $B_i(w_i)$ are monic polynomial.

Remark 2. In theorem 3, the assumption $\tau(F_j)<1, 1\leq i\leq n$ is crucial. For instance, if n = 1 and

$$G(w) = \sum_{k=1}^{\infty}\binom{2k}{k}w^{-k} = \sum_{k=1}^{\infty}\frac{(2k)!}{(k!)^2}w^{-k}$$

$$=\sqrt{\frac{w}{w-4}}-1,$$

then G(w) is holomorphic in the outside of the interval $[0.4]$ In view of Example 5, $\tau([0,4])=1$ and obviously G(w) is not a rational function.

5. Proof of Theorem 1.

In this section, we give the proof of Theorem 1, it is inspired by Avanissian and Gay [3].

Proof of Theorem 1.

By means of Theorem 2, there exists an analytic functional T, which is carried by L and of type {0}, such that $f(z)=\langle T_\zeta, \exp(\zeta z)\rangle=\tilde{T}(Z)$.

From the assumption, L is contained in $\prod_{i=1}^{n}\{\zeta_i : |e^{\zeta}-1|<1\}$.

Now consider the analytic functional $\check{T}$ defined as follows

$$\langle\check{T},h\rangle=\langle T_\zeta, h(-\zeta)\rangle, \quad h \in Q(-L:\{0\}).$$

Obviously, T is carried by (-L) and of type {0}.

From Proposition 1-(2), we get :

$$G_{\check{T}}(w)=(-1)^n \sum_{m\in N^n}\check{T}(-m)w_1^{-m_1}\dots w_n^{-m_n}$$

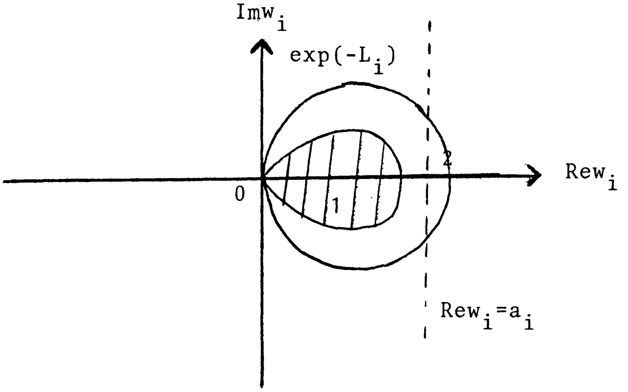
$$=(-1)^n \sum_{m\in N^n}T(m)w_1^{-m_1}\dots w_n^{-m_n}$$

$$=(-1)^n \sum_{m\in N^n} f(m) w_1^{-m_1},\dots w_n^{-m_n}$$

Remark that by means of the second assumption in Theorem 1, all $f(m)$, $m\in N^n$, belong to Z.
In virtue of Proposition 1-(1), $G_T^\vee$ is holomorphic in

$\prod_{i=1}^{n} \{C\setminus \exp(L_i)\}$. From the assumption upon L, $\exp(L_i)$ is contained

in $\{w_i\in C: |w_i-1|<1\}\cup\{0\}$.
So there exist $a_i>0$ $(1\leqq i\leqq n)$ such that $\exp(L_i)$
$\subset \{w_i\in C: |w_i-1|\leqq 1\}\cap\{w_i\in C: \mathrm{Re}\,w_i \leqq a_i\}$.(see Figure 3).



Call $F_i=\{w_i: |w_i-1|<1\}\cap\{w_i: \mathrm{Re}\,w_i\leqq a_i\}$, $1\leqq i\leqq n$.

By virtue of Proposition 2-(2) $\tau(F_i)<1$. Therefore $\tau(\exp(L_i))<1$.
Accordingly we can conclude that

$$G_T^\vee(w)=\frac{A(w_1\dots,w_n)}{B_1(w_1)\dots B_n(w_n)}$$

where $A(w_1, \dots, w_n)\in Z[w_1,\dots, w_n]$ and $B_i(w_i)$ are monic polynomials with integral coefficients.
The roots of $B_i(w_i)$ are algebraic integers which are contained in $\{w:\in C: |w_i-1|<1\}\cup\{0\}$ together with all their conjugate algebraic integers. But, in virtue of Proposition 1-(3), zero is not a root of $B_i(w_i)$ so that by means of C.R. Buck's lemma (See 3.2.5) in [3]), we can conclude that

$$B_i(w_i)=(w_i-1)^{m_i} \qquad (1\leqq i\leqq n).$$

Now, using the inversion formula of Proposition 1-(4),

$$f(-z)=(\tilde{\check{T}})(Z)=\left(\frac{-1}{2\pi i}\right)^n \int_{\partial(\exp(L_1))\times\dots\partial(\exp(L_n))} G_T^{\vee}(w_1,\dots,w_n)w_1^{-Z_1-1},\dots w_n^{-Z_n-1}\, dw_1\dots dw_n$$

$$=\left(\frac{-1}{2\pi i}\right)^n \int_{\partial(\exp(L_1))\times\dots\partial(\exp(L_n))} \frac{A(w_1,\dots, w_n)}{(w_1-1)^{m_i}\dots(w_n-1)^{m_i}}\, w_1^{-Z_1-1}\dots, w_n^{-Z_n-1}\, dw_1\dots dw_n$$

whence, by means of the residue theorem

$$f(-z)=P(z_1,\dots, z_n),\text{ a polynomial in } z_1,\dots, z_n.$$

But as $A(w_1,\dots, w_n)$ belongs to $Z[w_1,\dots, w_n]$, the coefficients of $P(z_1,\dots, z_n)$ are rational numbers.
Hence $f(z)$ is a polynomial with rational coefficients.
Finally, we give two examples.

Example 6. Suppose that $f(z)$ is holomorphic in the right half plane $\{z\in C:Re z>0\}$ and satisfies the following conditions :

$$|f(z)|\leqq C e^{\alpha|z|} \qquad (Re z>0) \qquad (1)$$

$$f(n)\in Z,\ n\in N. \qquad (2)$$

Since $a(z)=\alpha|z|$ and $\Gamma=\{z\in C:Re z>0\}$, $L=\Omega(a:\Gamma)$ is the same as in Example 1.
Therefore, if $\alpha<\log 2$, $f(z)$ is a polynomial with rational coefficients.

Example 7. Put $f(z)=\dfrac{1}{B(z,z)}=\dfrac{\Gamma(2z)}{\Gamma(z)^2}$,

where B and Γ are respectively the Bêta and Gamma functions.
This function has the following properties :

(1) $f(z)$ is holomorphic in $\{z:Re z>-\frac{1}{2}\}$, and hence also in $\{z:Re z>0\}$.

(2) By virtue of Stirling's formula, for any $\varepsilon>0$, there exists a constant $C_\varepsilon>0$, such that

$$|f(z)| \leq C_\varepsilon e^{(2\log 2)x+\varepsilon|y|}$$

and this for all $z=x+iy\in\{z:\mathrm{Re}z>0\}$.

(3) $f(n) = \binom{2n}{n}\in Z$.

In this case $\Gamma=\{z\in C:\mathrm{Re}z>0\}$ and $A(z)=2(\log 2)x$.
So $\Omega(a:\Gamma)=\{\xi\in R:\xi\leq 2\log 2\}$.
Since $L=\Omega(a:\Gamma)$ is not contained in $\{\zeta\in C:|e^{\zeta}-1|<1\}$ this function f(z) is not a polynomial.

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