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ON THE REPRESENTATION OF NORM ATTAINING POSITIVE OPERATORS ON $L^p[0,1]$

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Let $([0,1], \mathcal{B}, m)$ denote the unit interval, the Lebesgue measurable sets, and Lebesgue measure, respectively. We denote by L^p, $1 \leq p < \infty$, the space of all real-valued Lebesgue measurable functions on $[0,1]$ whose absolute p-th power are integrable. By $\mathcal{L}(L^p, L^r)$ we denote the Banach space of all bounded linear operators from L^p into L^r. An operator T is said to be positive, $T \geq 0$ if $Tf \geq 0$ for all $f \geq 0$.

A representation for operators on L^1 has been established by Kantorovič and Vulikh [5]. Using this result Ryff [8] presented the representation theorem for doubly stochastic operators.

An operator $T \in \mathcal{L}(L^p, L^r)$ is called a pseudo-integral operator if there is a map $y \longrightarrow \mu_y$ of $[0,1]$ into the space of bounded Borel measures on $[0,1]$ such that

1° if $B \in \mathcal{B}$ and $m(B)=0$, then $\mu_y(B)=0$ a.e.

2° for every $B \in \mathcal{B}$, the functions $y \longrightarrow \mu_y(B)$, $y \longrightarrow |\mu_y|(B)$ are Borel measurable

3° $L^p \subset L^1(|\mu_y|)$ for almost every $y \in [0,1]$

and

$$(Tf)(y) = \int f(x) \, \mu_y(dx) \qquad \text{a.e.}$$

for every f L^p. An operator T is a pseudo-integral operator if and only if T is order-continuous i.e. $0 \leq f_n \leq f \in L^p$ and $f_n \rightarrow 0$ a.e. implies $Tf_n \rightarrow 0$ a.e.. The pseudo-integral operators form a band (order-closed ideal) in the space of order -bounded operators. If $T \in \mathcal{L}(L^p, L^r)$ is positive, then T is a pseudo-integral operator (see Sourour [12]).

An operator $T \in \mathcal{L}(L^p, L^r)$ is called an integral operator, if there exists a measurable function $T(x,y)$ such that

$$(Tf)(y) = \int T(x,y) \, f(x) \, dx \qquad \text{a.e.}$$

for every $f \in L^p$. An operator $T \in \mathcal{L}(L^p, L^r)$ is an integral operator if and only if T maps order intervals into equimeasurable sets (Schachermayer [10], see also Schep [11]) . We recall that a set $H \subset L^r$ is called equimeasurable if for all $\varepsilon > 0$ there exists X_1 with $m([0,1] \setminus X_1) < \varepsilon$ such that $\{1_{X_1} h : h \in H\}$ is a relatively compact subset of L^∞ (cf.[2]).

The support of positive operator $T \in \mathcal{L}(L^p, L^r)$, supp T , is a maximal set $A \subset [0,1]$ (modulo zero Lebesgue measure sets) such that $T1_{A^c} = 0$ (cf. [3]).

Let

$$\mathcal{N} = \{T \in \mathcal{L}(L^p, L^r) : T \geq 0,\ T \text{ attains its norm at some } f \in L^p \text{ with } \operatorname{supp} f = \operatorname{supp} T\} .$$

Thus $\mathcal{N}$ is the set of all positive operators T such that there exists a function f of full support and T attains its norm at f. If $1 < r \leq p < \infty$,then the set $\mathcal{N}$ is norm dense in the positive part of $\mathcal{L}(L^p, L^r)$ ([4],Proposition 2) . The proof of this fact is a modification of Lindenstrauss's result [7] .

The purpose of this paper is to present a representation theorem for positive operators which attain their norm at a function of full support i.e. for operators from the set $\mathcal{N}$. For our aim we carry Ryff's representation of doubly stochastic operators to the case of positive norm attaining operators. The same method was been used to obtain certain properties of positive norm attaining operators on L^p ([3]) and the characterization of extreme positive contractions on l^p_n ([4]) .

We denote

$$M(T) = \{f : \|Tf\| = \|T\| \, \|f\|\} .$$

Note that if $T \geq 0$, then $f \in M(T)$ implies $|f| \in M(T)$, and the set $M(T)$ form a linear subspace of L^p if $1 < r \leq p < \infty$ ([3]).

Let $0 \leq f \in L^p$, $0 \leq g \in L^r$ be such that $\|f\| = \|g\| = 1$. We define $\tau : [0,1] \longrightarrow [0,1]$ and $\sigma : [0,1] \longrightarrow [0,1]$ by

$$\tau(x) = \int_0^x f^p \, dm \qquad\qquad \sigma(x) = \int_0^x g^r \, dm$$

The restricted mappings $\tau|_{\operatorname{supp} f}$ and $\sigma|_{\operatorname{supp} g}$ are increasing and onto $[0,1]$,thus invertible (modulo null sets).

<u>Theorem.</u> Let $1 < r \leq p < \infty$. A positive operator $T \in \mathcal{L}(L^p, L^r)$ with $\|T\| = 1$, supp $T = [0,1]$ is in $\mathcal{N}$ if and only if T admits a representation

$$(*)\qquad (Th)(y)=\begin{cases}\dfrac{1}{g^{r-1}(y)}\dfrac{d}{dy}\displaystyle\int_0^1 L(y,x)\,h(x)\,dx & y\in \operatorname{supp} g\\[2ex] 0 & y\notin \operatorname{supp} g\end{cases}$$

where $f\in M(T)$ is such that $\|f\|=1$, $f\geqslant 0$, $\operatorname{supp} f=[0,1]$, $g=Tf$, the kernel L is measurable and satisfies the following conditions:

a/. $L(0,x)=0$

b/. $L(y_1,\cdot)\leqslant L(y_2,\cdot)$ if $y_1<y_2$

c/. $L(1,x)=f^{p-1}(x)$

d/. $\int_0^1 L(y,x)\,f(x)\,dx = \sigma(y)$

e/. $\int_0^1 L(\sigma^{-1}(s),x)\,h(x)\,dx$ as a function of $s\in \operatorname{supp} g$ is absolutely continuous for every $h\in L^p$.

Proof. Let $T\in \mathcal{N}$ with $\|T\|=1$, $T\geqslant 0$. Let $f\in M(T)$ be such that $\|f\|=1$, $f\geqslant 0$, $\operatorname{supp} f=[0,1]$. Put $g=Tf$. Note that $\|Tf\|=1$, $Tf\geqslant 0$, $T^*(Tf)^{p-1}=f^{p-1}$ and $(Tf)^{r-1}\in M(T^*)$ (see [3]).

The operator

$$P = V\,T\,U$$

where
$$(U\,h)(x)=f(x)\,h(\tau(x)) \qquad ,h\in L^p$$
$$(V\,k)(s)=\frac{k(\sigma^{-1}(s))}{g(\sigma^{-1}(s))} \qquad ,k\in L^r$$

is doubly stochastic (i.e. $P\geqslant 0$, $P1=1$, $P^*1=1$). The operator U is an isometry on L^p and V is a coisometry on L^r (see [6]).

By the result of Ryff [8] we have

$$(Ph)(s) = \frac{d}{dt}\int_0^1 K(s,t)\;h(t)\,dt$$

where K is measurable on $[0,1]\times[0,1]$ and satisfies :

1/. $K(0,t)=0$

2/. $\int_0^1 K(\cdot,t)\,h(t)\,dt$ is absolutely continuous for every $h\in L^1$

3/. $s=\int_0^1 K(s,t)\,dt$

4/. $K(s_1,\cdot)\leqslant K(s_2,\cdot)$ if $s_1<s_2$

5/. $K(1,t)=1$.

Using the above representation we obtain

$$(Th)(y) = \begin{cases} \dfrac{1}{g^{r-1}(y)} \dfrac{d}{dy} \displaystyle\int_0^1 K(\sigma(y), \tau(x))\, f^{p-1}(x) h(x)\, dx & y \in \operatorname{supp} g \\ 0 & y \notin \operatorname{supp} g \end{cases}$$

and we get (*) putting $L(y,x) = K(\sigma(y), \tau(x))\, f^{p-1}(x)$. Clearly 1/. , 4/. , and 5/. implies a/. , b/. , and c/. Using 4/. we get

$$\int_0^1 L(y,x) f(x)\, dx = \int K(\sigma(y), \tau(x)) f^p(x)\, dx = \int K(\sigma(y),t)\, dt = \sigma(y).$$

If $h \in L^p$, then by Hölder's inequality, $\dfrac{h(\tau^{-1}(t))}{f(\tau^{-1}(t))} \in L^1$. Hence by 3/. a function

$$\xi(s) = \int_0^1 L(\sigma^{-1}(s),x) h(x)\, dx = \int_0^1 K(s, \tau(x)) f^{p-1}(x) h(x) dx =$$

$$\int_0^1 K(s,t) \frac{h(\tau^{-1}(t))}{f(\tau^{-1}(t))}\, dt \quad \text{is absolutely continuous.}$$

Now let $0 \le f \in L^p$, $0 \le g \in L^r$ be such that $\|f\| = \|g\| = 1$, and suppose a measurable function $L(y,x)$ satisfies conditions a/. — e/. . It is not diffcult to see that a function $K(s,t)$ such that $L(y,x) = K(\sigma(y), \tau(x)) f^{p-1}(x)$ satisfies 1/. — 5/. and that the operator $(Ph)(s) = \frac{d}{ds} \int K(s,t)\, h(t)\, dt$ is doubly stochastic. We define T by (*).

Let $h \in L^p$. We have $\| Th \|_r^r =$

$$\int_{\operatorname{supp} g} \left| \frac{1}{g^r(y)} \frac{d}{dy} \int_0^1 L(y,x)\, h(x)\, dx \right|^r g^r(y)\, dy =$$

$$\int_{\operatorname{supp} g} \left| \frac{1}{g^r(y)} \frac{d}{dy} \int_0^1 K(\sigma(y), \tau(x))\, f^{p-1}(x) h(x)\, dx \right|^r g^r(y)\, dy =$$

$$\int_0^1 \left| \frac{d}{ds} \int_{\operatorname{supp} f} K(s, \tau(x))\, f^{p-1}(x)\, h(x)\, dx \right|^r ds =$$

$$\int_0^1 \left| \frac{d}{ds} \int_0^1 K(s,t) \frac{h(\tau^{-1}(t))}{f(\tau^{-1}(t))}\, dt \right|^r ds =$$

$$\int_0^1 \left| P\left(\frac{h(\tau^{-1}(t))}{f(\tau^{-1}(t))} \right) \right|^r ds \leqslant \int_0^1 P\left(\left| \frac{h \; \tau^{-1}(t)}{f \; \tau^{-1}(t)} \right|^r \right) ds =$$

$$\int_0^1 \frac{d}{ds} \int_0^1 K(s,t) \left| \frac{h(\tau^{-1}(t))}{f(\tau^{-1}(t))} \right|^r dt\, ds =$$

$$\int_0^1 \left[K(1,t) - K(0,t)\right] \left|\frac{h(\tau^{-1}(t))}{f(\tau^{-1}(t))}\right|^r dt = \int_0^1 \left|\frac{h(\tau^{-1}(t))}{f(\tau^{-1}(t))}\right|^r dt =$$

$$\int_0^1 |h(x)|^r \, f^{p-r}(x)\,dx \leqslant \|h\|_r^r \, \|f\|_p^{p-r} = \|h\|_r^r \quad .$$

The first inequality follows from properties of doubly stochastic operators. The second inequality is a consequence of Hölder's inequality. Hence $T \in \mathcal{L}(L^p, L^r)$ and $\|T\| \leqslant 1$. Clearly $Tf=g$, $\|T\|=1$, $f \in M(T)$.

Remark. It is easy to see that, we can write an analogous theorem without the assumption $\operatorname{supp} T = [0,1]$.

A doubly stochastic operator P can be represented by

$$(Ph)(s) = \int_0^1 h(t) \; p(s,dt)$$

where a function $p(\cdot,\cdot) : [0,1] \times \mathcal{B} \longrightarrow \mathbb{R}_+$ satisfies

(i') for each $s \in [0,1]$ $p(s,\cdot)$ is a probability measure on $\mathcal{B}$

(ii') for each $B \in \mathcal{B}$ $p(s,B)$ is a measurable function of s

(iii') $\int_0^1 p(s,B)\,dx = m(B)$, $B \in \mathcal{B}$

(see, e.g. [1]) .

If $1 < r \leqslant p < \infty$, $T \in \mathcal{N}$ with $\|T\|=1$, $T \geqslant 0$, then there exist $0 \leqslant f \in L^p$ and $0 \leqslant g \in L^r$ such that $\|f\| = \|g\| = 1$, $f \in M(T)$, $g=Tf$, $\operatorname{supp} T = \operatorname{supp} f$. Using arguments similar to those in the proof of Theorem we obtain

$$(**) \qquad (Th)(y) = \int_{\operatorname{supp} f} \frac{h(x)}{f(x)} \; q(y,dx)$$

where a function $q(x,B) : [0,1] \times \mathcal{B} \longrightarrow \mathbb{R}_+$ satisfies the conditions:

(i) for each $y \in [0,1]$ $q(y,\cdot)$ is a probability measure on

(ii) for each $B \in \mathcal{B}$ $q(y,B)$ is a measurable function of y

(iii) $\int_0^1 g^r(y) \; q(y,B)\,dy = \int_B f^p(x)\,dx$.

Conversely, if we have $f \in L^p$, $g \in L^r$ with $\|f\| = \|g\| = 1$, $f \geqslant 0$, $g \geqslant 0$ and a function $q(x,B)$ satisfies (i)—(iii) then the formula (**) define $T \in \mathcal{N}$ such that $\|T\|=1$, $f \in M(T)$, $\operatorname{supp} T = \operatorname{supp} f$.

Let μ, ν be probability measures on $([0,1], \mathcal{B})$. We say that a measure λ defined on $([0,1]\times[0,1], \mathcal{B}\otimes\mathcal{B})$ is doubly stochastic with respect to μ and ν if

$\lambda(A\times[0,1]) = \nu(A)$ and $\lambda([0,1]\times B) = \mu(B)$

$A,B\in\mathcal{B}$. The relation

$$\lambda(A\times B) = \int 1_A P1_B \quad dm$$

determines a one-to-one correspondence between the set of all doubly stochastic measures with respect to m and m ([1] , see also [9]). Therefore , analogously, for every $T\in\mathcal{N}$ with $\|T\|=1$ the formula

$$(***) \qquad \lambda(A\times B) = \int 1_A g^{r-1} T(1_B f) \; dy$$

defines a doubly stochastic measure with respect to μ and ν , where $f\in M(T)$, $\|f\|=1$, $f\geqslant 0$, supp f = supp T , $g=Tf$, $d\mu=f^p\,dm$, $d\nu=g^r\,dm$.

Conversely, let $0\leqslant f\in L^p$, $0\leqslant g\in L^r$ with $\|f\|=\|g\|=1$. We define probability measures μ , ν by $d\mu=f^p\,dm$, $d\nu=g^r\,dm$. Let λ be a doubly stochastic measure with respect to μ and ν. Then (***) determines an operator T in $\mathcal{N}$ such that $\|T\|=1$, $f\in M(T)$, supp T = supp f .

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