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On strongly connected T_0 spaces

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ON STRONGLY CONNECTED T_O SPACES (1)

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In questa nota individuiamo classi di spazi T_0 non-T_1 aventi espansioni connesse massimali T_1 in alcuni casi,non-T_1 in altri.Viene individuata anche un'ampia classe di spazi T_0 connessi che non hanno espansioni connesse massimali T_1.

We recall that a topological space (S,τ) is *maximal connected* if it is connected and every strictly finer topology $\tau' \supsetneq \tau$ is a disconnected topology.

(S,τ) is *strongly connected* if it is connected and a maximal connected topology τ' exists which is finer than τ.

Several authors (see [4],[6],[7],[8],[13],[17],[18]) have studied properties of maximal connected spaces and have given necessary conditions for (S,τ) to be maximal connected,sometimes assuming that (S,τ) has further topological properties. A sufficient condition for (S,τ) to be a maximal connected space can be found in [8].The only characterization of maximal connectedness we know can be found in[17] and it concerns topologies in which arbitrary intersections of open subsets are open too.

Frequently maximal connectedness has been investigated in connection with separation axioms.

It is well known that all maximal connected spaces are T_0 but not necessarily T_1. Thomas [17] provided examples of T_1 maximal connected spaces while Guthrie and Stone [9],Wage [9],Simon [14] and El'kin [4] proved that T_2 maximal connected topological spaces exist;in particular the Euclidean topology on the reals is strongly connected [9],[14] and maximal connected Hausdorff spaces (S,τ) of arbitrary infinite cardinality can be found which have dispersion character $\Delta(S) = |S|$ [4].

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On the other hand there exist connected Hausdorff spaces that are not strongly connected [1],[8] ;a class of such spaces is provided by connected T_2 spaces with a dispersion point.

If S is a finite set,it is easily seen that each connected topology on S is strongly connected [17];furthermore it is trivially true that every maximal connected topology on S is T_0 but not T_1 if S has finite cardinality greater than one.

Here we shall single out a class of connected T_0 topologies on an infinite set S which have no finer T_1 (maximal) connected topologies;moreover we shall determine some class of strongly connected T_0 spaces with infinitely many points,whose topologies have,respectively,a finer T_0 or T_1 maximal connected topology.

All the spaces we shall consider have infinitely many points and verify the T_0 axiom.

Given a topological space (S,τ) and a point $x \in S$,we shall denote by $\mathfrak{J}(x)$ the filter of neighbourhoods of x and by $\tau(x)$ its local base of open subsets containing x. clX (intX) will denote the closure (the interior) of $X \subseteq S$ with respect to τ while,if τ' is any other toplogy on S, we shall denote by $cl_{\tau'}X$ ($int_{\tau'}X$) the closure (the interior) of X in τ'. We shall denote by CX the complement S - X of X in S.

We shall call (proper) expansion of τ each topology on S which is strictly finer than τ.

If $X \subseteq S$ is a(non-open) subset of S,the topology $\tau(X) = \{A \cup (A' \cap X) \ / \ A,A' \in \tau\}$ generated by $\tau \cup \{X\}$ is (strictly) finer than τ;such a topology,already considered in [10],was first called simple extension of τ by X in [12] and later simple expansion of τ by X in [6]; infinite expansions were considered in [2].

It is easily seen [3] that a connected space (S,τ) is maximal connected iff the simple expansion $\tau(X)$ is disconnected whenever X is a non-open subset of S.

Now let us denote by $\tau°$ the family of non-trivial open subsets of the topological space (S,τ).

DEFINITION 1. We say that (S,τ) is a hyperconnected topological space if any two non-trivial open subsets in τ have a non-empty intersection.

A hyperconnected space (S,τ) is called principal if $\bigcap \tau° \neq \emptyset$,non-principal otherwise.

DEFINITION 2. A space (S,τ) is ultraconnected if any two non-trivial closed subsets in τ have a non-empty intersection.

An ultraconnected space is called principal if $\bigcap_{A\in\tau°} CA \neq \emptyset$,non-principal otherwise.

It is well known that every hyperconnected or ultraconnected space is connected but is not Hausdorff,and that no ultraconnected space is a T_1 space.

Furthermore no principal hyperconnected space is T_1 and each T_0 principal hyperconnected (ultraconnected) space contains just one point x_0 that belongs to all non-trivial open (closed) subsets.

Now we note that each T_0 non-T_1 space has a MNT_1 (maximal non-T_1) expansion (see [5],[16]); on the other hand all T_0 maximal non-T_1 spaces, which are antiatoms in the lattice of topologies on a set S, are disconnected spaces if S has at least three points.

Given a T_0 non-T_1 topology τ on S and a T_1 expansion $\tau' \geq \tau$, then a MNT_1 topology exists between τ and τ' iff τ' is the disbrete topology δ. Hence every connected non-T_1 topology has a disconnected non-T_1 expansion. The converse is not necessarily true; indeed, if we denote by ν the cofinite topology, a connected T_0 non-T_1 space (S,τ) has a connected T_1 expansion iff $\tau\vee\nu$, the least T_1 topology containing τ, is connected.

So let us give the following definition.

DEFINITION 3. (S,τ) is a T_1-disconnected space if $\tau\vee\nu$ is a disconnected topology on S.

Trivially all disconnected spaces are T_1-disconnected and a T_1 space is T_1-disconnected iff it is disconnected.

PROPOSITION 1. Let (S,τ) be a connected space.
(S,τ) is T_1-disconnected iff τ has non-empty finite open sets.

Proof. The open sets of $\tau\vee\nu$ can be written as follows

$$\bigcup_{i\in I} \dot{A}_i \quad \text{where } A_i \in \tau ,\ \dot{A}_i \subseteq A_i \text{ and } A_i - \dot{A}_i \text{ is finite.}$$

Assume that (S,τ) is T_1-disconnected and let $P,Q \in \tau\vee\nu$ be non-empty open subsets such that $P \cap Q = \emptyset$, $P \cup Q = S$.

If $P = \bigcup_{i\in I} \dot{A}_i$ and $Q = \bigcup_{j\in J} \dot{B}_j$ then the subsets $A = \bigcup_{i\in I} A_i$ and $B = \bigcup_{j\in J} B_j$ are non-empty open subsets in τ and they cover S.

Moreover it follows from $\emptyset = P\cap Q = \bigcup_{i,j} (\dot{A}_i \cap \dot{B}_j)$ that $\dot{A}_i \cap \dot{B}_j = \emptyset$ for all i,j and consequently each intersection $A_i \cap B_j$ is a finite subset of S. On the other hand $i' \in I$ and $j' \in J$ exist such that $A_{i'} \cap B_{j'} = \emptyset$: otherwise we should have $A \cap B = \emptyset$ and (S,τ) would be disconnected.

$A_{i'} \cap B_{j'}$ is actually the finite open subset we had to find in τ.

Conversely, if U is a non-empty finite open subset in τ, then $CU \in \nu$ and the open sets U, $CU \in \tau\vee\nu$ form a non-trivial subdivision of $(S,\tau\vee\nu)$.

COROLLARY 1. Every T_0 non-T_1 connected space has a connected expansion which is T_1-disconnected.

Proof. If $x \neq y$ are two points in S such that $\tau(x) \subseteq \tau(y)$, then $\tau(\{y\})$ must be connected : otherwise we could find two open sets $A,B\in\tau$ which determine the non-trivial open subdivision $A\cup\{y\}$, B of S in $\tau(\{y\})$;it would follow from $x\in A\cup B$ that $y\in A\cup B$ and consequently $A\cup B = S$ and $A\cap B = \emptyset$, a contradiction. The statement is now an immediate consequence of proposition 1.

COROLLARY 2. No non-principal hyperconnected space is T_1*-disconnected.*

Proof. It follows trivially from proposition 1 since a non-principal hyperconnected space has no finite open subset.

We remark that no T_0 connected T_1-disconnected space has a maximal connected T_1 expansion.

Moreover all T_0 non-T_1 maximal connected spaces are T_1-disconnected (by corollary 1) and the class of T_0 connected spaces which have no T_1 connected expansion contains some principal hyperconnected spaces (Prime ideal topology) and some non-principal ultraconnected spaces (Divisor topology) but contains no non-principal hyperconnected space (see [15] for the topologies we mentioned above and for further examples we shall refer to later).

Finally we remark that each connected non-T_1 topology is coarser than a connected non-T_1 topology which has no T_1 maximal connected expansion.

Now we are going to formulate a simple condition that implies maximal connectedness.

LEMMA 1. If (S,τ) *is a connected space and each non-open subset* $X\subseteq S$ *is closed, then* (S,τ) *is maximal connected.*

Proof. Straightforward.

PROPOSITION 2. Let (S,τ) *be a* T_0 *non-*T_1 *hyperconnected space.*

If (S,τ) *is not* T_1*-disconnected, then an expansion* τ' *of* τ *exists which is a maximal connected and hyperconnected (hence maximal hyperconnected)* T_1 *non-*T_2 *topology.*

If (S,τ) *is principal hyperconnected and* $\{x\} = \bigcap\tau^\circ$ *, then the particular point topology containing the open set* $\{x\}$ *is a non-*T_1 *maximal connected and hyperconnected expansion of* τ.

Proof. Since (S,τ) is hyperconnected, we can consider an ultrafilter υ containing all non-empty open sets in τ.
The topology $\tau' = \upsilon\cup\{\emptyset\}$ is then hyperconnected and it satisfies the T_1 axiom if υ is a non-principal ultrafilter, i. e. τ' contains no non-trivial finite open subset.
If othrwise υ is a principal ultrafilter (of course τ is principal hyperconnected in this case) and x is the limit point of υ, then τ' is just a particular point top-

ology.
In both cases (S,τ) is a maximal connected space by lemma 1.
We complete the proof noticing that a non-principal ultrafilter υ containing all non-empty open sets of τ exists iff τ has no finite element.

Example 1. there are several hyperconnected spaces on an infinite set S that are described in [15] :

the Particular point topology is maximal connected (see also [17]) and consequently maximal hyperconnected;

the Overlapping interval topology and the Prime ideal topology are principal hyperconnected topologies;

the Right order topology on the real line R is a non-principal hyperconnected topology.

Now we notice that every T_0 principal ultraconnected space (S,τ) is strongly connected and strongly ultraconnected, since it is coarser than the excluded point topology whose closed point is x, where $\{x\} = \bigcap_{A\in\tau^\circ} CA$.

In order to investigate strong connectedness of a T_0 non-principal ultraconnected space (S,τ) we first remark that such a space contains no closed point and has infinitely many non-isolated points; indeed all closed sets in τ must have infinitely many points whence no point may be closed and the subset of non-isolated points, which is of course closed, is necessarily infinite.

At last the preceding remarks allow us to prove the following.

PROPOSITION 3. *If (S,τ) is a non-principal ultraconnected T_0 space, then an ultraconnected (of course non-principal and non-T_1) expansion τ' of τ exists.*

Proof. Let y be a non-isolated point in τ and $\tau' = \tau(\{y\})$. Trivially $F' \subseteq S$ is closed in τ' iff it can be written in the form $F' = F_1 \cup (F_2 - \{y\})$ for some F_1, F_2 closed subsets in τ.
Let us consider two non-empty closed sets in τ', $F' = F_1 \cup (F_2 - \{y\})$, $G' = G_1 \cup$ $\cup (G_2 - \{y\})$ and let i,j be two integers in $\{1,2\}$ such that F_i and G_j are non-empty; of course $F_i \cap G_j \neq \emptyset$ and $F_i \cap G_j \cap C\{y\} \neq \emptyset$, whence $F' \cap G' \neq \emptyset$.
So τ' is an ultraconnected expansion of τ.

COROLLARY 3. *All T_0 hyperconnected spaces are strongly hyperconnected and strongly connected, while no non-principal ultraconnected T_0 space may be either strongly ultraconnected or maximal connected.*

The excluded point topology is the only ultraconnected T_0 topology which is maximal connected.

Proof. It follows from propositions 2 and 3 since an ultraconnected expansion of a non-principal ultraconnected topology is necessarily non-principal, while a principal ultraconnected expansion of a principal ultraconnected T_0 topology is coarser than an excluded point topology.

The different behaviour of hyperconnectedness and ultraconnectedness toward maximality is illustrated by the following example.

Example 2. The Left order topology τ on $S = \{x \in Z \,/\, x > 1\}$ provides us an example of a principal hyperconnected and a non-principal ultraconnected topology that has a non-principal ultraconnected expansion, the Divisor topology, which is neither finer nor coarser then the maximal hyperconnected expansion of τ constructed in proposition 2.

Now let (S,τ) be a T_0 topological space whose isolated points form a dense subset D and such that for every non-isolated point x it results $(\bigcap \tau(x)) \cap D \neq \emptyset$; furthermore let us assume that a point x_0 exists whose neighbourhoods all contain D.

Obviously (S,τ) is a connected, non-T_1 and T_1-disconnected space.

We recall that a topological space is *submaximal* if every dense subset is open.

It is well known that τ has a submaximal connected expansion τ' (see [6]) which is non-T_1 since τ is T_1-disconnected.

Furthermore we have the following.

LEMMA 2. *The subset of isolated points of* τ' *is* D *and each subset of* $C = S - D$ *is closed in* τ'.

Proof. If x is a non-isolated point in τ, then every $U \in \tau(x)$ intersects D. Since $\tau' = \tau(\{A_i \subseteq S \,/\, A_i \supseteq D\})$ we have that every $U' \in \tau'(x)$ intersects D and consequently x is not isolated in τ'.
Moreover every subset of S containing D is dense in τ' and then it is open since τ' is submaximal.

Now we set $C_0 = C - \{x_0\}$ and we consider a function $f : C_0 \to D$ such that $f(x) \in \bigcap \tau(x)$ for every $x \in C_0$.

If we construct the expansion τ'' of τ' by the family of subsets $\{\{x, f(x)\}/x \in C_0\}$ then we have the following.

PROPOSITION 4. (S,τ'') *is a maximal connected non-*T_1 *space.*

Proof. If A and B are two non-trivial open sets in τ'' which cover S and if $x_0 \in B$, then we have $D \subseteq B$ and consequently A and B are not disjoint sets since D is dense in τ''. So τ'' is a connected topology and it is non-T_1 since τ was T_1-disconnected.

Now let $X \subseteq S$ be non-open in τ''; of course we have $X \cap C \neq \emptyset$.
If $x_0 \notin X$, then a point $x \in X \cap C$ exists such that $f(x) \notin X$ and $\tau''(X)$ is not connected since $\{x\} = \{x, f(x)\} \cap X$ is open in $\tau''(X)$.
If $x_0 \in X$, then $D - X$ is non-empty since all subsets containing $D \cup \{x_0\}$ are open in τ' (and consequently in τ''). We have obviously $U = D - X \in \tau$ and , if $Y = f^{-1}(X \cap D)$, $V = (C_0 - Y) \cup f(C_0 - Y) \in \tau''$ whence $A = U \cup V \in \tau''$.
Since $B = S - A = Y \cup X$ belongs to $\tau''(X)$ we realize that A and B form an open non-trivial subdivision of S in $\tau''(X)$ i. e. $\tau''(X)$ is not connected.

Proposition 4 assures that the class of topological spaces to which it referred contains only T_0 strongly connected spaces which have at least one maximal connected non-T_1 expansion.

The following example shows that such a class contains topological spaces that are neither hyperconnected nor ultraconnected.

Example 3. Let S be the union of two disjoint infinite sets $S = N \cup N'$ and let τ be the topology on S whose open sets are the subsets of S that are contained or contain N'.

Surely (S,τ) is a T_0 submaximal connected T_1-disconnected space.

If we fix any point $x_0 \in N$ and put $N_0 = N - \{x_0\}$, each function $f : N_0 \to N'$ allows us to construct a maximal connected non-T_1 expansion of τ following proposition 4.

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