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M-Structure (a survey)

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Abstract: In this talk the basic definitions, some fundamental theorems as well as some directions of applications of M-structure theory have been presented.

1. The basic definitions

The aim of M-structure theory is, roughly speaking, to describe how a given Banach space behaves like a space of continuous functions. This is done by defining objects (subspaces, operators) in arbitrary Banach spaces which in the case of CK-spaces are suitable to characterize these spaces. The fundamental definitions of the theory are due to Cunningham [16] and Alfsen-Effros [1,2].

Definition: Let X be a real Banach space, $J \subset X$ a closed subspace

(i) J is called an M-summand (resp. L-summand) if there is a closed subspace $J^\perp$ such that $X = J \oplus J^\perp$ algebraically and $\|x + x^\perp\| = \max \{ \|x\|, \|x^\perp\|\}$ (resp. $\|x+x^\perp\| = \|x\| + \|x^\perp\|$) for $x \in J$, $x^\perp \in J^\perp$.

(ii) J is called an M-ideal if J^π, the annihilator of J in X', is an L-summand.

(iii) An operator $T: X \to X$ is called a multiplier if for every extreme functional p on X there is an $a_T(p) \in \mathbb{R}$ such that $p \circ T = a_T(p)p$. $Z(X)$ (the centralizer of X) denotes the collection of all multipliers.

Examples:

1. $\{o\}$ and X are always L-summands (M-summands), $Z(X)$ always contains $\mathbb{R}\,\mathrm{Id}$.
2. If $X = CK$, then
   - X contains only $\{o\}$ and X as L-summands
   - the M-summands of X are precisely the subspaces $\{f \mid f \in CK, f|_A = o\} =: J_A$, where $A \subset K$ clopen

- the M-ideals of X are precisely the subspaces J_A, where $A \subset K$ closed
- $Z(X)$ contains just the operators M_h: $f \mapsto hf$, where h runs through X.

3. Let X be the self-adjoint part of a W*-algebra A. Then the M-ideals of X are just the self-adjoint parts of the closed two-sided ideals in A, and the operators in $Z(X)$ are precisely the operators $a \mapsto z a$ $(z = z^*$, z in the center of $A)$. In particular, $[K(H)]_{sa}$ is an M-ideal in $[B(H)]_{sa}$ (H Hilbert space); this has been the first interesting M-ideal in the literature (Dixmier [20]).

2. Some fundamental theorems

Since the beginning of M-structure theory many authors have contributed to this field. Most of the results are concerned with special aspects, and there is no hope to give a survey here (see, however, section 3). There are some frequently used theorems which apply to arbitrary situations, the most important of them (in the author's opinion) are the following:

- The characterization theorem for M-ideals
  Alfsen and Effros [1,2] have shown that it is possible to characterize M-ideals without using the dual space by means of an intersection property:

  A closed subspace J of X is an M-ideal iff $J \cap B_1 \cap B_2 \cap B_3 \neq \emptyset$ for every collection of three open balls B_1, B_2, B_3 such that $B_1 \cap B_2 \cap B_3 \neq \emptyset$, $B_i \cap J \neq \emptyset$ $(i=1,2,3)$.
- The abstract Dauns-Hofmann theorem
  This theorem (which is also due to Alfsen and Effros; cf. also [21]) relates the notions of multipliers and M-ideals. It states that the multipliers correspond to those bounded functions on the extreme boundary of the dual unit ball which are continuous with respect to a topology defined by means of the M-ideals of the space.
- The function module representation theorem. Cunningham [16] has shown that the above example 2, where we described the multiplier of CK-spaces, is typical in the following sense:
  every X can be regarded as a space of vectorvalued functions over a compact Hausdorff space K such that the $T \in Z(X)$ correspond to the multiplication operators associated with the

continuous functions on K .

The L-M-theorem

This theorem (due to the author [3]) states the surprising fact that a space X cannot have nontrivial (i.e. different from $\{o\}$ and X) L-summands and M-summands at the same time.

More generally: If one extends the definition of L-summands and M-summands to those of L^p-summands (where the relevant norm condition is $\|x + x^{\perp}\|^p = \|x\|^p + \|x^{\perp}\|^p$) then for at most one p there can exist non-trivial L^p-summands.

3. Some applications of M-structure theory

I. Approximation theory

On the one hand, M-ideals have interesting approximation theoretical properties (they are proximal in a very strong sense). On the other hand, $K(H)$ is an M-ideal in $B(H)$, and these two facts motivated a number of authors ([15,29,30,31,32]) to approximate operators on a Hilbert space by compact operators. In order to have similar techniques for more Banach spaces the following problem has been of interest: For what Banach spaces X is it true that $K(X)$ is an M-ideal in $B(X)$. This problem is far from being solved, partial answers can be found in [24,26,27,35,36, 45].

One can regard the nice behaviour of $K(H)$ in $B(H)$ also under another viewpoint. Since $B(H)$ is the bidual of $K(H)$ it is interesting to investigate those spaces X such that X is an M-ideal in its bidual. Spaces with this property have been treated in [25,37].

II. M-Structure and L^1-preduals

CK-spaces are the most simple examples of spaces for which the dual is an L^1-space. These spaces have been studied intensively during the last decade, and it is not surprising that M-structure plays a rôle in this theory. Using M-structure methods several authors have been able to give new characterizations for known classes of L^1-preduals ([22,38,39,40]) or to define and investigate interesting new classes ([17,41,42]).

Recently the author [9] has obtained a theorem that states that all L^1-preduals share a certain symmetry property, and the formulation and the proof depend heavily on M-structure methods.

III. Vector-valued Banach-Stone theorems

A Banach space X is called to have the Banach-Stone property if $C(M,X) \cong C(N,X)$ always implies that M and N are homeomorphic (for compact Hausdorff spaces M,N). Clearly $\mathbb{R}$ has the Banach-Stone property by the classical Banach-Stone theorem. Vector-valued Banach-Stone theorems are due to Jerison, Cambern and Sundaresan ([10-14,33,47]). The author has shown that the most far-reaching results can be obtained by using M-structure theory. To be more precise:

III_1: Suppose that $Z(X) = \mathbb{R}\, Id$. Then X has the Banach-Stone property [5]
(already this result contains most of the Banach-Stone theorems which have been obtained without using M-structure theory).

III_2: Suppose that X is a Banach space such that $Z(X)$ is finite-dimensional. Then X can uniquely be written as
$X = \prod_{i=1}^{k} X_i^{r_i}$, where the X_i are pairwise not isometrically isomorphic and have one-dimensional centralizer.
X has the Banach-Stone property iff $\min_{i=1,\dots k} r_i = 1$ [4].
(this generalizes all other Banach-Stone theorems).

III_3: There is a method (described in [6,7,8] by which one can obtain Banach-Stone theorems involved one wants to. The precise formulation (which de[illegible]s on the function module representation theorem) is som[illegible]at lengthy and is therefore omitted here.

III_4: Finally, it has recently be shown in [8] that in a sense every "reasonable" vector-valued Banach-Stone theorem can be proved by using M-structure methods.

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