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THE PRECOMPACTNESS - LEMMA

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There is a lemma on precompactness in duality systems of vector spaces which appeared in a paper of Grothendieck's in 1951 and earlier in 1951 in an even more general but less memorable form in a paper of Kakutani's. Apart from its intrinsic elegance and simplicity I found it always useful to have this lemma in mind when dealing with questions concerning precompactness in locally convex spaces. Though it has found its way into some textbooks ([7], II p. 203 and [4], p. 200 where it is called "le théorème de précompacité réciproque") I observed that it is rarely used, widely unknown, and its basic character in the theory of locally convex spaces is not acknowledged; I actually think it deserves to be taught in an early part of lectures on this topic. In this paper I shall give proofs of two known theorems which became rather simple by having this lemma at one's disposal, namely P. Dierolf's version of the Orlicz-Pettis-theorem on vector measures and Randtke's result on Schwartz-spaces $L(E,F)$. I am grateful to Eric Thomas who drew my attention to the existence of Kakutani's paper during the conference.

1. <u>The lemma.</u> Let E be a (real or complex) vector space and C an absolutely convex subset of E. Then the Minkowski-gauge-functional

$$m_C(x) := \inf \{ \lambda \geqq 0 \mid x \in \lambda C \} \in [0,\infty]$$

defines a semi-norm on span $C = \{x \in E \mid m_C(x) < \infty\}$. A subset $A \subset E$ is called m_C-*precompact*, or simply C-*precompact*, if it is precompact in the semi-normed space $[\![C]\!] := (\text{span } C, m_C)$, i. e.: For every $\varepsilon > 0$ there is a finite set $A_\varepsilon \subset$ span C (which can be even chosen to be in A) such that $A \subset A_\varepsilon + \varepsilon C$. If $\langle E_1, E_2 \rangle$ is a duality system of vector spaces (not necessarily separating), $A \subset E_1$ any subset, and

$$A^\circ := \{y \in E_2 \mid |\langle a,y\rangle| \leq 1 \text{ for all } a \in A\}$$

the absolute polar of A, then the relation

$$m_{A^\circ}(y) = \sup_{a\in A} |\langle a,y\rangle|$$

holds for all $y \in E_2$.

PRECOMPACTNESS-LEMMA (for dual systems): *Let $\langle E_1,E_2\rangle$ be a dual system of vector spaces, $A \subset E_1$ and $B \subset E_2$. Then the following statements are equivalent:*

(1) A *is* B°*-precompact*

(2) B *is* A°*-precompact.*

Proof: Every $b \in B$ operates on A by the duality bracket $\langle\cdot,\cdot\cdot\rangle$ and it is easily seen that

$$B \subset \mathcal{C}(A, m_{B^\circ})$$

(continuous scalar-valued functions) is uniformly equicontinuous and pointwise bounded. The Arzelà-Ascoli-theorem implies that B is precompact with respect to the uniform convergence on A, i.e. with respect to m_{A° by the afore-mentioned relation. ∎

A function $f : E \times F \to \mathbb{K}$ (scalar field) is called *totally bounded* on $A \times B \subset E \times F$ if for every $\varepsilon > 0$ there are finite partitions $A = A_1 \cup \ldots \cup A_n$ and $B = B_1 \cup \ldots \cup B_m$ such that for all $i=1,\ldots,n$ and $j=1,\ldots,m$

$$|\ f(a_1,b_1) - f(a_2,b_2)\ | \leq \varepsilon$$

holds for all $a_1,a_2 \in A_i$ and $b_1,b_2 \in B_j$; in particular: f is uniformly bounded. It is immediate that (1) and (2) together imply that $\langle\cdot,\cdot\cdot\rangle$ is totally bounded on $A \times B$ and vice-versa. So (1) and (2) are equivalent to

(3) *The duality bracket* $\langle\cdot,\cdot\cdot\rangle$ *is totally bounded on* $A \times B$.

If Σ_1 is a cover of E_1 by $\sigma(E_1,E_2)$-bounded sets and τ_{Σ_1} the topology on E_2 of uniform convergence on all $A \in \Sigma_1$ (and Σ_2 a cover of E_2 of the same sort) then the lemma implies that all $A \in \Sigma_1$ are τ_{Σ_2}-precompact if and only if all $B \in \Sigma_2$ are τ_{Σ_1}-

-precompact. This less esthetic and less sharply focussed consequence of the precompactness-lemma was already often used in the literature.
As a first (well-known) application take two dual systems $\langle E_1, E_2\rangle$ and $\langle F_1, F_2\rangle$ and an operator $T: E_1 \to F_1$ with an existing dual $T': F_2 \to E_2$ which means

$$\langle Tx, y\rangle_{F_1, F_2} = \langle x, T'y\rangle_{E_1, E_2}$$

for all $(x,y) \in E_1 \times F_2$. If $A \subset E_1$ and $B \subset F_2$, the following result holds true:

COROLLARY: $T(A)$ *is* $B°$-*precompact if and only if* $T'(B)$ *is* $A°$-*precompact.*

For normed spaces E_1 and F_1 (and their dual spaces), the unit balls $A \subset E_1$ and $B \subset F_2 = F_1'$ this is Schauder's theorem.

Proof: If $T(A)$ is $B°$-precompact, the precompactness-lemma implies that B is $(T(A))°$-precompact. The definition (by covers!) implies that $T'(B)$ is $T'(T(A)°)$-precompact and since $T'(T(A)°) \subset A°$ the result follows. An alternative proof uses the new duality bracket on $E_1 \times F_2$

$$\langle x, y\rangle_{E_1, F_2} := \langle Tx, y\rangle_{F_1 F_2} = \langle x, T'y\rangle_{E_1, E_2}$$

By condition (3) of the lemma $T(A)$ being $B°$-precompact is equivalent to $\langle \cdot, \cdot\cdot\rangle_{E_1, F_2}$ being totally bounded on $A \times B$, which in turn is equivalent to $T'(B)$ being $A°$-precompact. ∎

Precompactness of sets of operators - the idea of collective compactness from the point of view of the precompactness-lemma was studied in [1], including results on the ε-product of locally convex spaces.

2. Kakutani's general version of the lemma. The proof of the lemma actually did not use any linearity arguments (except for the boundedness of the duality-bracket on $A \times B$). So the result holds in the more general situation of two sets X and Y and a bounded map $f: X \times Y \to \mathbb{K}$. Define d_X and d_Y by

$$d_Y(x_1,x_2) := \sup_{y\in Y} | f(x_1,y)-f(x_2,y) | < \infty$$

for all $x_1,x_2 \in X$ and

$$d_X(y_1,y_2) := \sup_{x\in X} | f(x,y_1)-f(x,y_2) | < \infty$$

for all $y_1,y_2 \in Y$ then the proof of the following result is just the same as the one for the special case of dual systems:

PRECOMPACTNESS-LEMMA (general form): *Let* X,Y *be sets and* $f:X \times Y \to \mathbb{K}$ *a bounded function. Then the following are equivalent:*

(1) (X,d_Y) *is a precompact semi-metric space.*

(2) (Y,d_X) *is a precompact semi-metric space.*

(3) f *is totally bounded on* $X \times Y$.

It is rather an exercise to find an elementary argument showing that (1) implies (3), the other implications then being obvious. Hence it is interesting to notice that this lemma easily implies the Arzelà-Ascoli-theorem: To see this take a set X with a precompact uniformity τ and a uniformly equicontinuous, uniformly bounded family $F = \{ f(\cdot,y) \mid y \in Y \}$ of functions on X . Then the identity map

$$(X,\tau) \to (X,d_Y)$$

(d_Y defined as before) is uniformly continuous and whence (X,d_Y) is a precompact semi-metric space; the lemma shows that (Y,d_X) is precompact which readily means that F is precompact in the normed space $(\mathcal{C}_b(X), \|\cdot\|_\infty)$. So the general precompactness-lemma is somehow the core of the Arzelà-Ascoli-theorem - while the lemma for duality systems represents the core of Schauder's theorem.

3. P. Dierolf's Orlicz-Pettis-theorem. Which are the polar topologies τ on a dual system $\langle E_1,E_2\rangle$ such that every $\sigma(E_1,E_2)$-measure is a τ-measure? Reformulated (and generalized to arbitrary infinite index sets I instead of the natural numbers) this is asking the following question: Which are the $\sigma(E_2,E_1)$-bounded sets $B \subset E_2$ such that every $\sigma(E_1,E_2)$-subfamily summable family $(x_i)_{i\in I}$ in E_1 is Cauchy-summable with respect to m_{B° . By a nice observation of

A. Robertson's the latter is equivalent to the $B°$-precompactness of the set

$$\{ \sum_{i \in J} x_i \mid J \subset I \text{ finite} \}$$

Define
$$m_o := \{ \alpha : I \to \mathbb{K} \mid \alpha(I) \text{ finite} \} \subset \ell_\infty(I)$$

$$\ell_1 := \ell_1(I)$$

then it is easy to see that there is a one-to-one correspondence between $\sigma(E_1,E_2)$-subfamily-summable families $(x_i)_{i \in I}$ and $\sigma(m_o,\ell_1)$ - $\sigma(E_1,E_2)$-continuous operators $T : m_o \to E_1$ by

$$T(\alpha) := \sum_{i \in I} \alpha(i)\, x_i$$

$$x_i := T(e_i)$$

(e_i being the characteristic function of the set $\{i\}$; the family $(e_i)_{i \in I}$ is $\sigma(m_o,\ell_1)$-subfamily-summable). Denoting

$$A := \{ \sum_{i \in J} e_i \mid J \subset I \text{ finite} \}$$

the problem is reduced to the question: When is $T(A)$ $B°$-precompact for all $T \in L((m_o,\sigma(m_o,\ell_1)),(E_1,\sigma(E_1,E_2)))$?

This is the setting of the precompactness-lemma (more precisely: the corollary mentioned in 1.): $T(A)$ is $B°$-precompact if and only if $T'(B)$ is $A°$-precompact. But it is immediately checked that $m_{A°}$ and the usual norm are equivalent norms on ℓ_1 . So defining the cover

$$\mathcal{OP} := \{ B \subset E_2 \mid \underset{S \in L((E_2,\sigma(E_2,E_1)),\ (\ell_1,\sigma(\ell_1,m_o)))}{\forall} S(B) \text{ rel. norm-compact} \}$$

some straightforward arguments show the following result of P. Dierolf's.

THEOREM: *Let* $\langle E_1,E_2 \rangle$ *be a dual system and* Σ *a cover of* E_2 *by* $\sigma(E_2,E_1)$*-bounded sets. Then the following statements are equi-*

valent:

(1) $\Sigma \subset \mathcal{OP}$

(2) *Every* $\sigma(E_1,E_2)$ *- subfamily-summable family is* τ_Σ*-subfamily-summable.*

Schur's lemma states that norm-compact = $\sigma(\ell_1,m_o)$-compact in ℓ_1, so applying the Alaoglu-Bourbaki-theorem it is readily seen that all compatible topologies are coarser than $\tau_{\mathcal{OP}}$ hence satisfy (2). To see non-compatible topologies being involved, take a Banach-space G and the dual pair $\langle G',G\rangle$. Then, by the theorem, the subfamily-summable families coincide for $\sigma(G',G)$ and the norm-topology if and only if

$$L((G,\sigma(G,G')),(\ell_1,\sigma(\ell_1,m_o))) = K(G,\ell_1)$$

(compact operators); by the closed graph theorem this is equivalent to $L(G,\ell_1) = K(G,\ell_1)$ - a result of P. Dierolf and Ch. Swartz [3]; the latter condition holds e.g. for $G = c_o$ or a Grothendieck-space. Actually the arguments used apply in more general situations: Take for example G an inductive limit of a sequence of Banach-spaces G_n with $L(G_n,\ell_1) = K(G_n,\ell_1)$ then the subfamily-summable families are the same for $\beta(G',G)$ and $\sigma(G',G)$ (use the fact that every bounded set in G is in the closure of a bounded set in some G_n).

4. Randtke's theorem. A locally convex space is called a Schwartz-space if for every (absolutely convex) neighbourhood U of zero there is another one, V, such that V is U-precompact. If $\langle E_1,E_2\rangle$ is a dual pair and Σ a cover of E_2 by $\sigma(E_2,E_1)$-bounded sets (they can be assumed to be absolutely convex and $\sigma(E_2,E_1)$-closed as well) then $(E_1)_\Sigma := (E_1,\tau_\Sigma)$ is a Schwartz-space (by the precompactness-lemma and the bipolar theorem) if for every $A \in \Sigma$ there is a $B \in \Sigma$ such that A is B-precompact.

Let E and F be separated locally convex spaces (different from $\{0\}$), and Σ a cover of E by bounded, closed, absolutely convex sets; $L_\Sigma(E,F)$ denotes $L(E,F)$, equipped with the topology of uniform convergence on all elements of Σ.

THEOREM: $L_\Sigma(E,F)$ *is a Schwartz-space if and only if* E'_Σ *and* F *are Schwartz-spaces.*

Proof: Since E'_Σ and F can be considered as subspaces of $L_\Sigma(E,F)$ one implication is immediate. Now assume E'_Σ and F being Schwartz and consider the dual system

$$\langle L(E,F)\ ,\ E\otimes F'\rangle\ .$$

The topology of uniform convergence on all $A\in\Sigma$ is the topology on $L(E,F)$ coming from the cover

$$\{\ \overline{\Gamma\ (A\otimes U^\circ)}\ \mid A\in\Sigma\ ,\ U\in\mathcal{U}_F(0)\ \}$$

of $E\otimes F'$ (the closure taken with respect to $\sigma(E\otimes F',\ L(E,F))$ and Γ points at the absolutely convex hull). Now, given $A\in\Sigma$ and $U\in\mathcal{U}_F(0)$ there are, by assumption, $B\in\Sigma$ and $V\in\mathcal{U}_F(0)$ such that A is B-precompact and V is U-precompact — equivalently: U° is V°-precompact. It follows that there is a $\lambda>0$ such that $A\subset\lambda B$ and $U^\circ\subset\lambda V^\circ$ and that for every $\varepsilon>0$ there are finite $A_\varepsilon\subset A$ and $C_\varepsilon\subset U^\circ$ such that

$$A\subset A_\varepsilon+\varepsilon B\quad\text{and}\quad U^\circ\subset C_\varepsilon+\varepsilon V^\circ$$

whence $$A\otimes U^\circ\subset A_\varepsilon\otimes C_\varepsilon\ +\ (2\lambda\varepsilon+\varepsilon^2)\ B\otimes V^\circ\ ;$$

this implies that $\overline{\Gamma\ A\otimes U^\circ}$ is $\overline{\Gamma\ B\otimes V^\circ}$-precompact. Thus $L_\Sigma(E,F)$ is a Schwartz-space. ∎

The proof is quite natural in the sense that it treats $L_\Sigma(E,F)$ somehow as a (tensor)-product of E'_Σ and F .

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