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Geometry of Banach spaces and solvability of nonlinear equations

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Deep characterizations of reflexivity (or weak compactness of subset) of Banach spaces (or locally convex spaces) due to Banach-Bourbaki, Šmulian, James and Mackey are well-known. Further characterizations of reflexivity of Banach spaces have been obtained by means of (i) proximal properties of subsets and subspaces, (ii) separation properties of convex sets, (iii) duality mappings (or support mappings) and differentiability of the norms. We refer the reader to [3], [4], [5], [8], [9], [10] for extensive literature in these topics. Let X be a normed linear space, X^* its dual space, $\langle , \rangle$ a pairing between X^* and X. Let $B_1(0)$ be a closed unit ball in X, τ a canonical mapping of X into X^{**}, $J : X \to 2^{X^*}$ a duality mapping defined by

$$J(u) = \{u^* \in X^* : \langle u^*, u\rangle = \|u\|^2, \|u^*\| = \|u\|\}.$$

It is well known that for each $u \in X$ $J(u)$ is non-empty weakly* compact subset of X^*. The mapping J is single-valued $\Leftrightarrow$ X is smooth (i.e. the $\|\cdot\|$ of X is Gâteaux differentiable on $S_1(0) = \{u \in X : \|u\| = 1\}$) $\Leftrightarrow$ J is continuous from the strong topology of X to the weak* topology of X^* (see [3]). According to the Bishop-Phelps theorem [1] the set of all linear continuous functionals of $S_1^*(0) = \{u^* \in X^* : \|u^*\| = 1\}$ which attain their norms on $S_1(0)$ is norm-dense in $S_1^*(0)$. Hence we shall say that for a given $u_0^* \in S_1^*(0)$ sequences $(u_n^*) \subset S_1^*(0)$, $(u_n) \subset S_1(0)$ have the Bishop-Phelps property if $u_n^* \to u_0^*$ and $\langle u_n^*, u_n \rangle = 1$ for each n.

Theorem 1. Let X be a Banach space.

Then: (i) If X, X^* are both smooth, then X is reflexive if and only if J^{-1} is continuous from the strong topology of X^* into the weak topology of X;
(ii) If X^* is smooth, then X is reflexive if and only if $\tau(B_1(0))$ is sequentially weakly* closed in X^{**}.

Theorem 2. Let X be a Banach space.

Then X is reflexive if and only if the following condition is satisfied: For a given $u_0^* \in S_1^*(0)$ and the sequences $(u_n^*) \subset S_1^*(0)$, $(u_n) \subset S_1(0)$ having the Bishop-Phelps property there exists at least one subnet $(\tau(u_n))_{\alpha \in I}$ of the sequence $(\tau(u_n))$ such that its weak* limit point is a $\sigma(X^*, X)$-continuous functional on X^*.

According to Wulbert [15] we shall say that a normed linear space X admits nearest points if for each closed subset $E \subset X$ the set $\{u \in X :$ there is a point $v \in E$ such that $\|u - v\| = \inf \|u - z\| : z \in E\}$ is dense in X. Wulbert [15] has proved that a Banach space X admits nearest points if either a) X is (2 R)-space of Ky Fan and Glicksberg [7] (in particular uniformly rotund space), or b) X is uniformly smooth and (H)-space.

<u>Definition 1.</u> Let X, Y be normed linear spaces. We shall say that a mapping $G : X \to Y$ has the property (B), if G is onto and there exists a constant $\alpha > 0$ such that for each $v \in Y$ there exists $u \in X$ such that $G(u) = v$ and $\alpha \|u\| \leq \|v\|$.

<u>Definition 2.</u> Let X, Y be normed linear spaces, M an algebraically open subset of X, $F : M \to Y$. We shall say that F has an approximation property (AP) on M if for each fixed $u \in M$ there exists a positively homogeneous mapping $G_u : X \to Y$ having the property (B) such that for given $h \in X$ there is a constant $\delta = \delta(u_o, h) > 0$ such that $0 < t < \delta \Rightarrow$

$$\| F(u_o + th) - F(u_o) - G_u(th) \| \leq \alpha_u \| th \| ,$$

where α_u is a constant from the Definition 1.

Remark 1. Let X, Y be normed linear spaces, $M \subset X$ an algebraically open subset of X, $F : M \to Y$ a mapping having the one-sided Gâteaux differential $V_+ F(u, \cdot)$ for each $u \in M$. If $V_+ F(u, \cdot)$ has the property (B) for each fixed $u \in M$, then F has (AP) on M. In particular, F has this property, when X, Y are both complete, F possesses the Gâuteaux derivative $F'(u)$ on M and $F'(u)$ is onto for each (fixed) $u \in M$.

Theorem 3. Let X, Y be normed linear spaces, $F : X \to Y$ a mapping having the approximation property on X. Moreover, assume that one of the following three conditions is satisfied : (i) Y is a Banach space having the nearest points and $F(X)$ is closed; (ii) Y is reflexive and $F(X)$ is sequentially weakly closed; (iii) F is sequentially weakly continuous, $F(0) = 0$ and $E(c) = \{u \in X : \|F(u)\| \leq c\}$ is relatively weakly compact for each $c \geq 0$.

Then $F(X) = Y$.

Let us remark that the results of Theorem 3 are related to that of Pochožajev [12], [13] Zabrejko and Krasnoselskij [14]. For the further results in so called normal solvability see for instance Browder [2] and Downing and Kirk [6].

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