Open relations

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The letters X and Y will stand for topological spaces, G and H for topological groups, E and F for topological vector spaces over R or C. The letter Γ will stand for a relation in $X \times Y$, or in $G \times H$, or in $E \times F$. Given $A \subset X$, $\Gamma(A)$ denotes the image set $\{y \in Y : (x,y) \in \Gamma \text{ for some } x \in A\}$. Γ is said to be open if the image of each open set is open; Γ is open at a point $y \in Y$ if for each open set U in X, $y \in \Gamma(U)$ implies $y \in \operatorname{Int} \Gamma(U)$. Γ is said to be nearly open if the image of each open set U is nearly open, i.e. $\Gamma(U) \subset \operatorname{Int} \overline{\Gamma(U)}$; Γ is nearly open at $y \in Y$ if for each open subset U of X, $y \in \Gamma(U)$ implies $y \in \operatorname{Int} \overline{\Gamma(U)}$. The set of all points at which Γ is nearly open will be denoted by $NO(\Gamma)$.

Theorem 1. Let Γ be a surjective relation.

(i) If X is second-countable and Y is second category, then $NO(\Gamma)$ is non-empty.

(ii) If X is second-countable and Y is Baire, then $NO(\Gamma)$ is dense in Y.

(iii) If G is separable or Lindelöf, H is second category, and Γ is a subgroup or $G \times H$, then Γ is nearly open.

(iv) If F is second category and Γ is a vector subspace of $E \times F$, then Γ is nearly open.

(v) If E and F are locally convex, F is barrelled, and Γ is a vector subspace of $E \times F$, then Γ is nearly open.

Consider the following condition strictly related to Banach's Open Mapping and Closed Graph Theorems:

(B) If Γ is nearly open, then Γ is open.

Theorem 2. If X is locally compact and Γ is a closed subset of $X \times Y$, then (B) holds.

Γ is said to be separating if for each two points $x_i \in X$ there exist open sets U_i in X containing x_i $(i = 1,2)$ such that $\overline{\Gamma(U_1)} \cap \Gamma(U_2) = \emptyset$. This implies that Γ is an injective relation which is a closed subset of $X \times \Gamma(X)$, and that $\Gamma(X) \in T_2$.

Theorem 3 [10]. If X is Čech-complete and Γ is separating and surjective, then (B) holds.

Theorem 3 partially supports Conjecture from [5].

Corollary [2]. If X is Čech-complete , $Y \in T_2$ and f is a bijection of X onto Y, then (B) holds (for $\Gamma = f$) provided either Y is Čech-complete or f is continuous.

The following two examples answer some questions from [6].

Example 1 [2]. There exists a separable complete metric space X and a continuous nearly open surjection $f : X \to [0,1]$ which is not open.

Example 2 [9]. There exists a separable complete metric space X, a separable second category metric space Y and a nearly continuous nearly open bijection f of X onto Y having a closed graph, which is neither continuous nor open.

Theorem 4 [8,10]. If G is Čech-complete and Γ is a closed subgroup of $G \times H$, then (B) holds.

Some partial results in the direction of Theorem 4 were previously obtained e.g. in [4,3,1] (the last under the assumption that Γ is a continuous function). For locally convex spaces the result is in [7] (proof based on duality theory). Čech--completeness cannot be replaced with completeness in the left (or two-sided) uniformity.

In the following X and Y stand for uniform spaces, and U (V) stand for members of the uniformities on X (resp. Y). Γ is said to be uniformly open if $\forall U\ \exists V\ \forall x,\ \Gamma(U(x)) \supset V(\Gamma(x))$; Γ is uniformly nearly open if $\forall U\ \exists V\ \forall x,\ \overline{\Gamma(U(x))} \supset V(\Gamma(x))$.

Consider the condition

(B′) If Γ is uniformly nearly open, then Γ is uniformly open.

Theorem 5 [10]. If X is uniformly Čech-complete and Γ is a closed subset of $X \times Y$, then (B′) holds.

For complete metric space the result is in [4]. Theorem 5 answers some question from [6].

The notion of uniform Čech-completeness (UČC) is studied in [10]; X is UČC if there exists a countable complete family of uniform covers of X. If X is UČC, then X is complete and (topologically) Čech-complete and paracompact. If X is uniformly locally compact, then X is UČC. If a uniform space X is metrizable, then X is UČC iff X is complete. If a topological space X is Čech-complete and paracompact, then X is UČC with respect to the finest uniformity on X. X is UČC iff there exists a perfect uniformly continuous mapping of X onto a complete metric space. A closed subspace of a UČC space is UČC. Countable product of UČC spaces is UČC. A topological group G is UČC with respect to the two-sided uniformity on G iff G is Čech-complete; G is UČC with respect to the left uniformity on G iff G is Čech-complete and left-complete. Atopological vector space E (over a number field) which is UČC (with respect to the translation-invariant uniformity on E) admits a complete metric.

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