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# Remarks on dimensions of graphs

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# REMARKS ON DIMENSIONS OF GRAPHS

Jiří Vinárek

## 1. Preliminaries

The well-known Dushnik - Miller dimension of partly ordered sets (see [DM]) was shown by Ore([O]) to coincide with the necessary number of linearly ordered factors in a product $\prod L_i$ into which the given poset can be fully embedded. It is a particular case of a characteristic of objects based on representations of products of subdirectly irreducibles.

Recall a definition of a subdirectly irreducible (SI) object for a productive hereditary class $\underline{C}$ of digraphs (i.e. a class closed to categorical products and full subgraphs): A $\underline{C}$-graph (i.e. a digraph $A \in \underline{C}$) is SI iff for every full subgraph $m : A \longrightarrow \prod_{i=1}^{n} A_i$ such that all $p_i m$ are onto ($p_i$ are projections) at least one $p_i m$ is an isomorphism. (This is a special case of the general categorical definition of a SI object - see e.g. [PV].)

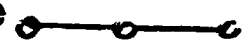
One can see easily that under the assumption of productivity and hereditarity of a class $\underline{C}$, every

$$\underline{C}X = ( \{ R \subset X^2 \ ; \ (X,R) \in \underline{C} \}, \cap )$$

is a complete meet semilattice. A digraph $A = (X,R)$ is called meet irreducible (MI) iff for $R = \bigcap_{i=1}^{n} R_i$ at least one $R_i = R$. One can see easily (cf. [PV]) that every SI is MI.

Now, three types of dimensions based on MI and SI can be defined : Let $A = (X,R)$ be an object of $\underline{C}$. Then a meet dimension $m\text{-dim}_{\underline{C}}(X,R) = \min \{ n \ ; \ \exists R_1, \ldots, R_n, (X_i, R_i)$ are MI for $i = 1, \ldots, n$ and $R = \bigcap_{i=1}^{n} R_i \}$, a product dimension $p\text{-dim}_{\underline{C}} A = \min \{ n \ ; \ A$ is a full subgraph of $\prod_{i=1}^{n} A_i$ with $A_i$ SI$\}$, and a subdirect dimension $s\text{-dim}_{\underline{C}} A = \min \{ n \ ; \ A$ is a full subgraph of $\prod_{i=1}^{n} A_i$ with $A_i$ SI and $p_i m$ onto ($p_i$ are projections, $m$ is an embedding)$\}$, i.e. s-dim is the smallest number of factors in a subdirect representation of A.

Remark. The original Dushnik - Miller dimension was m-dim of posets. The product dimension of graphs was stu[illegible] b[illegible] L.Lovász, J.Nešetřil, A.Pultr etc. (see e.g. [[illegible]], [[illegible]], [$Tr_1$], [$Tr_2$]).

As we mentioned, for $\underline{C}$ a class of [illegible] reflexive posets, there is $p\text{-}dim_{\underline{C}} \equiv m\text{-}dim_{\underline{C}}$ (and also $\equiv s\text{-}dim_{\underline{C}}$). Another example is the class of all the antireflexive antisymmetric digraphs (where $m\text{-}dim_{\underline{C}} \equiv p\text{-}dim_{\underline{C}} \equiv s\text{-}dim_{\underline{C}} \leq$ $\leq 2$). But in the general case, these three dimensions can be different. One can see easily that $p\text{-}dim_{\underline{C}} \equiv$ $\equiv s\text{-}dim_{\underline{C}}$ iff the subdirect irreducibility is hereditary in $\underline{C}$. (An example of non-validity of this equality are bipartite graphs where  is SI but  is not.)

Notation. Denote $\underline{P}$ a class of all the antireflexive posets, $\underline{Q}$ a class of all the digraphs $(X,R)$ such that $\text{card}(R \cap R^{-1}) \leq 1$ and if $(x,y)$ and $(y,x) \in R$ then $\{(x,z), (z,x)\} \cap R \neq \emptyset$ implies $z = x$. (Actually, $\underline{Q}$ contains antireflexive antisymmetric digraphs with possible one isolated loop added.)

## 2. Digraphs

Definition. A class $\underline{C}$ of digraphs is called trivial if every $\underline{C}$-graph has at most one vertex.

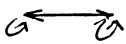
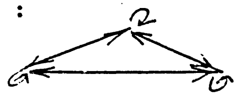
The aim of this chapter is to prove the following

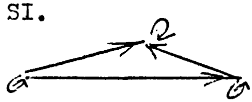
Theorem 1. Let $\underline{C}$ be a productive hereditary class of digraphs. If $p\text{-}dim_{\underline{C}} \equiv m\text{-}dim_{\underline{C}} \equiv s\text{-}dim_{\underline{C}}$ then either $\underline{C}$ is trivial or $\underline{P} \subset \underline{C} \subset \underline{Q}$.

<u>Lemma 1.</u> If s-$\dim_{\underline{C}}$ A $\leq$ m-$\dim_{\underline{C}}$ A for any A $\in$ $\underline{C}$ then $\underline{C}$ contains no digraph with two loops.

<u>Proof.</u> Let G be a maximal $\underline{C}$-graph with two vertices and two loops.

Consider three cases :

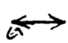
1\. G =  . Then  is in $\underline{C}$, it is MI but it is not SI.

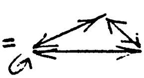
2\. G =  . Then  is in $\underline{C}$, it is MI but it is not SI.

3\. G =  . Then  is in $\underline{C}$, it is MI but it is not SI.

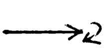
In all these cases an existence of an object which is MI but not SI contradicts the assumption s-$\dim_{\underline{C}}$ $\leq$ m-$\dim_{\underline{C}}$.

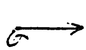
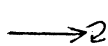
<u>Lemma 2.</u> If s-$\dim_{\underline{C}}$ A $\equiv$ m-$\dim_{\underline{C}}$ A for any A $\in$ $\underline{C}$ then  $\notin$ $\underline{C}$.

<u>Proof.</u> Suppose G =  $\in$ $\underline{C}$. Then

H =  $\in$ $\underline{C}$.

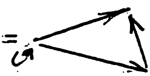
By Lemma 1, H is maximal hence MI. But on the other hand, H is a full subgraph of $G^2$ and therefore it is not SI which is a contradiction.

<u>Lemma 3.</u> If s-$\dim_{\underline{C}}$ $\equiv$ p-$\dim_{\underline{C}}$ $\equiv$ m-$\dim_{\underline{C}}$ then  $\notin$ $\underline{C}$,  $\notin$ $\underline{C}$.

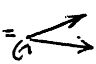
<u>Proof.</u> a/ Suppose G =  , H =  $\in$ $\underline{C}$. Then

$K$ =  is a full subgraph of $G \times H$ hence $K \in \underline{C}$ and it is not SI. But according to Lemma 1 and Lemma 2, $K$ is MI which is a contradiction.

b/ Suppose $G \in \underline{C}$, $H \notin \underline{C}$. Let $K$ be a maximal $\underline{C}$-graph containing $G$ as a full subgraph. (Such a graph exists because e.g.  is a $\underline{C}$-graph containing $G$ as a full subgraph.)

/i/ Suppose $K$ =  . Then $L$ =  is a full subgraph of $K \times G$ hence $L$ is a subdirectly reducible $\underline{C}$-graph. But one can see easily (according to previous lemmas) that $L$ is MI which is a contradiction.

/ii/ Suppose $K$ =  . Then $M$ =  is a full subgraph of $K \times G$ hence $M$ is a subdirectly reducible $\underline{C}$-graph. According to the maximality of $K$ and the assumption $H \notin \underline{C}$, $M$ is MI.

/iii/ Suppose $K$ =  . Then $K$ is MI but it is not SI which is a contradiction.

c/ Using the same technique as in b/ one can prove that also under the assumption $G \notin \underline{C}$, $H \in \underline{C}$ one obtains a contradiction.

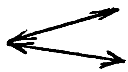
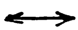
<u>Proposition 1.</u> Let $\underline{C}$ be a productive hereditary class of digraphs. If s-$\dim_{\underline{C}} \equiv$ p-$\dim_{\underline{C}} \equiv$ m-$\dim_{\underline{C}}$, $G = (X,R) \in \underline{C}$

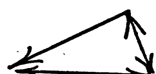
then for every $Y \subseteq X$ such that card $Y = 3$ there is card $(R \cap Y \times Y) \leq 3$.

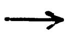
Proof.

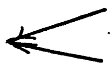
1. If Y contains a loop vertex of G then the assertion follows from Lemmas 2 and 3.

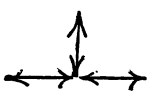
2. Suppose there exists an antireflexive C-graph with 3 vertices and more than 3 edges. Let G be a maximal C-graph with these properties.

a/ G =  . Then H =  is a subgraph of G ×  hence it is meet reducible and K =  is a C-graph. Therefore, L =  =

=  $\wedge$  $\wedge$  $\wedge$ 

is a C-graph, m-dim$_{C}$ L = 4 but L is a full subgraph of G ×  and hence s-dim$_{C}$ L = 2.

b/ G =  . Then again L =  is a C-graph and m-dim$_{C}$ L = 4, s-dim$_{C}$ L = 2.

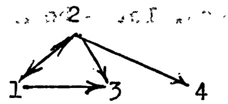
c/ G =  . Then M =  is a full subgraph of $G^2$ hence it is a subdirectly reducible C-graph. But according to the maximality of G, M is also maximal (and hence MI) which is a contradiction.

d/ G =  . Then  is a full subgraph of

$G \times \longleftrightarrow$ hence it is a subdirectly reducible C-graph; it must be also meet reducible and therefore also

 is a C-graph. By a similar technique, one can prove that  is a C-graph. Hence, all the tournaments with 3 vertices are meet reducible in C.

Denote

A = 


Since A is a full subgraph of $G^2$ it is a C-graph and s-$\dim_C$ A = 2. Therefore, m-$\dim_C$ A = 2 and $A = B \wedge C$ .

According to a/, b/, c/ neither B nor C have edges (3,1), (3,2), (4,2).

/i/ Suppose B has both edges (3,4) and (4,3). Then C contains none of edges (3,4) , (4,3) and a subdirectly reducible two-point discrete graph is a full subgraph of C which is a contradiction.

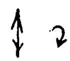
/ii/ Suppose B has only one of edges (3,4) and (4,3). Then B contains a tournament with three vertices as a full subgraph which is a contradiction with the meet reducibility of all the tournaments with 3 vertices.

e/ For the case G =  or G =  one can use a similar technique as in d/. Q.E.D.

<u>Proposition 2.</u> Let <u>C</u> be a productive hereditary class of digraphs. If s-dim$_{\underline{C}}$ ≡ p-dim$_{\underline{C}}$ ≡ m-dim$_{\underline{C}}$ then every <u>C</u>-graph is antisymmetric.

<u>Proof.</u> Suppose the contrary. Then <u>C</u> contains a symmetric graph G with two vertices. By Lemma 1 and Lemma 2, G = $\longleftrightarrow$ . Take a maximal <u>C</u>-graph H with three vertices containing G as a full subgraph.

Consider two cases :

a/ H has a loop. According to Proposition 1, H =  .
Take a maximal <u>C</u>-graph K with 4 vertices containing H as a full subgraph. According to Proposition 1, the fourth vertex of K cannot be connected with both vertices of the symmetric edge by an edge. Hence, H contains a discrete graph $D_2$ with two vertices as a full subgraph which is a contradiction with the assumption p-dim$_{\underline{C}}$ ≡ s-dim$_{\underline{C}}$ (because $\cdot\ \cdot = \longleftrightarrow \times \cdot$).

b/ H has no loop. Then H contains $D_2$ as a full subgraph and it is a contradiction with the assumption p-dim$_{\underline{C}}$ ≡ ≡ s-dim$_{\underline{C}}$. Q.E.D.

<u>Proposition 3.</u> Let <u>C</u> be a productive hereditary class of digraphs. If s-dim$_{\underline{C}}$ ≡ p-dim$_{\underline{C}}$ ≡ m-dim$_{\underline{C}}$ then either <u>C</u> is trivial, or <u>C</u> ⊃ <u>P</u>.

<u>Proof.</u> Suppose that <u>C</u> is not trivial. Since <u>C</u> is productive

and hereditary, it suffices to prove that $\underline{C}$ contains all the antireflexive linear orderings $L_1$, $L_2$, ...

Suppose that there exists an n such that $L_n \in \underline{C}$, $L_{n+1} \notin \underline{C}$. Consider three cases :

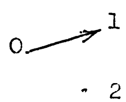
/1/ $n = 0$. Then according to Proposition 2, $\underline{C}$ contains no digraphs with proper edges. Since $\underline{C}$ is not trivial , there are the following possibilities :

a/ $\underline{C} = \underline{SET}$ (the system of all the discrete graphs) . But for $D_3$ (a discrete graph with three vertices ) there is $s\text{-}dim_{\underline{C}}\ D_3 = 2$, $m\text{-}dim_{\underline{C}}\ D_3 = 1$ which is a contradiction.

b/ $\underline{C} = \underline{SET}_{\circlearrowleft}$ ( the system of all the digraphs with at most one loop and with no proper edge ) . But then $s\text{-}dim_{\underline{C}}\ \circlearrowleft\ : =$ $= 2$, $m\text{-}dim_{\underline{C}}\ \circlearrowleft\ : = 1$ which is a contradiction.

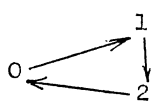
/2/ $n = 1$. Take

G =

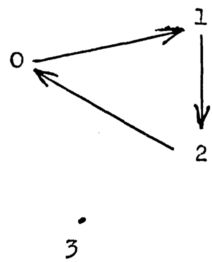


Then $s\text{-}dim_{\underline{C}}\ G = 2 \Rightarrow m\text{-}dim_{\underline{C}}\ G = 2$ . Therefore, there exist IH $\underline{C}$-graphs $H_1, H_2$ such that $G = H_1 \wedge H_2$. We can suppose that $H_1$ has no loop. Then

$H_1$ =



Take

$K =$ 

Then $s\text{-}dim_{\underline{C}}\, K = 2 \Rightarrow m\text{-}dim_{\underline{C}}\, K = 2$ and there exist MI $\underline{C}$-graphs $K_1$ , $K_2$ such that $K = K_1 \wedge K_2$ . We can assume that $K_1$ has no loop. Since $s\text{-}dim_{\underline{C}} \equiv p\text{-}dim_{\underline{C}}$ is supposed and $D_2$ is not SI, $K_1$ has to be a tournament. But there is no tournament with 4 vertices which does not contain $L_2$ as a full subgraph; it is a contradiction.

/3/ $n \geq 2$. Then take $G = L_{n+1} \setminus \{(n-1, n), (n, n+1)\}$ . G is a full sbgraph of $L_n^2$ ; hence, G is a $\underline{C}$-graph and $s\text{-}dim_{\underline{C}}\, G = 2$. Suppose $m\text{-}dim_{\underline{C}}\, G = 2$, $G = G_1 \wedge G_2$ where $G_1$ and $G_2$ are MI $\underline{C}$-graphs. Since $p\text{-}dim_{\underline{C}} \equiv s\text{-}dim_{\underline{C}}$ , neither $G_1$ nor $G_2$ contains $D_2$ as a full subgraph. Every vertex of G is an initial or an end vertex of some edge ; thus, neither $G_1$ nor $G_2$ contains a loop. Hence, $G_1$ and $G_2$ are tournaments. Since $G_1 \neq L_{n+1}$, $(n+1, n-1)$ and $(n+1, n)$ are edges of $G_1$ and $(n-1, n)$, $(n, n+1)$ are edges of $G_2$. Thus, $G_2 = L_{n+1}$ which is a contradiction. Q.E.D.

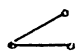
This finishes also the proof of Theorem 1.

## 3. Undirected graphs

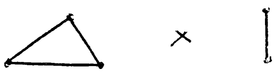
In this part, we are going to study dimensions in subclasses of a class $\underline{G}$ of all the undirected graphs without loops.

Denote $\underline{G}_1$ the system of all the graphs of a degree less or equal to 1.

Proposition 4. Let $\underline{C}$ be a productive hereditary subclass of $\underline{G}$. Then $s\text{-dim}_{\underline{C}}\ A \geq m\text{-dim}_{\underline{C}}\ A$ for every $A \in \underline{C}$ iff either $\underline{C}$ is trivial, or $\underline{C} = \underline{SET}$, or $\underline{C} = \underline{G}_1$ .

Proof. 1. Suppose that  $\in \underline{C}$ or  $\in \underline{C}$.

Since  is a full subgraph of

 ,

in both these cases  $\in \underline{C}$ and $m\text{-dim}_{\underline{C}}\ D_3 = 3$ while $s\text{-dim}_{\underline{C}}\ D_3 = 2$ which is a contradiction. Hence, either $\underline{C}$ is trivial, or $\underline{C} = \underline{SET}$, or $\underline{C} = \underline{G}_1$ .

2\. a/ If $\underline{C} = \underline{SET}$ then $m\text{-dim}_{\underline{C}} \equiv 1$.

b/ $\underline{G}_1$ has only two SI graphs :  and  .

MI $\underline{G}_1$-graphs are just graphs with 2n vertices and n-1 or n edges and graphs with 2n+1 vertices and n edges.

One can see easily that $m\text{-}dim_{\underline{C}}\, A \leq s\text{-}dim_{\underline{C}}\, A$ for every $A \in \underline{C}$.

<u>Theorem 2.</u> Let $\underline{C}$ be a productive hereditary subclass of $\underline{G}$. Then $s\text{-}dim_{\underline{C}} \equiv m\text{-}dim_{\underline{C}}$ iff $\underline{C}$ is trivial.

<u>Proof</u> follows directly from Proposition 4 because <u>SET</u> and $\underline{G}_1$ does not satisfy the condition $s\text{-}dim_{\underline{C}} \equiv m\text{-}dim_{\underline{C}}$.

<u>Theorem 3.</u> Let $\underline{C}$ be a productive hereditary subclass of $\underline{G}$. Then $p\text{-}dim_{\underline{C}} \equiv m\text{-}dim_{\underline{C}}$ iff $\underline{C}$ is trivial.

<u>Proof.</u> Suppose $p\text{-}dim_{\underline{C}} \equiv m\text{-}dim_{\underline{C}}$. Then $m\text{-}dim_{\underline{C}}\, A \leq s\text{-}dim_{\underline{C}}\, A$ for every $A \in \underline{C}$. According to Proposition 4, there are three possibilities :

/i/ $\underline{C}$ is trivial - the assertion holds trivially.

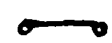
/ii/ $\underline{C}$ = <u>SET</u>. Then $m\text{-}dim_{\underline{C}}\, D_3 = 1$ while $p\text{-}dim_{\underline{C}}\, D_3 = 2$ which is a contradiction.

/iii/ $\underline{C} = \underline{G}_1$. Then $m\text{-}dim_{\underline{C}}$  $= 1$ while $p\text{-}dim_{\underline{C}}$  $= 2$ which is a contradiction. Q.E.D.

For a graph $G$ denote (similarly as in [NP$_1$]) $SP(G)$ the system of all the full subgraphs of $G^k$ where $k$ is a non-negative integer. Denote by $K_n$ the complete (anti-reflexive) graph with $n$ vertices.

<u>Theorem 4.</u> Let $\underline{C}$ be a productive hereditary subclass of $\underline{G}$.

Then s-dim$_{C}$ $\equiv$ p-dim$_{C}$ iff either $C = SET$, or $C = SP(K_n)$ for some n.

Proof. If $C = SET$ or $C = SP(K_n)$ then the assertion holds. If $C \neq SP(K_n)$ then there exists a SI $C$-graph which contains $D_2$ as a full subgraph. Since s-dim$_{C}$ $\equiv$ $\equiv$ p-dim$_{C}$, $D_2$ is SI. Hence, C does not contain  and $C = SET$. Q.E.D.

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